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The Equivalence Myth of Quantum Mechanics—Part II

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To axiomatize a theory is to define a set-theoretical predicate
Patrick Suppes (1957, p. 249).

The author endeavours to show two things: first, that Schrödinger's (and Eckart's) demonstration in March (September) 1926 of the equivalence of *matrix mechanics*, as created by Heisenberg, Born, Jordan and Dirac in 1925, and *wave mechanics*, as created by Schrödinger in 1926, is not foolproof; and second, that it could not have been foolproof, because at the time matrix mechanics and wave mechanics were neither mathematically nor empirically equivalent. That they were is the Equivalence Myth. In order to make the theories equivalent and to prove this, one has to leave the historical scene of 1926 and wait until 1932, when von Neumann finished his magisterial edifice. During the period 1926–32 the original families of mathematical structures of matrix mechanics and of wave mechanics were stretched, parts were chopped off and novel structures were added. To Procrustean places we go, where we can demonstrate the mathematical, empirical and ontological equivalence of 'the final versions of' matrix mechanics and wave mechanics.

The present paper claims to be a comprehensive analysis of one of the pivotal papers in the history of quantum mechanics: Schrödinger's equivalence paper. Since the analysis is performed from the perspective of Suppes' *structural view* ('semantic view') of physical theories, the present paper can be regarded not only as a morsel of the internal history of quantum mechanics, but also as a morsel of applied philosophy of science. The paper is self-contained and presupposes only basic knowledge of quantum mechanics. For reasons of length, the paper is published in two parts; Part I appeared in the previous issue of this journal. Section 1 contains, besides an introduction, also the paper's five claims and a preview of the arguments supporting these claims; so Part I, Section 1 may serve as a summary of the paper for those readers who are not interested in the detailed arguments.

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1. Structures

Before we can prove the mathematical equivalence of two physical theories, we need to be clear about the meaning of 'mathematical equivalence' and of 'physical theory'. We characterise these terms in (a); and we present the mathematical structures of matrix mechanics in (b) and those of wave mechanics in (c), as expounded in Part I, Sections 3 and 4, respectively.

(a) The *structural view* on physical theories, founded by Patrick Suppes, is that a physical theory **T** essentially is a class of set-theoretical structures.[1] The structural view dissolves a number of difficulties that plagued the linguistic view on physical theories of the logical-positivists (Rudolf Carnap *cum suis*). One prominent virtue of the structural view lies in the fact that the classes of set-theoretical structures can be formulated and discussed informally but rigorously. In a time-honoured abuse of language of Tarskian origin a set-theoretical structure $\mathfrak{U} \in \mathbf{T}$ is also called a *model*. Each structure (model) $\mathfrak{U} \in \mathbf{T}$ belongs to the same *species of structure* or *family of structures* or *type of models*, specified by a predicate. What precisely is meant by 'a family of structures' etc. will become sufficiently clear in the applications below. To describe a physical system by theory **T** means constructing an adequate model $\mathfrak{U}$ that qualifies as a member of **T**. Suppose the set $\mathfrak{D}$, called a *data structure*, consists of measurement results of an experiment concerning a particular physical system that **T** is supposed to deal with. **T** is *empirically adequate* iff each relevant data structure is *embeddable* in **T**, which means: isomorphic to some substructure[2] of some $\mathfrak{U} \in \mathbf{T}$. This substructure is then called an *empirical substructure* $[\mathfrak{U}]_{\mathrm{emp}}$ of that model $\mathfrak{U}$. Model $\mathfrak{U}$ has various empirical substructures iff various non-isomorphic data structures are embeddable in $\mathfrak{U}$.

The physical system under consideration is supposed to be an *instantiation* of the stipulated *ontological substructure* $[\mathfrak{U}]_{\mathrm{ont}}$ of $\mathfrak{U}$; the meaning of 'instantiation' is the same as in the assertion that Schrödinger is an instantiation of *homo sapiens*.

The 'mathematical equivalence' or 'formal equivalence' or 'formal identity' or 'identity in content' or 'exact mathematical equivalence' or 'fundamental identity from the mathematical standpoint' or 'strict equivalence' of matrix mechanics and wave mechanics (terms used by the Myth disseminators mentioned in the Part I, Section 1) is now readily construed as the statement that a matrix-mechanical structure and a wave-mechanical structure that describe the same physical system are *isomorphic* [Ludwig (1968, p. 32), Wick (1995,

[1] (Suppes, 1957; Suppes, 1960, Chap. 12); Torretti (1990, pp. 109–130) reports on a voyage through the awesome maze of Balzer *et al.* (1987); Van Fraassen (1991, pp. 4–15) presents a very short introduction. The name 'semantic view' is misleading because it carries linguistic connotations whereas the core idea of the structural view is to achieve a non-linguistic characterisation of scientific theories. Parenthetically, the existence of a class of set-theoretical structures is troublesome when based on standard Zermelo–Fraenkel set theory.

[2] We non-standardly take the notion of 'a substructure of $\mathfrak{U}$' to include '(substructure of) a contraction of $\mathfrak{U}$'; a *contraction (expansion)* of a structure $\mathfrak{U}$ results when slots are deleted from (added to) $\mathfrak{U}$.

p. 26).] Since we shall suppose that the values in the data structures and in the empirical substructures are always suitably gauged, we define the *empirical equivalence* of matrix mechanics and wave mechanics as the identity of their empirical substructures; we define the *ontological equivalence* as the isomorphism of their ontological substructures. The principal set of phenomena that matrix mechanics and wave mechanics were intended to describe, which we therefore need to consider, were the measured frequencies and intensities of the atomic spectra:

$$\mathfrak{D}(\mathrm{E}, I, I') = \{\langle m, n, k, \nu_k(m,n), I_k(m,n)\rangle \in I \times I' \times \mathbb{N}_3 \times \mathbb{R} \times [0, \infty]\}, \quad (1)$$

where the Index-sets $I, I' \subset \mathbb{N}$ of finite cardinality contain the labels of all the measured levels, and E is an Element of the Periodic System. For instance, in the case of the Balmer series we have $I = \{2\}$ and $I' = \{3, 4, 5, 6\}$, where the labels are identified by the putative embeddablility relation as Bohr's principal quantum numbers.

We have to know the matrix-mechanical and wave-mechanical structures precisely in order to consider the purported isomorphism between them. Clearly these structures are not spelled out in the papers of the founding fathers. So we have to reconstruct them from their papers (up until March 1926). Our Postulates of Part I, Sections 3 and 4 make this a fairly straightforward task. The pay-off will be that the holes in the equivalence-proof, as well as suggestions about how to fix some of them, will be staring us in the face.

(b) The Postulates of matrix mechanics (Part I, Section 3) are (intended to be) satisfied by the following mathematical structure:

$$\begin{aligned}\mathfrak{M}'_{d,N} = \langle &\mathrm{Prop}_{\mathrm{mx}}, \mathsf{C}_{\mathrm{mx}}, \mathfrak{A}(\mathsf{C}_{\mathrm{mx}}), \mathcal{F}(\mathsf{C}_{\mathrm{mx}}), \{\mathsf{H}, \mathsf{L}\}, \\ &\partial\mathcal{F}(\mathsf{C}_{\mathrm{mx}}), \partial_t\mathcal{F}(\mathsf{C}_{\mathrm{mx}}), T_{\mathrm{BJ}}, L_{\mathrm{mx}}, \sigma_{\mathrm{mx}}(\mathsf{H}), \Omega_{\mathrm{mx}}, \{\mathsf{E}, \mathsf{B}\}, \mathrm{Const}\rangle. \quad (2)\end{aligned}$$

To single out one member of this family of structures, one needs to specify: the number of particles of the atomic system ($N \in \mathbb{N}$); the number of degrees of freedom ($Nd, d \in \mathbb{N}_3$); the charges and the masses of the N particles and the total charge and mass of the composite system which together constitute the ordered $(2N + 2)$-tuple $\mathrm{Prop}_{\mathrm{mx}} \in \mathbb{R}^{2N+2}$ (the properties of the particles); C_{mx}, which consist of non-denumerably many canonical matrices because of the time-dependence (for each $t \in \mathbb{R}$ there are Nd canonical pairs); the Hamiltonian matrix $\mathsf{H} \in \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$; and the set $\{\mathsf{E}, \mathsf{B}\}$ of the vector–matrix-valued functions that solve matrix-versions of the (Lorentz-covariant!) Maxwell equations (Postulate M7).[3] The problem of how to characterise exactly the elements of the putative matrix algebra $\mathfrak{A}(\mathsf{C}_{\mathrm{mx}})$ (Postulate M1), and the set $\mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ of matrix-valued functions (Postulate M2), is relegated to Section 2. The subset $\{\mathsf{H}, \mathsf{L}\} \subset \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ consisting of the Hamiltonian matrix and the Lagrangian matrix is set apart for emphasis (Postulate M3). The set $\partial\mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ is the set of all first-order partial matrix-derivatives of all matrix-valued functions

[3] Pauli took the Galilean-covariant (!) Coulomb-potential in the Hamiltonian matrix for the hydrogen atom [Part I, Section 3, consequence (m1)], thereby avoiding any contact with these vector-matrix-valued fields.

in $\mathcal{F}(\mathsf{C}_{\mathrm{mx}})$: $2Nd$ partial matrix-derivatives, at a fixed time, for each $\mathsf{F} \in \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$. The set $\partial_t \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ contains all time derivative matrices of all elements of $\mathcal{F}(\mathsf{C}_{\mathrm{mx}})$. The set $T_{\mathrm{BJ}} \subset \mathbb{R} \times \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ contains the solutions of the Born–Jordan equation (I.11) for each $\mathsf{F} \in \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$; a solution is a set of ordered pairs $\langle t, \mathsf{F}(t) \rangle$ for all $t \in \mathbb{R}$ (implied by Postulates M2 and M5). The set $\sigma_{\mathrm{mx}}(\mathsf{H})$ contains the diagonal elements of the diagonalised H (Postulate M3). The set L_{mx} contains the orbital angular momentum values [consequence (m2)], and the set Ω_{mx} contains all ordered quintuples $\langle m, n, k, \nu_k(m, n), I_k(m, n) \rangle$, wherein the frequency and the intensity are defined by Bohr's frequency condition (I.4) and the matrix-mechanical formula (I.6), respectively (Postulates M4 and M6). Finally, the set Const$\subset \mathbb{R}$ is a finite set of physical constants: Planck's constant $\hbar$, Coulomb's constant κ, Sommerfeld's (fine structure) constant α and other required parameters that are not regarded as particle properties (and are therefore not in $\mathrm{Prop}_{\mathrm{mx}}$).

The frequencies and intensities per direction of polarisation and the charges and masses of the electron are measurable physical magnitudes; they were experimentally determined at the time. Thus the substructure

$$[\mathfrak{M}'_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}_{\mathrm{mx}}, \Omega_{\mathrm{mx}}, L_{\mathrm{mx}} \rangle \tag{3}$$

is the empirical substructure of matrix mechanics. The inference ('to the best explanation') that particles really exist was resisted by Heisenberg and Pauli, who doubted in particular 'the reality of particles'.[4] On the other hand, Born regarded the existence of particles as inescapable in the light of the atomic collision experiments that were performed in Göttingen by his friend James Franck; Jordan took Born's side.[5] So it is difficult to decide whether $\mathrm{Prop}_{\mathrm{mx}}$ not in $[\mathfrak{M}'_{d,N}]_{\mathrm{ont}}$ (Heisenberg, Pauli) or $\mathrm{Prop}_{\mathrm{mx}}$ in $[\mathfrak{M}'_{d,N}]_{\mathrm{ont}}$ (Born, Jordan). Since neither choice will undermine any of the claims defended in this paper, and since closer analysis has revealed that neither Heisenberg nor Pauli was an anti-realist (Regt, 1993, pp. 118–121, 116–117), we guardedly put the particle properties in the ontological substructure:

$$[\mathfrak{M}'_{d,N}]_{\mathrm{ont}} = \langle \mathrm{Prop}_{\mathrm{mx}}, \sigma_{\mathrm{mx}}(\mathsf{H}), \Omega_{\mathrm{mx}} \rangle. \tag{4}$$

We have put the set of energy values $\sigma_{\mathrm{mx}}(\mathsf{H})$ in $[\mathfrak{M}'_{d,N}]_{\mathrm{ont}}$ because matrix mechanics took the energies to be as least as fundamental as the frequencies. Heisenberg said in 1972: 'It was extremely important for the interpretation to say that the eigenvalues of the Schrödinger equation are not only frequencies —they are actually energies'. [Heisenberg (1977, p. 269), Beller (1983, pp. 480–481)].

(c) The Postulates of wave mechanics (Part I, Section 4) are (intended to be) satisfied by the following mathematical structure:

[4] Heisenberg (1926, pp. 989, 991), Beller (1983, pp. 475–479), Miller (1986, pp. 139–154). Concerning the claim that Heisenberg was an anti-realist De Regt (1993, p. 118) however declares: 'I submit that closer analysis reveals that this claim is false.'

[5] Beller (1985); on the subject of the unobservable atoms Jordan (1938, p. 91) writes that 'there is no longer any possible doubt concerning their real existence'.

$$\mathfrak{S}'_{d,N} = \langle \mathbb{R}^{Nd}, L^2(\mathbb{R}^{Nd}), \mathbb{E}^3, \mathrm{Prop}_{\mathrm{wv}}, \mathsf{C}_{\mathrm{mx}}, \mathfrak{A}(C_{\mathrm{wv}}), \mathcal{F}(C_{\mathrm{wv}}), \partial\mathcal{F}(C_{\mathrm{wv}}), \{\widehat{H}\}, L_{\mathrm{wv}}, \sigma_{\mathrm{wv}}(\widehat{H}), \{\phi_n\}, \Omega_{\mathrm{wv}}, \{\mathbf{E}, \mathbf{B}\}, \mathrm{Const}\rangle. \quad (5)$$

To single out one family member, one needs to specify: the number of 'particles' of the atomic system ($N \in \mathbb{N}$), the number of spatial degrees of freedom ($d \in \mathbb{N}_3$), the Hamiltonian wave-operator $\widehat{H} \in \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ and the electromagnetic fields $\{\mathbf{E}, \mathbf{B}\}$ involved. The *Euclidean structure* $\mathbb{E}^3$, referring to the spatial canvas which Schrödinger needed in order to paint his picture of the atomic world with smeared matter-charge densities, is defined as:

$$\mathbb{E}^3 := \langle \mathbb{R}^3, T(\mathbb{R}^3), d_{\mathrm{Eucl}}\rangle, \quad (6)$$

where $d_{\mathrm{Eucl}} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty]$ is the Euclidean distance-function, leading to the Pythagorean theorem, and $T(\mathbb{R}^3)$ is the standard topology of open balls, licensing the use of familiar topological concepts like *continuity*, which Schrödinger uses all over the place, *e.g.* Schrödinger's quotation that opens Section I.5. (Actually, $\mathbb{E}^3$ is isomorphic to each spatial slice of the Galilean space–time manifold of classical mechanics.) We henceforth follow the time-honoured tradition of confusing the structure $\mathbb{E}^3$ and its base set $\mathbb{R}^3$; we shall use $\mathbb{E}^3$ to remind ourselves of the fact that it is assumed to represent physical space. The set $\mathrm{Prop}_{\mathrm{wv}} \in (\mathbb{E}^3 \times \mathbb{R})$ contains N+1 charge and N+1 matter densities, thereby extending Postulate W5 to many-electron systems, defined as follows. The charge density $\rho_j : \mathbb{E}^3 \to \mathbb{R}$ for the jth electron is defined by integrating over all $3(N-1)$ configuration-coordinates save the ones referring to the jth electron ($\mathbf{q}_j \mapsto \mathbf{r}$):

$$\rho_j(\mathbf{r}) := e\int_{-\infty}^{+\infty} |\psi(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)|^2 d\mathbf{q}_N d\mathbf{q}_{N-1} \ldots \mathbf{q}_{j+1} d\mathbf{q}_{j-1} \ldots \mathbf{q}_1. \quad (7)$$

The total charge density $\rho_{\mathrm{total}} : \mathbb{E}^3 \to \mathbb{R}$ of the compound system *we* define as:

$$\rho_{\mathrm{total}}(\mathbf{r}) := \sum_{j=1}^{N} \rho_j(\mathbf{r}), \quad (8)$$

in contrast to Schrödinger's original definition of ρ_{total} as the function $e\psi^*\psi : \mathbb{R}^{Nd} \to \mathbb{R}$ defined on configuration space (Schrödinger, 1926e; Schrödinger, 1927, p. 120). And *mutatis mutandis* with regard to the matter densities.[6] The canonical wave-operators $\widehat{P}_k, \widehat{Q}_j \in C_{\mathrm{wv}}$ are unbounded self-adjoint operators. So they cannot be defined everywhere.[7] But they do however have dense domains (Prugovecki, 1981, pp. 224–226, Theorem 4.11). Examples of dense domains on which all elements of the canonical wave-operator algebra $\mathfrak{A}(\mathsf{C}_{\mathrm{mx}})$ are defined are: the set $C_0^\infty(\mathbb{R}^{Nd})$ of all C^∞-functions having compact support, and the Schwartz space $S(\mathbb{R}^{Nd})$ of C^∞-functions that are rapidly decreasing (Emch, 1983, p. 272). The singleton set $\{\widehat{H}\} \subset \mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ is mentioned explicitly

[6] With these definitions we need not waste time on an irrelevant but ineradicable objection to Schrödinger's interpretation, levelled for example by Heisenberg (1926, p. 992), (1929, p. 493), *viz.* that ψ is defined on configuration space rather than on physical space.

[7] Prugovecki (1981, p. 195), Hellinger–Toeplitz Theorem 2.10.

due to the importance of the Hamiltonian wave-operator for the characterisation of the atomic system under consideration (Postulate W3). The set $\partial\mathcal{F}(\mathsf{C}_{\mathrm{mx}})$ is the set of first-order partial wave-operator-derivatives. The set Ω_{wv} contains, just like Ω_{mx}, frequencies and intensities, defined by Bohr's frequency condition (4) and the wave-mechanical formula (I.38), respectively (Postulates W4 and W6). The set *Const* of physical constants and parameters is identical to the one in matrix mechanics.

Schrödinger emphasised that in wave mechanics there is no need to go to matrix-valued electromagnetic fields with non-commuting components or to reformulate 'the Maxwell–Lorentz equations'; on the contrary, wave mechanics meshes quite nicely with classical electrodynamics due to the presence of the charge densities in $\mathrm{Prop}_{\mathrm{wv}} \in \mathfrak{S}'_{d,N}$. Schrödinger declared (1926c; 1927, pp. 59–60):

> the mechanical field scalar [= wave-function] is perfectly capable of entering the Maxwell–Lorentz equations between the electro-magnetic field vectors, as the 'source' of the latter; just as, conversely, the electro-dynamical potentials enter into the coefficients of the wave equation, which defines the field scalar.

This meshing of wave mechanics with classical electrodynamics was the fuel on which Schrödinger was to travel a few miles, for example arriving at a putative explanation of why atoms in stationary states do not radiate [see MacKinnon (1977, pp. 19–20, 24–29) and Mehra and Rechenberg (1987, pp. 797–799)].

Schrödinger's ontological claim that (unobservable) microphysical systems exist in space and consist of smeared charge-matter densities is now expressed succinctly by: the ontological substructure

$$[\mathfrak{S}'_{d,N}]_{\mathrm{ont}} := \langle \mathbb{E}^3, \mathrm{Prop}_{\mathrm{wv}}, \{\phi_n\}, \Omega_{\mathrm{wv}}, L_{\mathrm{wv}} \rangle \tag{9}$$

is *instantiated.* In contradistinction to Heisenberg [see end of part (b)], Schrödinger regarded energy as an 'abstract idea', as a derivative magnitude; fundamental for wave mechanics are the frequencies (Schrödinger, 1926c; Schrödinger, 1927, pp. 59, 141). Schrödinger asked rhetorically: 'Is it quite certain that the conception of energy, indispensable as it is in macroscopic phenomena, has *any* meaning in micro-mechanical phenomena other than the number of vibrations in h seconds?'[8] Hence $\sigma_{\mathrm{wv}}(\widehat{H}) \notin [\mathfrak{S}'_{d,N}]_{\mathrm{ont}}$. The empirical substructure of $\mathfrak{S}'_{d,N}$ is of course:

$$[\mathfrak{S}'_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}_{\mathrm{wv}}, \Omega_{\mathrm{wv}}, L_{\mathrm{wv}} \rangle. \tag{10}$$

To summarise parts (b) and (c): according to the structural view, the physical theories called matrix mechanics and wave mechanics are now defined as follows:

$$\mathbf{MM}' := \left\{ \langle \mathfrak{M}'_{d,N}, [\mathfrak{M}'_{d,N}]_{\mathrm{emp}}, [\mathfrak{M}'_{d,N}]_{\mathrm{ont}} \rangle \mid d \in \mathbb{N}_3, N \in \mathbb{N}, \ldots \right\}, \tag{11}$$

$$\mathbf{WM}' := \left\{ \langle \mathfrak{S}'_{d,N}, [\mathfrak{S}'_{d,N}]_{\mathrm{emp}}, [\mathfrak{S}'_{d,N}]_{\mathrm{ont}} \rangle \mid d \in \mathbb{N}_3, N \in \mathbb{N}, \ldots \right\}, \tag{12}$$

[8] Quoted in MacKinnon (1980, p. 41).

where the dots stand for the Postulates specifying the slots in $\mathfrak{M}'_{d,N}$ (2) and in $\mathfrak{S}'_{d,N}$ (5), respectively. It is not standard to mention ontological substructures explicitly, but it is requisite to achieve a faithful construal of physics, as the present examples of matrix mechanics and wave mechanics bear witness.

Define $\mathfrak{D}_s(\mathrm{H, Balmer})$ as the s-th set of measured frequencies of the Balmer series in the Hydrogen spectrum that was considered to be reliable by the community of atomic physicists at the time; where $s \in \{1, 2, \ldots, S\}$ such that $S \in \mathbb{N}$ is the number of relevant experiments performed at the time. So

$$\mathfrak{D}_s(\mathrm{H, Balmer}) := \{\langle 2, n, k, \nu(2,n)\rangle \in \{2\} \times \{3,4,5,6\} \times \mathbb{N}_3 \times (0,\infty)\}. \quad (13)$$

At the time when the equivalence proof appeared, the entire empirical adequacy of matrix mechanics **MM**′ (11) and wave mechanics **WM**′ (12) resided in the fact that $\mathfrak{D}_s(\mathrm{H, Balmer})$ was embeddable in some $\mathfrak{M}'_{3,1} \in \mathbf{MM}'$ and in some $\mathfrak{S}'_{3,1} \in \mathbf{WM}'$, respectively. Matrix mechanics and wave mechanics were *actually empirically equivalent* in the narrow sense that they were both able to embed one, and only one, type of data structure, which contained precisely *four* frequencies. (At this point in time the Bohr–Sommerfeld model of an atomic system was still empirically superior to both matrix mechanics and wave mechanics.) This fact, the derived quantisation of orbital angular momentum, and the coinciding energy values of a few toy systems (harmonic oscillator, rigid rotator) constituted the mentioned *explanandum* (in Section I.5) for which an equivalence-proof was supposed to furnish the *explanans* (Schrödinger, 1926c; Schrödinger, 1927, pp. 45–46).

We have now characterised matrix mechanics **MM**′ (11) and wave mechanics **WM**′ (12) such as they were in March 1926 in the structural framework with sufficient precision for the purposes of the present paper. For convenience we list the empirical and ontological substructures together:

$$[\mathfrak{M}'_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}_{\mathrm{mx}}, \Omega_{\mathrm{mx}}, L_{\mathrm{mx}}\rangle, \quad (14)$$

$$[\mathfrak{M}'_{d,N}]_{\mathrm{ont}} := \langle \mathrm{Prop}_{\mathrm{mx}}, \sigma_{\mathrm{mx}}(\mathrm{H}), \Omega_{\mathrm{mx}}\rangle, \quad (15)$$

$$[\mathfrak{S}'_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}_{\mathrm{wv}}, \Omega_{\mathrm{wv}}, L_{\mathrm{wv}}\rangle, \quad (16)$$

$$[\mathfrak{S}'_{d,N}]_{\mathrm{ont}} := \langle \mathbb{E}^3, \mathrm{Prop}_{\mathrm{wv}}, \{\phi_n\}, \Omega_{\mathrm{wv}}, L_{\mathrm{wv}}\rangle. \quad (17)$$

2. Equivalence Wrecked

Having defined the mathematical structures of matrix mechanics and of wave mechanics as they were in March 1926, we show that matrix mechanics and wave mechanics are: (a) neither mathematically, (b) nor empirically equivalent (Claims IA and IB); and point out in (c) that the *Anschaulichkeit* of wave mechanics was *not mere 'interpretation' of one of two equivalent 'formalisms'* ('formalism' provisionally identified as the employed class of mathematical structures), but was firmly rooted in the wave-mechanical 'formalism' and was not, and could not be, rooted in the matrix-mechanical 'formalism' (Claim II).

(a) One does not need to be a mathematical genius to see that a matrix-mechanical structure $\mathfrak{M}'_{d,N} \in \mathbf{MM}'$ (11) and a wave-mechanical structure

$\mathfrak{S}'_{d,N} \in \mathbf{WM}'$ (12), both describing the same physical system with the 'same' Hamiltonian etc. are manifestly not isomorphic:

$$\mathfrak{M}'_{d,N} \not\cong \mathfrak{S}'_{d,N}. \tag{18}$$

To begin with, their number of slots (elements) differ. Six further features of the structures evidently cause trouble for constructing a fully-fledged isomorphism.

(i) The time-evolution in matrix mechanics, codified by the solution $T_{\mathrm{BJ}} \in \mathfrak{M}'_{d,N}$ (2) of the Born–Jordan equation, has no counterpart in wave mechanics.

(ii) The substructure consisting of configuration-space, the space of wave-functions and the eigenvibrations:

$$\langle \mathbb{R}^{Nd}, L^2(\mathbb{R}^{Nd}), \{\phi_n\}\rangle, \tag{19}$$

has no counterpart in the matrix-mechanical structure $\mathfrak{M}'_{d,N}$ (2). Matrix mechanics lacks a 'state space'. von Neumann was aware of this. Beller's observation that nothing in matrix mechanics corresponds to the eigenvibrations, is a corollary of the absence of a state space [von Neumann (1932, p. 14, fn 18), Beller (1983, p. 480)]. Finally we quote Bohr from a letter to Kronig (in 1926):

> [...] in the wave mechanics we possess now the means of picturing a single stationary state. In fact, this is the very reason for the advantage which wave mechanics exhibits when compared to the matrix method.[9]

The absence of states in matrix mechanics was not a mathematical oversight of the founding fathers. On the contrary, Heisenberg counted the abolition of such unobservable relics from the old quantum theory, wherein (stationary) states were identified with electron orbits, as a personal victory; he originally even intended to eliminate the 'indirectly observed' electron paths in Charles Wilson's cloud chamber experiments! (Beller, 1983, pp. 479–480, 472). In 1972 Heisenberg reflected upon the formation of concepts in quantum mechanics and admitted [Heisenberg (1977, pp. 268, 269); (cf. Beller, 1983, p. 481)].

> But what was this state of the atom? How could it be described? It could not be described by referring to an electronic oribit. So far it could be described only by stating an energy and transition probablities; but there was no picture of the atom. [...] we did not know what the word 'state' could mean.

(iii) The wave-mechanical Euclidean structure $\mathbb{E}^3$ (3), which is the mathematical structure satisfying all microphysical talk in wave-mechanics about physical space, has no matrix-mechanical counterpart.[10]

(iv) The electromagnetic substructure $\{\mathsf{E}, \mathsf{B}\}$ in $\mathfrak{M}'_{d,N}$ is not isomorphic to its wave-mechanical counterpart $\{\mathbf{E}, \mathbf{B}\}$ in $\mathfrak{S}'_{d,N}$, if only because the components of the latter commute whereas the components of the former do not (Mehra and Rechenberg, 1982, p. 88).

[9] Quoted in Beller (1996, p. 555).

[10] The directions of polarisation, which were recognised in matrix mechanics as 'observable', do not require the full-blooded Euclidean structure $\mathbb{E}^3$.

(v) Schrödinger's confession (Part I, p. 20) about the possible non-existence of wave-operators implies at the very least a hole in the proof of the isomorphism of the canonical algebras $\mathfrak{A}(C_{\text{wv}})$ and $\mathfrak{A}(\mathsf{C}^0_{\text{mx}})$. One may try to cover this hole by the *verbum nauseabilum* 'technicality', but even for those of us having a strong stomach, such a cover-up will ultimately be an obstacle to discerning and appreciating the crucial shift in mathematical perspective that will let us fix this hole once and for all (cf. Section 4).

(vi) There are finitely many canonical wave-operators in C_{wv} ($2Nd$ to be precise), but there are non-denumerably many canonical matrices in C_{mx} due to the time-dependence of the matrices. This makes a bijection, and hence an isomorphism, between C_{wv} and C_{mx} impossible.

These observations establish Claim IA of the mathematical non-equivalence of matrix mechanics and wave mechanics as they stood in March 1926.

(b) Claim IB of empirical non-equivalence,

$$[\mathfrak{M}'_{d,N}]_{\text{emp}} \neq [\mathfrak{S}'_{d,N}]_{\text{emp}}, \tag{20}$$

is grounded in at least the following fact.

Suppose a charge-detector occupies a volume $\Delta \subset \mathbb{R}^3$ and the wave-function of an electron is ψ. Then wave mechanics predicts that a quantity of charge of

$$e\int_{\Delta} \rho(\mathbf{r})d^3r = e\int_{\Delta} |\psi(\mathbf{r})|^2 d^3r \tag{21}$$

will be detected. This value will in general be smaller than e, whereas there is nothing in matrix mechanics to suggest that it predicts anything different from precisely e for every spatial volume Δ. So there is conceivably an *experimentum crucis* that will decide which of the two theories is empirically correct: $\text{Prop}_{\text{mx}} \neq \text{Prop}_{\text{wv}}$ implies inequality (20). If one were to remove the charges and masses Prop_{mx} from $[\mathfrak{M}_{d,N}]_{\text{emp}}$, then matrix mechanics would either be empirically more limited than wave mechanics whenever the wave-mechanical predictions were confirmed, or would empirically stand up whenever the wave-mechanical predictions were refuted. On both counts the conclusion (20) of empirical non-equivalence follows.

So far, of course, charge measurements of electrons had (and have) always resulted in (integer multiples of) the electron charge e, never any other amount; this fact threatens the empirical *adequacy* of wave mechanics. Schrödinger recognised this threat *ab initio*—and ultimately with a vengeance. The threat was to increase after wave mechanics had been enriched with *the time-dependent Schrödinger equation* (in June 1926), which inevitably led to the inevitable spreading of the charge-matter densities over time.

Further support for Claim IB was to come very soon, in the Summer of 1926, from Max Born's wave-mechanical treatment of atomic collision experiments, an issue about which matrix mechanics remained and had to remain mute. Even without Born's 'interpretation of the wave-function', Born's wave-mechanical model ('Born approximation') resulted in an increase in the empirical content of wave mechanics; this increase was not matched by a similar increase in the empirical content of matrix mechanics. [11] In January 1926, *before* Schrödinger's first paper on wave mechanics had appeared, Born and Wiener had already stated: 'The matrix analysis fails in the case of a-periodic motions' (Born and Wiener, 1926, p. 214, my translation).

(c) On the one hand, Schrödinger drew the strongest possible conclusion from his equivalence proof [Schrödinger (1926c),Schrödinger (1927), my italics].

> If the two theories—*I might reasonably have used the singular*—, should be tenable in the form just given, i.e. for more complicated systems as well, then every discussion of the superiority of the one over the other has only an illusory object, in a certain sense. *For they are completely equivalent from the mathematical point of view, and it can only be a question of the subordinate point of convenience of calculation.*

But on the other hand Schrödinger pointed out that the distinction 'mathematical equivalence' and 'physical equivalence' would collapse if one were to regard 'the task of physical theory as being merely a mathematical description (as economical as possible) of the empirical connections between observable quantities [. . .]' Schrödinger (1926c; 1927, p. 58); he rejected this task as being too shallow an aim for physics. Schrödinger then proceeded to argue in favour of the superiority of wave mechanics as a *physical* theory: wave mechanics is *anschaulich*, it paints figurative pictures on space–time of atomic reality, justifying the standard talk of physicists about atomic processes [Van der Waerden (1967, pp. 33, 231), Miller (1986, pp. 143–144), Mehra and Rechenberg (1987, pp. 772–804)] and wave mechanics harmonises perfectly with classical electromagnetic field theory. [12] Surely this is not 'only a question of the subordinate point of convenience of calculation'! It seems as if Schrödinger felt trapped in a version of G. E. Moore's Paradox, by thinking: matrix mechanics and wave mechanics are equivalent but I don't believe it. [13] (The fact that matrix mechanics and wave mechanics were not equivalent dissolves the paradox.)

Heisenberg declared the programme for quantum mechanics to be one 'in which only relations between observable quantities occur' [Heisenberg (1925) in Van der Waerden (1967, p. 261)]. At the time Heisenberg and Pauli were on a crusade to eliminate the unobservable electron orbits and their revolution-periods from atomic theory, because they had convinced themselves that clinging to *anschauliche Bilder* of atomic reality was responsible for the lack of theoretical success in atomic physics. [14] The fact that the matrices $\mathsf{P}_k(t)$ and $\mathsf{Q}_j(t)$ were still called 'the quantum analogues of momentum and of position', respectively, and the Born–Jordan equation was still referred to as an 'equation of motion', was at best a way of paying one's last respects to the sinking

[11] Born wanted to describe asymptotic scatter states as free particles. Pencil and paper will illustrate the problems that crop up whenever one tries to describe a free particle in matrix mechanics: how to relate the apparently continuous position of an electron in an accelerator beam to a constant, discrete matrix Q?

[12] As mentioned in **(c)**, Section 1; cf. Hanson (1963, pp. 113–114, 119).

[13] Moore's Paradox: It is raining but I don't believe it.

[14] Regt (1993, pp. 116–121); compare Miller (1986, pp. 139–154).

ship of classical physics—because these *discrete* matrices were, in logical-positivistic jargon, *theoretical terms*, not supposed to correspond to anything in atomic reality. None of this in wave mechanics. Position and momentum were still intended by Schrödinger to be *continuous* attributes of the smeared electron, no matter how difficult this proved to maintain on closer inspection. Physical reality according to wave mechanics was comparable to physical reality according to classical electrodynamics, the crucial difference being that electrons are neither point-particles nor tiny charged rigid balls moving in space and obeying the classical equation of motion, but are tiny jelly-like lumps of vibrating charged matter, described by wave-packets moving in space and obeying an appropriate wave-mechanical equation of motion [Schrödinger's time-dependent wave equation, discovered in June 1926 (Moore, 1989, pp. 217–218)]. Matrix mechanics abolished an *anschauliche* ontology for atomic reality, but wave mechanics provided one. Matrix mechanics proclaimed an ontological revolution, wave mechanics propounded a refinement of the classical space–time ontology. Bearing these insights in mind, let us return to our structural characterisations.

The mathematical structure satisfying Schrödinger's description of atomic reality is

$$[\mathfrak{S}'_{d,N}]_{\text{ont}} := \langle \mathbb{E}^3, \text{Prop}_{\text{wv}}, \{\phi_n\}, \Omega_{\text{wv}}, L_{\text{wv}} \rangle. \tag{22}$$

Thus all Schrödinger's talk about atomic reality in terms of the charge-densities in space and time is, whenever appropriately formalised, *just as rigorously made true (satisfied), in the model-theoretic (logical) sense, by any wave-mechanical structure* $\mathfrak{S}'_{d,N} \in \mathbf{WM}'$ *as is his talk about wave-functions and wave-operators.* The matrix-mechanical structure $\mathfrak{M}'_{d,N} \in \mathbf{WM}'$ on the other hand does not satisfy such microphysical talk and its creators did not intend it to. A wave-mechanical sentence like (Tomonaga, 1966, p. 62):

> The radial charge density at distance a_0 (Bohr radius) from the centre of the hydrogen atom in the ground-state equals $ea_0^2/\exp[1]$,

(*)

is not even a well-formed linguistic formula in the formal language of, and hence not made true by, an appropriate mathematical structure $\mathfrak{M}'_{3,1} \in \mathbf{MM}'$.

N. R. Hanson construed the Schrödinger–Eckart equivalence proof as having established 'the intertranslatability' of matrix mechanics and wave mechanics and thereby replaced, so to speak, the Equivalence Myth with the Intertranslatability Myth, which is widely believed in circles of professional historians of quantum mechanics [Hanson (1963, pp. 132–133), Jammer (1966, p. 271), MacKinnon (1980, pp. 14, 19, 48), Wessels (1980, pp. 61–71), Wessels (1981, p. 192)] But the equivalence proof does not provide a clue about how to translate a sentence like (*) into matrix-mechanical language. As another counterexample to the intertranslatability, take a sentence pertaining to atomic collision experiments which were performed at the time in Göttingen:

$$\text{'The scattered electron is detected in spatial region } \Delta.\text{'} \tag{23}$$

In wave mechanics the existence of a spatial region Δ is taken to be a consequence of the instantiation of the Euclidean structure $\mathbb{E}^3$ (6). But in matrix mechanics there is only 'the quantum-mechanical analogue of classical position' (whatever that means): one infinite matrix Q_k per Cartesian component k, which is a denumerable set of complex numbers. How should $\Delta \subset \mathbb{E}^3$ be related to $\mathsf{Q}_j(t)$? The answer is again that any relevant matrix-mechanical structure $\mathfrak{M}'_{3,1}$ simply does not satisfy sentences such as (23) whenever they are appropriately formalised. The purported element of physical reality *spatial volume* Δ has no counterpart in matrix mechanics. So small wonder that Heisenberg preferred to call $|Q_k(m,n)|^2$ 'radiation value tables', in order to avoid evoking any associations with *position* [Heisenberg (1926, p. 990), cf. Beller (1983, p. 482), Beller (1985, p. 340), d'Abro (1939, pp. 822–824)]. And even after the community of atomic physicists had embraced 'the statistical interpretation',[15] an assertion like 'the probability of finding an electron, having wave-function ψ, in region $\Delta \in B(\mathbb{E}^3)$ is 0.35', expressed in wave-mechanical language as (symbolically):

$$\text{'}P_\psi(\Delta) = \int_\Delta |\psi(\mathbf{q})|^2 d^3q = 0.35\text{'}, \tag{24}$$

remained inexpressible in matrix-mechanical language. Initially Heisenberg even repudiated the Born–Pauli position probabilities as being incoherent in matrix mechanics. For how could a probability measure over positions be defined if positions themselves are not defined, asked Heisenberg in a letter to Pauli (February 1927) (Beller, 1985, p. 342). Heisenberg had hit the nail on the head! A similar story can be told for momentum. This should not come as a surprise, because in the pre-von Neumann era, one simply lacked the mathematical resources to say in matrix-mechanical language what sentences like (23) and (24) express in wave-mechanical language.

Finally, we draw attention again to the fact that a state space is absent in matrix mechanics and that eigenvibrations are present in the structure of wave mechanics. We then see that Schrödinger was also hitting a nail on the head when he emphasised that the wave-functions 'do not form, as it were, an arbitrary and special "fleshy clothing" for the bare matrix skeleton, provided to pander to the need for *anschaulichkeit*'. (Schrödinger, 1927, p. 58, corrected translation).

To conclude, the ontological differences between matrix mechanics and wave mechanics are mathematically codified in their ontological substructures, as defined in (15) and (17), which are manifestly not isomorphic:

$$\langle \mathrm{Prop}_{\mathrm{mx}}, \sigma_{\mathrm{mx}}(\mathsf{H}), \Omega_{\mathrm{mx}} \rangle \not\cong \langle \mathbb{E}^3, \mathrm{Prop}_{\mathrm{wv}}, \{\phi_n\}, \Omega_{\mathrm{wv}}, L_{\mathrm{wv}} \rangle. \tag{25}$$

Thus the *anschaulichkeit* of wave mechanics was firmly anchored in all wave-mechanical structures $\mathfrak{S}'_{d,N} \in \mathbf{WM}'$ (12) and was not, and could not be, anchored in the matrix-mechanical structure $\mathfrak{M}'_{d,N} \in \mathbf{MM}'$ (12). This is the content of Claim II.

[15] First conceived by Born (1926)

3. The Moment Problem

In challenging the equivalence proof (Section 2) we have standardly taken 'mathematical equivalence' to mean 'isomorphism'. This rendition of 'mathematical equivalence' encapsulated various features of Schrödinger's proof: the purported isomorphism f_Φ (I.25) between the canonical matrix algebra and the canonical wave-operator algebra, the ensuing identical non-commutative structure of the two algebras, and the coinciding energy values, leading to the identity of the calculated frequencies of the atomic spectra. But in our analysis of Schrödinger's equivalence proof (I.5) we also encountered the mapping M (I.39) from the collection Φ_{wv} for all wave bases to the collection $\{(\mathsf{C}^0_{\mathrm{mx}})\}$ of all canonical matrix algebras, each element of which is putatively generated by a distinct canonical pair $\{\mathsf{P},\mathsf{Q}\} =: \mathsf{C}^0_{\mathrm{mx}}$ (remember that N and d were both set equal to 1 in Part I, Section 5). Although we learned to understand, by immersion in wave mechanics, that requiring the bijectivity of mapping M was a prerequisite for the desired equivalence, mapping M was omitted from our criticism on the basis of the standard definition of mathematical equivalence as isomorphism. We now remedy this omission by introducing the notion of *Schrödinger(S)-equivalence*, which charitably (but *stricto sensu* illegitimately) ignores the equivalence-wrecking features that Schrödinger also ignored and includes only those features that are explicitly aimed at by Schrödinger.

Definition. Matrix mechanics and wave mechanics are *S-equivalent* iff

$$\begin{gathered}(\mathsf{C}^0_{\mathrm{mx}}) \simeq (C_{\mathrm{wv}}),\ \sigma_{\mathrm{mx}}(\mathsf{H}) = \sigma_{\mathrm{wv}}(\widehat{H}),\\ \Omega_{\mathrm{mx}} = \Omega_{\mathrm{wv}},\ \text{mapping } M \text{ is bijective.}\end{gathered} \tag{26}$$

S-equivalence is weaker than the fully-fledged isomorphism in the sense that it requires some but not all elements (*albeit* the salient ones from an empirical point of view) in each of the wave-mechanical and matrix-mechanical structures to be isomorphic or identical; but S-equivalence is stronger than the fully-fledged isomorphism in the sense that it requires the bijectivity of mapping M (I.39). Because of Schrödinger's failure to demonstrate the algebraic isomorphism, the demonstration of S-equivalence fails too. But let us ignore *that* for the moment. This leaves us with the bijectivity of mapping M (I.39).

The invertibility of M was guaranteed, Schrödinger claimed in the last quotation of Part I, Section 5, by the unique solvability of the following denumerable system of Riemann-integral equations ('Schrödinger's moment problem'):

$$(\mathsf{Q}^n)_{jk} = \int_{-\infty}^{+\infty} q^n u_j(q) u_k(q)\,\mathrm{d}q, \tag{27}$$

where Schrödinger requires the functions u_j to be real, positive, twice differentiable everywhere and vanishing (asymptotically) for large $|q|$ (cf. Part I, Section 5). We argue that: (a) Schrödinger's appeal to the moment problem to prove the bijectivity of mapping M was in vain (claim IIIA), which makes the conclusion of S-equivalence a *non sequitur*; and that (b) by an appeal to von Neumann's unitary uniqueness theorem one can, in the territory charted by von

Neumann (1932), safely drive home the bijectivity of mapping M (claim IIIB), which parries the *non sequitur* charge.

(a) Before going into Schrödinger's moment problem, we observe that the mutual orthogonality of the u_j's (and the coincidence of the eigenvalues of H and $\widehat{H}$) is an immediate corollary of the fact that in the diagonal representation of H we have $f_u(\widehat{H}) = \mathsf{H}$, and therefore the mutual orthogonality need not be assumed, as Schrödinger did. An alternative way of establishing the orthogonality of the functions u_j would be to define $\mathsf{Q}^0 := 1$ and $\hat{Q}^0 := \hat{1}$.

The so-called 'moment problem' to which Schrödinger appealed, goes back to the 'Stieltjes power moment problem', formulated and solved in 1895 by the Dutch–French mathematician Thomas-Jan Stieltjes. The problem is to solve the following denumerable system of Stieltjes-integral[16] equations ($n \in \mathbb{N}$):

$$\int_0^{+\infty} x^n \, \mathrm{d}g(x) = a_n, \tag{28}$$

where $a_n \in [0, \infty]$ are given, and where the solution $g : \mathbb{R} \to \mathbb{R}$ is required to be a real non-decreasing function.[17] Schrödinger's problem (I.43) resembles a non-trivial modification of Stieltjes' power moment problem (28) from $[0, \infty]$ to $(-\infty, +\infty)$, propounded in 1920 by the German mathematician Hans Hamburger. 'Hamburger's power moment problem' concerns the following system of Stieltjes-integral equations ($n \in \mathbb{N}_0$, $s_n \in \mathbb{R}$; without loss of generality: $s_0 := 1$):

$$\int_{-\infty}^{+\infty} x^n \, \mathrm{d}g(x) = s_n, \tag{29}$$

where g is a real non-decreasing function on $\mathbb{R}$ having an infinite number of points of increase [$g(a) < g(b)$ for every interval (a, b) which contains such a point]. Whenever g is differentiable on $\mathbb{R}$, the system (29) can be written as a system of Riemann-integrals:

$$\int_{-\infty}^{+\infty} x^n g'(x) \, \mathrm{d}x = s_n. \tag{30}$$

Hamburger's Theorem (proved in 1920) states: the system (30) is solvable iff the Hankel matrix S, defined as $S_{jk} := s_{j+k}$, is positive definite (Akhiezer, 1965, p. 30, Theorem 2.1.1). Various sufficient and necessary conditions are known for the *unique* solvability of Hamburger's power moment problem (29), but they are rather cumbersome to apply (Akhiezer, 1965, pp. 41, 50, 64, 83, 84, 85, 87, 88). Schrödinger's power moment problem (I.43) becomes an instance of Hamburger's power moment problem (30) whenever we make the identifications: $s_n = (\mathsf{Q}^n)_{jk}$ and $g' = u_j u_k$. The latter identification is allowed for two reasons. First, since $u_j(x)u_k(x)$ vanishes for $|x| \to \infty$ and is differentiable everywhere,

[16]The concept of the Stieltjes-integral was going to pervade von Neumann's (1932) edifice; in modern expositions, like Prugovecki's (1981), it is replaced with the all-embracing Lebesgue-integral.

[17]Akhiezer (1965, p. v); the function g can only be determined up to an additive constant, due to the nature of the Stieltjes-integral. In general, setting the function g in $\mathrm{d}g(x)$ equal to the identity on $\mathbb{R}$ reduces the Stieltjes-integral to the Riemann-integral.

$u_j u_k$ is bounded and continuous, and therefore g' is Riemann-integrable by virtue of a theorem due to Lebesgue (proved in 1902) (Chae, 1995, p. 41, Theorem 5.1) If g' were not Riemann-integrable, then the Riemann-integral in equation (30) would not exist. Second, the differentiability and positivity of $u_j(x)u_k(x)$ everywhere imply that g exists and is non-decreasing everywhere, which is a condition under which Hamburger's power moment problem is defined. From the quotation in Part I, Section 5 we see that Schrödinger focuses on the case where $j = k$, such that we have to make the identifications $g' = u_j^2$ and $s_n = (\mathsf{Q}^n)_{jj}$; for each $j \in \mathbb{N}$ we have a new g' to determine from the sequence of jth diagonal elements of $\mathsf{Q}, \mathsf{Q}^2, \mathsf{Q}^3, \ldots$

We claim (IV) that Schrödinger's appeal to the power moment problem (30), the only extant candidate problem to which Schrödinger could have appealed (as far as the author has been able to establish) is in vain, for the following reasons.

First, the restriction to *positive* real wave-functions, to let u_j^2 determine u_j and to meet the requirement $g'(x) \geq 0$ (*vide supra*), is unacceptable. The wave-mechanical solutions of the problems that Schrödinger had solved himself (hydrogen atom, harmonic oscillator, rigid rotator with free and with fixed axes) were *not* positive wave-functions. For example, the solutions of the harmonic oscillator problem in one dimension were $\exp[-x^2/2]H_n(x)$, where H_n are the Hermite polynomials (Schrödinger, 1926b; Schrödinger, 1927, p. 31).[18]

Second, the restriction to *real* wave-functions, necessary to make contact with the power moment problem, is unacceptable too. For example, Schrödinger's solution of the problem of the rigid rotator with free axes contains spherical harmonics, which are *complex* functions (Schrödinger, 1926b; Schrödinger, 1927, p. 35). Subsequent developments in wave mechanics, in particular the discovery of the time-dependent wave equation, revealed that being complex is the generic case for wave-functions. A similar problem is that Q^n is in general complex, so s_n too is in general complex, whereas the power moment problem requires s_n to be real. (Pencil and paper will convince the reader that splitting a 'complex moment problem' in real and imaginary parts in the hope of reducing it into two real moment problems is of no avail, due to the identification $g' = u_j u_k$.)

Third, Schrödinger's moment problem (I.43) requires the functions u_j to fall off faster than any power in order to prevent the integrals from diverging. Not all solutions of the time-independent wave equation comply with this requirement. For example, the asymptotic scatter states which Born found in the summer of 1926 were of the form (in spherical coordinates): $\exp[ikz] + f_k(\theta, \varphi)\exp[ikr]/r$, which do not even fall off faster than r^{-1}, and by implication not faster than *any* power.

Fourth, suppose that Schrödinger's moment problem (I.43) is uniquely solvable and has solutions $\{u_j\}$. Then the momentum matrix P and its powers P^n, which matrix mechanics also has on offer, should be such that the u_j satisfy:

$$(-i\hbar)^n \int_{-\infty}^{+\infty} u_j(x) \frac{\partial^n u_k(x)}{\partial x^n}\, \mathrm{d}x = (\mathsf{P}^n)_{jk}, \tag{31}$$

[18] Schrödinger accepted a restriction that excludes all of his own results so far.

for all $j, k \in \mathbb{N}$, and at least for $n = 1$. Schrödinger should have averted this *prima facie* threat of overdetermination by appealing to the only available matrix-mechanical facts, namely that P is Hermitian and is canonically conjugated to Q. Schrödinger left it lurking in the dark.

Fifth, it is an open question whether all Hankel matrices built from the rows of Q and its powers Q^n, the problem that their elements are in general complex being set aside, are positive definite, which according to Hamburger's Theorem is necessary for the Hamburger power moment problem to be solvable.

We conclude that Schrödinger's appeal to the moment problem in order to establish the bijective nature of the mapping M (I.39) is in vain (claim IIIA). Consequently, concluding that matrix mechanics and wave mechanics are S-equivalent is a *non sequitur*.

(b) Let us drop Schrödinger's restriction of u_j to real, positive, twice differentiable wave-functions and return to $L^2(\mathbb{R})$ and forget about the moment problem. We next show that the mapping $M : \Phi_{\mathrm{wv}} \to \{(\mathsf{C}^0_{\mathrm{mx}})\}$ (I.39) is bijective (up to Lebesgue-equivalence). This is Claim IIIB. We surreptitiously enter von Neumann's (1932) edifice so that we feel free to appeal to Hilbert space; and we confine ourselves again to the case $N = 1$ and $d = 1$.

Proof. Theorem: all bases of a Hilbert space are related by a unitary transformation; and conversely each unitary transformation turns a basis into a basis (Prugovecki, 1981, p. 215, Theorem 4.4). So Φ_{wv} is equinumerous (of equal cardinality) to the set $\mathcal{U}[L^2(\mathbb{R})]$ of all unitary wave-operators. Consider a canonical pair of Hermitian matrices $\langle \mathsf{P}, \mathsf{Q}\rangle$. The set of all such pairs is obviously equipollent to $\{(\mathsf{C}^0_{\mathrm{mx}})\}$. Born and Jordan showed[19] that every unitarily transformed canonical pair of Hermitian matrices is a canonical pair of Hermitian matrices too (elementary exercise). Enter von Neumann, for the converse (not a particularly elementary exercise) is precisely the content of his unitary-uniqueness theorem, which states, formulated in all generality, that all irreducible solutions of the canonical commutation relations for linear, self-adjoint Hilbert-space operators are related by a unitary transformation [von Neumann (1931); see also Prugovecki (1981, pp. 342–347), Emch (1983, pp. 336–337)].[20] So $\{(\mathsf{C}^0_{\mathrm{mx}})\}$ is equipollent to the set $\mathcal{U}_{\mathrm{mx}}$ of all unitary matrices—conceived as sequence-operators, to anticipate the next Section. Thus it is sufficient to prove that the set $\mathcal{U}[L^2(\mathbb{R})]$ of all unitary wave-operators is equipollent to the set $\mathcal{U}_{\mathrm{mx}}$ of all unitary matrices.

Consider the Schrödinger–Eckart mapping f_ϕ (I.25), but now $\mathcal{U}[L^2(\mathbb{R})] \to \mathcal{U}_{\mathrm{mx}}$ (any choice for the basis $\{\phi_n\}$ will do since unitary wave-operators are bounded). For each unitary wave-operator $\hat{U} \in \mathcal{U}[L^2(\mathbb{R})]$, there is one matrix $f_\phi(\hat{U}) \in \mathcal{U}_{\mathrm{mx}}$ (up to Lebesgue-equivalence) which is provably unitary. So we only have to show now that, conversely, for each unitary matrix $\mathsf{U} \in \mathcal{U}_{\mathrm{mx}}$ there exists one unitary wave-operator $\hat{U} \in \mathcal{U}[L^2(\mathbb{R})]$. The existence is guaranteed by the inverse of the Schrödinger–Eckart mapping $f^{\mathrm{inv}}_\phi(\mathsf{U})$ (I.30) whenever in-

[19] Section 3, consequence (m4).

[20] Since the theorem shows there are just as many solutions of the canonical commutation relations as there are real numbers, the author finds the standard name 'uniqueness theorem' misleading.

equality (I.32) holds for all Schmidt-sequences. The question whether inequality (I.32) holds is the same as asking whether $\|\mathsf{Uc}\| < \infty$, where c is an arbitrary Schmidt-sequence. The answer is in the affirmative because multiplication by a unitary matrix preserves the norm: $\|\mathsf{Uc}\| = \|\mathsf{c}\| < \infty$. The obtained wave-operator $f_{\Phi}^{\text{inv}}(\mathsf{U})$ is unitary. Hence $\mathcal{U}_{\text{mx}}$ and $\mathcal{U}[L^2(\mathbb{R})]$ are equipollent. QED

4. Matrices as Operations

In our analysis of Schrödinger's equivalence proof (Part I, Section 5) we have encountered two major mathematical problems concerning the canonical matrices: what conditions they have to obey for corresponding wave-operators to exist (Problem 1); and what conditions they have to obey in order to generate an algebra (Problem 2).

In Part I, Section 1 we mentioned two sacred texts of quantum mechanics, one by Dirac (1930) and the other by von Neumann (1932); both promulgated the state-observable characterisation of quantum mechanics despite their remarkable differences. Spelling out the precise relation of matrix mechanics and wave mechanics to Dirac's quantum mechanics would be a project on its own. The difficulties of formulating a mathematically tractable version of Dirac's quantum mechanics that does justice to all its aspects are quite formidable; *contra* popular belief, the introduction of Laurent Schwartz's concept of a *distribution* in 1949 to define Dirac's 'improper δ-function' properly, and the rigging of Hilbert space to provide plane waves with a quantum-mechanical passport, do not resolve all difficulties. They resolve a number of them but leave others unresolved, e.g. how to define $\langle\delta_a|\delta_b\rangle$, where δ_a is Dirac's delta distribution. The most penetrating mathematical exploration to date into Dirac's quantum mechanics is the monograph *A Mathematical Introduction to Dirac's Formalism* by Eijndhoven and De Graaf (1986). The lesson to be drawn from this monograph is that a mathematically decent version of Dirac's quantum mechanics is far more intricate than von Neumann's edifice, a fact that is veiled superbly by Dirac's elegant *notation*. Into von Neumann's edifice we now stride.

The shift in mathematical perspective we have alluded to a few times, consists in seeing an infinite matrix as a partial specification of a linear operator acting in the Hilbert space $l^2(\mathbb{N})$ of Schmidt-sequences. In Part I, Section 5 we mentioned that in 1925 Kornel Lanczos made the first attempt to connect matrix mechanics to Hilbert's theory of quadratic forms and integral equations (cf. Lanczos, 1926; Van der Waerden, 1973), and Born and Wiener made the first attempt to carry over the view of matrices as operations from pure mathematics to matrix mechanics (Born and Wiener, 1926). By and large these attempts failed, mainly because there was still no appropriate theory of unbounded matrices. Leon Lichtenstein, Professor of Mathematics at Leipzig University, who commissioned the young Aurel Wintner to write a state-of-the-art monograph on infinite matrices, wrote in the Introduction to Wintner's *Spektraltheorie der Unendlichen Matrizen. Einführung in den analytischen Apparat der Quan-*

tenmechanik: 'A flawless, mathematically satisfactory theory of the quantum-theoretical matrices is at present still a *desideratum*' (Wintner, 1929, p. VII, my translation). Then von Neumann arrived on the scene.[21] The rest, as the saying goes, is history.

In an opening lecture on Functional Analysis the following symbolical slogan might be written on the blackboard:

$$\mathbb{OPERATOR} \quad = \quad \mathbb{OPERATION} \quad + \quad \mathbb{DOMAIN}.$$

The domain co-determines the properties of the operator. To look upon matrices as multiplicators of a Schmidt-sequence is particular to matrix mechanics. This is a consequence of the fact that the Hilbert space of (*the final version of*) matrix mechanics contains Schmidt-sequences: $\mathrm{Qs} \in l^2(\mathbb{N})$ for all $\mathrm{s} \in D(\mathrm{Q})$, where $D(\mathrm{Q}) \subset l^2(\mathbb{N})$ is a set of appropriate Schmidt-sequences. The matrix Q captures the 'operation' of a *sequence-operator* $\hat{q}$: it tells us what to do with an appropriate Schmidt-sequence s to obtain its image: $\hat{q} : \mathrm{s} \mapsto \hat{q}\mathrm{s} := \mathrm{Qs} \in l^2(\mathbb{N})$. The matrix Q does not contain any information about the domain of this $\hat{q}$. Set-theoretically speaking, $\hat{q}$ is a non-denumerable set of ordered pairs:

$$\langle \mathrm{s}, \hat{q}\mathrm{s} \rangle \in \hat{q} \subset l^2(\mathbb{N}) \times l^2(\mathbb{N}). \tag{32}$$

The vector space obtained by lumping together all first elements of these ordered pairs is by definition *the domain of* $\hat{q}$, denoted by $D(\hat{q})$. Then $D(\hat{q}) = D(\mathrm{Q})$. So $\hat{q}$ trivially contains all information about 'its' domain $D(\hat{q})$. Changing $D(\hat{q})$ will result in an operator $\hat{q}'$ which is different from $\hat{q}$ but will not result in a different matrix:

$$\langle \mathrm{b}_j | \hat{q}\mathrm{b}_k \rangle = \langle \mathrm{b}_j | \hat{q}'\mathrm{b}_k \rangle = \mathrm{b}_j^\dagger \mathrm{Q} \mathrm{b}_k, \tag{33}$$

for all bases $\{\mathrm{b}_k\} \subset D(\hat{q}) \cap D(\hat{q}')$. So much for matrices as sequence-operations. We turn to Problems 1 and 2, defined in the opening paragraph of this Section.

Let Γ_ϕ, which we shall refer to as a *Riesz–Fischer mapping*, map each wave-function $\psi \in L^2(\mathbb{R})$ to the Schmidt-sequence consisting of the expansion coefficients of ψ in the basis $\{\phi_n\}$:

$$\Gamma_\phi : L^2(\mathbb{R}) \to l^2(\mathbb{N}), \quad \psi \mapsto \Gamma_\phi(\psi) := \mathrm{c}, \quad \text{where } c_n := \langle \phi_n | \psi \rangle. \tag{34}$$

Notice that $\Gamma_\phi(\phi_n) = \mathrm{e}^n$: the basis $\{\phi_n\}$ is mapped onto the natural basis $\{\mathrm{e}^n\}$. The Riesz–Fischer mapping Γ_ϕ (34) is an isomorphism (up to Lebesgue equivalence).[22]

Let $S(\mathbb{R})$ be the Schwartz space of all rapidly decreasing C^∞-functions, which is a dense domain of the canonical wave-operator algebra $(C_{\mathrm{wv}}) \subset \mathcal{D}[L^2(\mathbb{R})]$, where the latter is the set of all densely-defined wave-operators. The solutions of the harmonic oscillator problem are such functions and form a basis for $L^2(\mathbb{R})$: $\varphi_n(x) := \exp[-x^2/2]H_n(x)$, where H_n are appropriately normalised Hermite polynomials. Because the Riesz–Fischer mapping Γ_φ (34) is an isomorphism, $\Gamma_\varphi[S(\mathbb{R})]$ is a dense subset of $l^2(\mathbb{N})$.

[21] Specifically von Neumann (1929a; 1929b; 1932).

[22] A deep theorem and one of the pillars of von Neumann's edifice; cf. Section 5.

With these mathematical facts in position, we leave it as an exercise for the reader to verify that Problem 1 is solved by requiring the canonical matrices to be Schrödinger–Eckart matrices[23] ($\mathrm{C}^0_{\mathrm{mx}} \subset \mathcal{M}_{\mathrm{SE}}$), in which case the inverse Schrödinger–Eckart mapping $f^{\mathrm{inv}}_{\varphi} : \mathrm{C}^0_{\mathrm{mx}} \to C_{\mathrm{wv}}$ (I.25) guarantees the unique existence (up to Lebesgue-equivalence) of wave-operators $f^{\mathrm{inv}}_{\varphi}(\mathrm{Q})$ and $f^{\mathrm{inv}}_{\varphi}(\mathrm{P})$ with dense domains $\Gamma^{\mathrm{inv}}_{\varphi}[D(\mathrm{Q})]$ and $\Gamma^{\mathrm{inv}}_{\varphi}[D(\mathrm{P})]$, respectively. Problem 2 is solved by $\Gamma_{\varphi}[S(\mathbb{R})]$, which is a common dense domain of the canonical matrices that are constructed by the Schrödinger–Eckart mapping $f_{\varphi} : C_{\mathrm{wv}} \to \mathrm{C}^0_{\mathrm{mx}}$ (I.25). The fact that Problems 1 and 2 are solved by conceiving an infinite matrix as a partial specification of a sequence-operator is the content of Claim IV—which is of course old news.

Corollary. The canonical wave-operator algebra $(C_{\mathrm{wv}}) \subset \mathcal{D}[L^2(\mathbb{R})]$ and the canonical matrix algebra $(\mathrm{C}^0_{\mathrm{mx}}) \subset \mathcal{M}_{\mathrm{SE}}$, conceived as operations of sequence-operators, having equinumerous generator sets C_{wv} and $\mathrm{C}^0_{\mathrm{mx}}$ are isomorphic.

5. Equivalence Salvaged

Gently twisting a famous saying of Otto Neurath, we say that constructing a scientific theory is like building a fleet of structures in the open sea. No wonder that the very first attempts of Schrödinger (and Eckart) to prove equivalence were not successful, notwithstanding all the stories of their success. These stories were told over and over again and created the Equivalence Myth. In the meantime work progressed in the open sea at an astounding pace [Mehra and Rechenberg (1982, pp. 196–301); (1987, pp. 684–771); and Jammer (1966, pp. 299–335)]. In this final Section we jump from 1926 to 1932, when von Neumann took control of the fleet and designed new ships. We leave the spelling out of the Postulates of the legitimate successors of matrix mechanics and wave mechanics as an exercise for the reader[24]; but in (a) we submit two families of structures of quantum mechanics as the legitimate successors of the structure families **MM′** (11) and **MW′** (12). In (b) we salvage the mathematical and empirical equivalence in the form of an Equivalence Theorem, which is the content of Claim V. Finally we indicate in (c) the position of the successor structures in the wider class of structures that defines orthodox quantum particle mechanics.

(a) The new type of matrix-mechanical structure for a physical system of $N \in \mathbb{N}$ particles in $d \in \mathbb{N}_3$ spatial dimensions is:

$$\mathfrak{M}_{d,N} := \langle \mathbb{P}l^2(\mathbb{N}), (\mathrm{C}^0_{\mathrm{mx}}), \mathcal{G}_{\mathrm{mx}}, \mathbb{P}\mathrm{s}, \mathcal{F}(\mathrm{C}^0_{\mathrm{mx}}), \sigma_{\mathrm{mx}}, \mathcal{K}_{\mathrm{mx}}, \mathrm{Prop}\rangle. \tag{35}$$

The new type of wave-mechanical structure for the same physical system is:

$$\mathfrak{S}_{d,N} := \langle \mathbb{P}\mathcal{L}^2(\mathbb{R}^{Nd}), (C_{\mathrm{wv}}), \mathcal{G}_{\mathrm{wv}}, \mathbb{P}\psi, \mathcal{F}(C_{\mathrm{wv}}), \sigma_{\mathrm{wv}}, \mathcal{K}_{\mathrm{wv}}, \mathrm{Prop}\rangle. \tag{36}$$

The classes of all these structures, for all $N \in \mathbb{N}$, all $d \in \mathbb{N}_3$, all Hamiltonians, etc. and their empirical and ontological substructures, constitute the final versions

[23] As defined in Part I, Section 2.

[24] See Prugovecki (1981, pp. 348–351) for the postulates of wave mechanics.

of matrix mechanics (**MM**) and wave mechanics (**WM**), respectively. A set Const of physical constants and parameters is identical for both structure families and is left implicit. The set Prop $\in \mathbb{R}^{2N+2}$ contains the masses and charges of the N particles and the total mass and charge of the composite N-particle system (their Properties), whose instantiation means that N microphysical entities ('point-like material particles'), having the properties mass m_j and charge q_j, really exist.

Both structures contain *state spaces*; the matrix-mechanical structure contains the projective Hilbert space $\mathbb{P}l^2(\mathbb{N})$ of Weyl-equivalence classes (rays) of Schmidt-sequences, and the wave-mechanical structure contains the projective Hilbert space $\mathbb{P}\mathcal{L}^2(\mathbb{R}^{Nd})$ of rays of Lebesgue-equivalent wave-functions. von Neumann showed that the Riesz–Fischer mapping (34) is a bijective morphism (up to Lebesgue-equivalence) (von Neumann, 1932, p. 58).[25] which immediately entails the isomorphism of the state spaces: $\mathbb{P}\mathcal{L}^2(\mathbb{R}^{Nd}) \simeq \mathbb{P}l^2(\mathbb{N})$.

In the so-called 'Schrödinger picture' the history of a physical system is described by the *time-dependent Schrödinger equation.*[26] Define ψ as the solution of the time-dependent Schrödinger equation, whenever it exists; $\psi \subset \mathbb{R}\times L^2(\mathbb{R}^{Nd})$ is a wave-function-valued function on $\mathbb{R}$: $\psi : t \mapsto \psi(t) = \hat{U}(t)\psi(0)$, where $\hat{U}(t)$ is a unitary wave-operator. The function ψ determines a continuous path in $L^2(\mathbb{R})$ (if no measurements are performed). The ensuing set $\mathbb{P}\psi$ of rays occurs in $\mathfrak{S}_{d,N}$ (36). Von Neumann showed that the unitary operators form a Lie group $\mathcal{G}_{\rm wv}$ of bounded operators. Each time-independent Hamiltonian wave-operator $\widehat{H}$, by assumption linear and self-adjoint, corresponds uniquely to a unitary group (Stone's Theorem), by means of $\hat{U}(t) = \exp[-it\widehat{H}/\hbar] \in \mathcal{G}_{\rm wv}$. For a time-dependent Hamiltonian wave-operator, $\widehat{H}(t)$ is related to $\hat{U}(t)$ by an integral equation. *Mutatis mutandis* for the Hamiltonian matrix $\mathsf{H}(t) \in \mathcal{M}_{\rm SE}$, Schmidt-sequence-valued function $\mathsf{s} \subset \mathbb{R} \times l^2(\mathbb{N})$, ensuing set of rays $\mathbb{P}\mathsf{s}$, and Lie group of unitary matrices $\mathcal{G}_{\rm mx} \subset \mathcal{M}_{\rm bnd}$. The isomorphism $\mathcal{G}_{\rm mx} \simeq \mathcal{G}_{\rm wv}$ is again established by a Schrödinger–Eckart mapping f_Φ (I.25). The isomorphism between $\mathbb{P}\psi$ and $\mathbb{P}\mathsf{s}$ is a composition of the identity on $\mathbb{R}$ and the Riesz–Fischer mapping: $I \circ \Gamma_\Phi$.[27]

[25] The isomorphism of the Schmidt-sequence space and the wave-function space is often (but not entirely appropriately) designated as the *Riesz–Fischer Theorem*, because an important ingredient of this isomorphism proof was discovered independently by the Hungarian mathematician Frédéric Riesz and the German mathematician Ernst Fischer, namely that the mapping (34), having domain $L^2[a,b]$, is bijective; cf. Jammer (1966, p. 330). In their chapters devoted to the equivalence of matrix mechanics and wave mechanics, both Mehra and Rechenberg (1982, pp. 636–684) and Hanson (1963, pp. 113–134) do not even mention the Riesz–Fischer Theorem. Both von Neumann himself (1932, p. 29), asserting a close connection between 'Schrödinger's original equivalence proof' and the present isomorphism, and Hughes (1989, p. 45), asserting that Schrödinger established equivalence 'by virtue of this isomorphism', could not, historically speaking, be more off target.

[26] Schrödinger (1926e; 1927, p. 104), final founding paper.

[27] The relation between the 'Heisenberg picture' and the Schrödinger picture is spelled out, for example, by Ludwig (1968, pp. 66–69) and Prugovecki (1981, pp. 293–298). Both pictures give rise to identical empirical substructures (*vide infra*). Prugovecki (1981, p. 296) asserts that Schrödinger's equivalence proof establishes the empirical equivalence of the Heisenberg picture and the Schrödinger picture. There is alas nothing of the kind in Schrödinger's equivalence

The isomorphism $(\mathrm{C}^0_{\mathrm{mx}}) \simeq (C_{\mathrm{wv}})$ between the canonical algebras of (time-independent) matrices and wave-operators was addressed in Section 4. One of von Neumann's many achievements was his generalised formulation of the eigenvalue problem in Hilbert space and another deep theorem: the spectral resolution theorem for unbounded self-adjoint operators, which entailed the unique solvability of the generalised eigenvalue problem. [28] The spectral resolution theorem states that each linear self-adjoint operator corresponds uniquely to a spectral family of projector-valued Borel functions on $B(\mathbb{R}^n)$. [29] A useful charm of a spectral family is the possibility of defining any Borel function of operators; the function $f(\hat{Q})$ has in general a spectrum different from $\sigma(\hat{Q})$ but does not have a different spectral family. [30] The sets $\mathcal{F}(C_{\mathrm{wv}}) \in \mathfrak{M}_{d,N}$ and $\mathcal{F}(\mathrm{C}^0_{\mathrm{mx}}) \in \mathfrak{S}_{d,N}$ contain all the Borel functions of the canonical elements that the physicist wishes to consider; they stand by construction in one-to-one correspondence by means of the Schrödinger–Eckart mapping f_Φ (I.25), up to Lebesgue-equivalence.

The sets σ_{mx} and σ_{wv} contain the *spectra* of all matrices in $\mathcal{F}(\mathrm{C}^0_{\mathrm{mx}})$ and wave-operators in $\mathcal{F}(C_{\mathrm{wv}})$, respectively; one proves that $\sigma_{\mathrm{mx}} = \sigma_{\mathrm{wv}}$, that is $\sigma\hat{A} = \sigma\mathsf{A}$, where $\mathsf{A} = f_\Phi(\hat{A})$, for all $\mathsf{A} \in \mathcal{F}(\mathrm{C}^0_{\mathrm{mx}})$ and all $\hat{A} \in \mathcal{F}(C_{\mathrm{wv}})$.

(b) The story of the inception of the Born–Pauli probability measure has been told by others in detail (Wessels, 1981; Beller, 1990). Succinctly, Born and Pauli *deleted* $\mathrm{Prop}_{\mathrm{wv}}$, the set of all charge and matter densities, from the wave-mechanical structure $\mathfrak{S}'_{d,N}$ (5), *retained* the Euclidean structure $\mathbb{E}^3$, *added* a probability measure over $B(\mathbb{R}^3)$, *defined* $|\psi(\mathbf{q})|^2$ as a position probability-density and *defined* $|c_n|^2$, where $c_n \in \mathbb{C}$ is the expansion coefficient of ψ in the energy basis, as the probability of finding the energy value E_n. The standard gloss of saying that Born, together with Pauli, provided a 'different interpretation of the same formalism', is an awkward and misleading way of formulating the matter at hand (cf. Section 6). It seems better to say that the expansion of the wave-mechanical structure with a set of Kolmogorovian probability structures:

$$\mathcal{K}_{\mathrm{wv}} := \{\langle \mathbb{R}^n, B(\mathbb{R}^n), P_n\rangle \mid n \in \mathbb{N}\}, \tag{37}$$

where P_n is a Kolmogorovian probability measure defined by a wave function and n commuting wave-operators, defined in the usual way, constitutes a radical change in the mathematical structure of wave mechanics, and therefore, according to the structural view of scientific theories, a radical change in wave mechanics itself. [31] Despite the mathematical fact that any $\mathcal{K}_{\mathrm{wv}}$ is explicitly definable from the other mathematical entities of wave mechanics, the change

proof, if only because the wave-mechanical evolution equation (the time-dependent Schrödinger equation) had yet to be discovered.

[28] Hanson's claim that equivalence only holds for bounded operators (1961, p. 417) is one of his many mistakes; this particular one is based on confusing the boundedness of operators with the separability of Hilbert space.

[29] Von Neumann (1929a; 1929b); cf. Prugovecki (1981, p. 250) Theorem 6.3 for a modern exposition.

[30] Cf. Prugovecki (1981, pp. 270–284); even extensions to functions of non-commutative operators are possible.

[31] In full concurrence with Hanson (1961, p. 421). But then Hanson goes on to claim (1961, p. 421) that Born showed the equivalence of matrix mechanics and wave mechanics and therefore

in mathematical structure counts as radical, for it would ultimately codify the transition from a deterministic to an indeterministic view of microphysical reality. And *mutatis mutandis* with regard to $\mathcal{K}_{\mathrm{mx}} \in \mathfrak{M}_{d,N}$. Von Neumann defined the probability measure $P_A : B(\mathbb{R}) \mapsto [0,1]$, where $P_A(\Delta)$ is the probability of finding a value in Borel set Δ of one physical magnitude A, as the *expectation-value* $\langle\psi|\hat{E}^A(\Delta)\psi\rangle$ of the spectral family member $\hat{E}^A(\Delta)$ (von Neumann, 1932, pp. 200–201). All matrix-mechanical probability measures and wave-mechanical probability measures pertaining to the same n magnitudes provably coincide as a consequence of the isomorphism between the state spaces and the operator algebras.

Hence the empirical substructures of $\mathfrak{M}_{d,N}$ and $\mathfrak{S}_{d,N}$, defined as

$$[\mathfrak{M}_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}, \sigma_{\mathrm{mx}}, \mathcal{K}_{\mathrm{mx}}\rangle \quad \text{and} \quad [\mathfrak{S}_{d,N}]_{\mathrm{emp}} := \langle \mathrm{Prop}, \sigma_{\mathrm{wv}}, \mathcal{K}_{\mathrm{wv}}\rangle, \tag{38}$$

respectively, are identical. This demonstrates the empirical equivalence of **MM** and **WM**.

We state the content of Claim V in the form of an *Equivalence Theorem* in the following concise way, which summarises parts (a) and (b) of the present section:[32]

$$\mathfrak{M}_{d,N} \simeq \mathfrak{S}_{d,N} \quad \text{and} \quad [\mathfrak{M}_{d,N}]_{\mathrm{emp}} = [\mathfrak{S}_{d,N}]_{\mathrm{emp}}. \tag{39}$$

We emphasise that only with the spectral families of the canonical matrices, conceived as sequence-operators, is it possible to express the position and the momentum probabilities in matrix language. That is why the mathematical and the empirical equivalence of matrix mechanics and wave mechanics, even expanded with the sets of Kolmogorovian probability structures $\mathcal{K}_{\mathrm{mx}}$ and $\mathcal{K}_{\mathrm{wv}}$, *could not have been proven* in March 1926, three years *before* von Neumann developed his spectral theory of unbounded self-adjoint operators. Furthermore, the proved *mathematical* equivalence between the structures $\mathfrak{M}_{d,N}$ and $\mathfrak{S}_{d,N}$ entails that (almost) any sentence is expressible and satisfied in the formal language of the matrix-mechanical structures iff it is expressible and satisfied in the formal language of the wave-mechanical structures.[33] For example, sentence (24) is expressed in matrix-mechanical language as (symbolically):

$$\text{‘}\langle \mathsf{s}|\mathsf{E}^{\mathsf{Q}}(\Delta)\mathsf{s}\rangle = 0.35\text{’}, \tag{40}$$

where $\mathsf{E}^{\mathsf{Q}}(\Delta) \in \mathcal{M}_{\mathrm{bnd}}$ is the appropriate projective matrix from the spectral family of Q.

deserves the credit, whereas in fact Born widened the gulf! It was von Neumann who then showed how to bridge the widened gulf as, *infra*, e.g. (40).

[32] Hill (1961, p. 427), one of the early advocators of distribution theory for quantum theory, makes the point that wave mechanics extended by distribution theory is not equivalent to matrix mechanics. (Exercise: find the inconsistency in the last but one paragraph of Hill (1961, pp. 427–428). Hill's denial of the equivalence of matrix mechanics and wave mechanics 'even within the bounds of von Neumann's formulation' is a howler.)

[33] We say 'almost', because a wave-mechanical sentence like 'the value of the wave function ψ at a point $\mathbf{r} \in \mathbb{R}^3$ is $0.24 + 0.11i$', or like Tomonaga's locution (23), is not translatable into a matrix-mechanical language. This derives from the fact that the Hilbert spaces of the wave-functions and the Schmidt-sequences have quite distinct elements, which live on different floors of the Cantor hierarchy, notwithstanding the isomorphism of the structures they form (Hilbert space).

(c) Let *orthodox quantum particle mechanics* (**QM**) be the family of mathematical structures satisfying von Neumann's postulates. **QM** is far from exhausted by the structure families **MM** and **WM** just discussed. Let $\mathfrak{Q}_N$ denote an element of **QM** for a system of $N \in \mathbb{N}$ particles, the empirical and ontological substructures being set aside for the moment. We confine ourselves to an outline of what $\mathfrak{Q}_N$ looks like. Structure $\mathfrak{Q}_N$ results when the slots in $\mathfrak{M}_{d,N}$ or $\mathfrak{S}_{d,N}$ are replaced with more abstract, encompassing mathematical objects that have the slots of $\mathfrak{M}_{d,N}$ and of $\mathfrak{S}_{d,N}$ as instances. This spells out the precise relation of the final versions of matrix mechanics and wave mechanics to von Neumann's axiomatised orthodox quantum particle mechanics.

The state space becomes the convex set $S(\mathcal{H})$ of von Neumann's *state operators*, acting on *any* Hilbert-space $\mathcal{H}$.[34] So mixed states are allowed. von Neumann defined the concomitant probability $P_A(\Delta)$ for a mixed state $\widehat{W} \in S(\mathcal{H})$ as $\mathrm{Tr}(\widehat{W}\widehat{A})$. Finite-dimensional Hilbert spaces are allowed, like the two-dimensional spin space $\mathbb{C} \times \mathbb{C}$ of two-element sequences; Pauli's spinor-space, e.g. $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ for a one particle system; and quaternion Hilbert space is still another example. The path in a Hilbert space becomes a path in $S(\mathcal{H})$, defined by the state operator-valued function $\widehat{W} : \mathbb{R} \to S(\mathcal{H}), t \mapsto \widehat{W}(t)$, defined as the solution of von Neumann's generalisation of the time-dependent Schrödinger equation [Neumann (1932, p. 350), Prugovecki (1981, p. 396)]. And instead of the canonical physical magnitudes position and momentum and functions of these, other magnitudes are allowed that are not functions of the canonical magnitudes, e.g. spin, the quantum-mechanical magnitude *par excellence*—remember that most Hamiltonians, pertaining to microphysical systems, actually employed by physicists contain spin-dependent terms. As to magnitudes, any set $\mathcal{O}(\mathcal{H})$ of densely-defined, linear, self-adjoint operators suffices; $\mathcal{O}(\mathcal{H})$ may be, but need not be, an algebra. Taking stock we obtain:

$$\mathfrak{Q}_N := \langle S(\mathcal{H}), \mathcal{O}(\mathcal{H}), \mathcal{G}, \widehat{W}, \sigma, \mathcal{K}, \mathrm{Prop} \rangle, \tag{41}$$

where σ is the set of all spectra $\sigma(\widehat{A})$ of operators $\widehat{A} \in \mathcal{O}(\mathcal{H})$ and $\mathcal{K}$ a set of Kolmogorov probability structures of interest concerning operators from $\mathcal{O}(\mathcal{H})$. As the empirical substructure one takes standardly:[35]

$$[\mathfrak{Q}_N]_{\mathrm{emp}} := \langle \mathrm{Prop}, \sigma, \mathcal{K} \rangle. \tag{42}$$

To summarise part (c):

$$\mathbf{MM} \subset \mathbf{QM} \supset \mathbf{WM}, \quad \mathbf{MM} \cap \mathbf{WM} = \varnothing, \quad \mathbf{MM} \cup \mathbf{WM} \subset \mathbf{QM}. \tag{43}$$

Figure 1 depicts Theorem (90) in a Venn-diagram.

Which substructure of $\mathfrak{Q}_N$ should be deemed the ontological substructure $[\mathfrak{Q}_N]_{\mathrm{ont}}$ depends heavily on one's philosophical view of science. A 'structural realist' tends to include as much as possible in $[\mathfrak{Q}_N]_{\mathrm{ont}}$; an 'entity realist'

[34] Neumann (1932, p. 316), where they are called 'statistical operators'; 'density operator' is another (less appropriate but omnipresent) name.

[35] More elegantly, the properties mass, charge, etc. can be subsumed in σ by means of introducing super-selection rules; $\mathcal{H}$ then becomes a super-selected sector of some gigantic Hilbert-space.

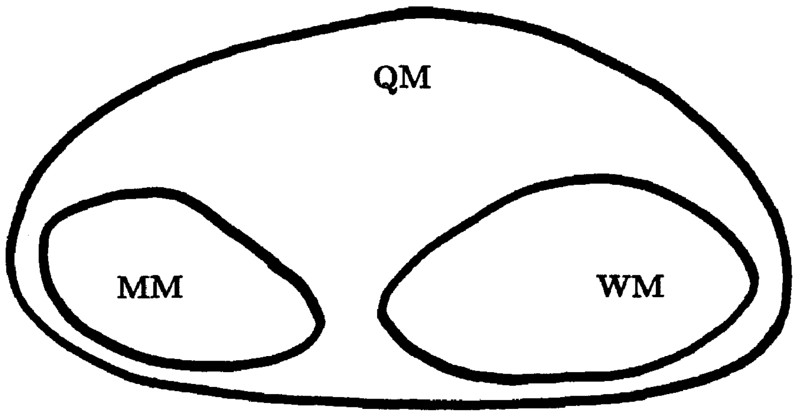

Fig. 1. *Venn diagram of Theorem (44).*

will include Prop in $[\Omega_N]_{\mathrm{ont}}$, whose instantiation is then taken to mean that microphysical entities and their (superselected) properties really exist; a 'positivist' tends to exclude as much as possible from $[\Omega_N]_{\mathrm{ont}}$; and an 'idealist', or even a 'solipsist' can be a physicist by putting $[\Omega_N]_{\mathrm{ont}} = \varnothing$ and taking $[\Omega_N]_{\mathrm{emp}}$ as an *experiential* substructure. The general element of **QM** is of the type

$$\langle \Omega_N, [\Omega_N]_{\mathrm{emp}}, [\Omega_N]_{\mathrm{ont}} \rangle. \tag{44}$$

6. Aftermath: Formalism and Interpretation

The conclusions of this investigation are in fact the five claims summarised in Part I, Section 1. The author is sceptical about making inductive inferences to conclusions of a more general nature from these claims, which pertain to one case study. We end by a remark on the formalism/interpretation distinction.

After wave mechanics appeared on the scene Heisenberg surreptitiously introduced the well-known formalism/interpretation distinction to physics [Heisenberg (1926, p. 994); (1929, pp. 493, 495)]. His reason for the introduction of this distinction, or so we speculate, was to resolve a dilemma: on the one hand he wrote that he loathed *anschauliche* wave mechanics and thought of it as 'Mist',[36] whereas on the other hand Heisenberg, ardent problem-solver that he was (Cassidy, 1992), immediately recognised the advantages of solving a linear partial differential equation over diagonalising an infinite unbounded matrix. This dilemma was not particular to Heisenberg; all the matrix mechanics (Born, Jordan, Dirac, Pauli) were seduced by Schrödinger's differential equation—and all would succumb. To accept or not to accept wave mechanics, that was the

[36]Moore (1989, p. 221) translates 'Mist' as 'bullshit', Cassidy (1992, p. 215) as 'crap', and Beller (1996, p. 549) as 'rubbish'; cf. Beller (1983, p. 489) and (1990, p. 574)

question. Heisenberg's resolution of this dilemma was to slice away everything from the wave-mechanical structure $\mathfrak{S}'_{d,N}$ (5) that was not necessary to calculate the frequencies and intensities, and call it 'interpretation', and call the *contracted* structure[37] 'formalism'. When Born and the other founding fathers of matrix mechanics *expanded* the wave-mechanical structure by Kolmogorovian measure structures, and *changed* its empirical substructure, the entire matrix division sung in close harmony that these moves accomplished merely a different 'interpretation' of wave mechanics, leaving its 'formalism' unaltered. So when Heisenberg declared publicly that he accepted the 'formalism', repudiated '*Schrödinger's* interpretation' and adopted '*Born's* interpretation', he turned everything upside down, according to the structural view. What Heisenberg really had been doing, or so it seems, without even a whisper of protest from his matrix allies, presumably because their conceptual faculties were clouded by their desire for Schrödinger's differential equation, was to baptise as 'formalism' what the matrix mechanics accepted of $\mathfrak{S}'_{d,N}$ (5), to baptise as 'Schrödinger's interpretation' what they amputated and to baptise as 'Born's interpretation' what they sewed on. These were the Procrustean methods of Heisenberg *cum suis*, viewed through Suppesian spectacles.

Heisenberg's surreptitious introduction of the formalism/interpretation distinction carried the day and has nestled itself in the standard vocabulary of the working physicist. How this distinction is to be understood exactly is not clear. (Did Heisenberg pick up a tan from the philosophical Sun of Logical Positivism that was rising above the Weimar Republic and whence the distinction springs?) But then again, the meaning of many concepts of the working physicist's standard vocabulary is not clear (examples: causality, explanation, theory, evidence, measurement, observation). The variety of ways in which the word 'interpretation' is currently used, the practice of 'interpreters' of quantum mechanics included, is bewildering to a mind that craves for clarity.

The formalism/interpretation distinction should of course ultimately be understood as a way of encoding the philosophically problematical relation between what we internally construct, in our minds or on paper, and the external world, which in some sense exists independently of us. Now let $\mathbf{T}$ be a physical theory and S a physical system. In the structural framework 'formalism of $\mathbf{T}$' can be identified, if one insists of having this particular notion, with the entire structure $\mathfrak{U} \in \mathbf{T}$, which is regarded as 'a model of S', and 'interpretation of $\mathbf{T}$' can be identified with the juxtaposition of: (i) the embeddability relation between the empirical substructures $[\mathfrak{U}]_{\mathrm{emp}}$ of $\mathfrak{U}$ and the data structures obtained in experiments pertaining to S; and (ii) the relation between these quantitative data structures and our qualitative sense impressions as expounded in Suppes' measurement theory; and (iii) the inscrutable relation between the ontological substructure $[\mathfrak{U}]_{\mathrm{ont}}$ of $\mathfrak{U}$ and S. On the basis of this construal of the formalism/interpretation distinction, our conclusion, argued for *supra*, of Heisenberg *c. s.* using Procrustean methods follows: they did not leave the formalism of

[37] For the meaning of 'contraction' and 'expansion' see footnote 3.

wave mechanics unaltered and merely change its interpretation. Rather, they changed the formalism and its interpretation.

We do not wish to defend that no construal of the notion 'formalism' is possible which justifies Heisenberg's way of putting things. Perhaps a wider notion than the evident one we explicitly used *supra* will accomplish this, like one including everything which is, in an appropriate sense, 'implicitly definable' in terms of a few 'basic' structures. But we do wish to emphasise that in the structural framework the notions 'formalism' and 'interpretation' are redundant. The relations between what we internally construct (theory **T**) and what exists 'independent' of our constructions (physical system S) and what we experience (our observations), are in the structural framework delicately taken care of, employing purely standard set-theoretical nomenclature, by the points (i), (ii) and (iii) mentioned above. The notions 'formalism' and 'interpretation' arguably are conceptual zombies that escaped with Heisenberg's diabolical help the coffin of Logical Positivism. Even brandishing Occam's razor suffices to chase them away from the structural framework.

Formalism/interpretation distinction, rest in peace.

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