Exercise 8 **Graphs and Algorithms**

Probabilistic method: First moment methods

1. **List coloring.** Let G be a bipartite graph with n vertices (in total). For each vertex v you are given a list of colors $S(v)$, from which the color of v must be chosen. Show that it is possible to find a valid coloring of G, if $|S(v)| > \log_2 n$ for each v.

   [Hint: Consider all the colors $\cup_v S(v)$ and split them into two random sets A and B.]

2. Show that there exist graphs with Chromatic $\chi(G) \geq n/(3 \log n)$, but which have no clique of size more than $3 \log n$. So having large cliques is not necessarily the reason why a graph needs many colors to be colored.

   [Hint: Consider a random graph $G(n, 1/2)$, and show that the expected number of independent sets of size more $\geq 3 \log n$ is much less than 1.]

3. Show that there are (n, n) bipartite graphs that have at least $\Omega(n^{4/3})$ edges, but do not have a $K_{2,2}$.

   [Hint: Construct an appropriate random graph, and do some alteration.]

4. Let $R(k, \ell)$ denote the smallest number n such that for the complete graph K_n, no matter how the edges are colored red or blue, it contains either a red K_k or a blue K_ℓ. Show that $R(k, \ell)$ satisfies $R(k, \ell) \leq R(k, \ell - 1) + R(k - 1, \ell)$. Together with the base case $R(\ell, 2) = R(2, \ell) = \ell$, show that this implies that $R(k, \ell) \leq \binom{k+\ell-1}{k-1}$.

   [Hint: Consider the complete graph on $R(k, \ell - 1) + R(k - 1, \ell)$ vertices and show that no matter how its edges are colored red and blue, it must contain a red K_k or blue K_ℓ.