A Simple Proof of the Existence of a Planar Separator

Sariel Har-Peled[①]

July 12, 2011

1. Introduction. The ***planar separator theorem*** is a fundamental result about planar graphs [LT79]. Informally, it states that one can remove $O(\sqrt{n})$ vertices from a planar graph with n vertices and break it into "significantly" smaller parts. It is widely used in algorithms to facilitate efficient divide and conquer schemes on planar graphs. For further details on planar separators and their applications, see Wikipedia (`http://en.wikipedia.org/wiki/Planar_separator_theorem`).

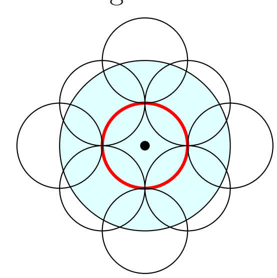
Here, we present a simple proof of the planar separator theorem. Most of the main ingredients of the proof are present in earlier work on this problem; see Miller *et al.* [MTTV97], Smith and Wormald [SW98], and Chan [Cha03]. Furthermore, the constants in the separator we get are inferior to known constructions [AST94].

Nevertheless, the new proof is relatively self contained and (arguably) significantly simpler than previous proofs. In particular, we prove the following version of the planar separator theorem.

Theorem 1.1 *Let $G = (V, E)$ be a planar graph with n vertices. There exists a set S of $4\sqrt{n}$ vertices of G, such that removing S from G breaks it into several connected components, each one of them contains at most $(9/10)n$ vertices.*

2. Construction and analysis. Given a planar graph $G = (V, E)$ it is known that it can be drawn in the plane as a ***kissing graph***; that is, every vertex is a disk, and an edge in G implies that the two corresponding disks touch (this is known as Koebe's theorem, see [PA95]). Furthermore, all these disks are interior disjoint.

Figure 1:



Let $\mathcal{D}$ be the set of disks realizing G as a kissing graph, and let P be the set of centers of these disks. Let d be the smallest radius disk containing $n/10$ of the points of P, where $n = |P| = |V|$. To simplify the exposition, we assume that d is of radius 1 and it is centered in the origin. Randomly pick a number $x \in [1, 2]$ and consider the circle C_x of radius x centered at the origin. Let S be the set of all disks in $\mathcal{D}$ that intersect C_x. We claim that, in expectation, S is a good separator.

Lemma 2.1 *The separator S breaks G into two subgraphs with at most $(9/10)n$ vertices in each connected component.*

Proof: The circle C_x breaks the graph into two components: (i) the disks with centers inside C_x, and (ii) the disks with centers outside C_x. Clearly, the corresponding vertices in G are disconnected once we remove S. Furthermore, a disk of radius 2 can be covered by 9 disks of radius 1, as depicted

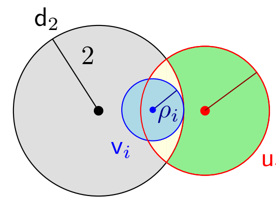
[①]Department of Computer Science; University of Illinois; 201 N. Goodwin Avenue; Urbana, IL, 61801, USA; `sariel@uiuc.edu`; `http://www.uiuc.edu/~sariel/`. Work on this paper was partially supported by a NSF AF award CCF-0915984.

in Figure 1. As such, the disk of radius 2 at the origin can contain at most $9n/10$ points of P inside it, as a disk of radius 1 can contain at most $n/10$ points of P. We conclude that there are at least $n/10$ disks of $\mathcal{D}$ with their centers outside C_x, and, by construction, there are at least $n/10$ disks of $\mathcal{D}$ with centers inside C_x. As such, once S is removed, no connected component of the graph $G \setminus S$ can be of size larger than $(9/10)n$. ∎

Lemma 2.2 *We have* $\mathbf{E}\big[|S|\big] \leq 4\sqrt{n}$*, where* $n = |V|$*.*

Proof: Consider a disk u_i of $\mathcal{D}$ of radius r_i centered at p_i. If u_i is fully contained in d_2 (the disk of radius 2 centered at the origin), then the circle C_x intersects u_i if and only if $x \in [\|\mathsf{p}_i\| - r_i, \|\mathsf{p}_i\| + r_i]$, and as x is being picked uniformly from $[1,2]$, the probability for that is at most $2r_i/|2-1| = 2r_i$. For reasons that would become clear shortly, we set $\rho_i = r_i$ and $\mathsf{v}_i = \mathsf{u}_i$ in this case.

Otherwise, if u_i is not fully contained in d_2 then the set $L_i = \mathsf{u}_i \cap \mathsf{d}_2$ is a "lens". Consider a disk v_i of the same area as L_i contained inside d_2 and tangent to its boundary. Clearly, if C_x intersects u_i then it also intersects v_i, see figure on the right. Furthermore, the radius of v_i is $\rho_i = \sqrt{\mathrm{area}(\mathsf{u}_i \cap \mathsf{d}_2)/\pi}$, and, by the above, the probability that C_x intersects v_i (and thus u_i) is at most $2\rho_i$.



Observe that as the disks of $\mathcal{D}$ are interior disjoint, we have that $\sum_i \rho_i^2 = \sum_i \mathrm{area}(\mathsf{u}_i \cap \mathsf{d}_2)/\pi \leq \mathrm{area}(\mathsf{d}_2)/\pi = 4$. Now, by linearity of expectation and the Cauchy-Schwarz inequality, we have that

$$\mathbf{E}\Big[|S|\Big] = \mathbf{E}\Big[|\mathcal{D} \cap C_x|\Big] = \sum_i \mathbf{Pr}\Big[\mathsf{u}_i \cap C_x \neq \emptyset\Big] \leq \sum_i \mathbf{Pr}\Big[\mathsf{v}_i \cap C_x \neq \emptyset\Big] \leq \sum_i 2\rho_i = 2\sum_i 1 \cdot \rho_i$$
$$\leq 2\sqrt{\sum_{i=1}^{n} 1^2}\sqrt{\sum_{i=1}^{n} \rho_i^2} \leq 2\sqrt{n}\sqrt{4} = 4\sqrt{n}.$$
∎

Now, putting Lemma 2.1 and Lemma 2.2 together implies Theorem 1.1.

Acknowledgments.. The author thanks Timothy Chan, Robert Krauthgamer and Günter Rote for useful comments on the manuscript.

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