Probabilistic method

1. Recall that a tournament is obtained from the complete graph K_n by directing each edge in one of the two possible directions. A tournament is called *transitive* if the directed edges (i, j) and (j, k) imply the directed edge (i, k). (Equivalently, there is some ordering of the vertices such that the endpoints of every directed edge (i, j) satisfy $i < j$.) Use the averaging principle to show that there exists a tournament on n vertices, which does not contain any transitive subtournament on $2 + 2\log_2 n$ vertices.

2. A Hamiltonian path in a directed graph is a path which visits each vertex exactly once. Show that there is a tournament on n vertices with at least $n!2^{1-n}$ Hamiltonian paths.

3. Consider the family of all pairs (A, B) of disjoint k-subsets of $\{1, 2, \ldots, n\}$. Say that a set $S \subseteq \{1, 2, \ldots, n\}$ *separates* (A, B) if $A \subseteq S$ and $B \cap S = \emptyset$. Show that there exists a family of at most $2k4^k \ln n$ sets S such that each pair (A, B) is separated by at least one of them.

   [Hint: Define some random sets, and pick a suitable number of them]

4. Let A be a matrix. A submatrix of A is obtained by deleting rows and column from A. A submatrix is called constant, if all its entries are equal.

   Show that for all $j \geq 2$, there exists an $n \times n$ $\{0, 1\}$-matrix (with $n = \lfloor 2^{j/2} \rfloor$) that has no constant $j \times j$ submatrix.