Algorithms and Complexity (AC), week 4

Marie Schmidt

based on slides by Jesper Nederlof

LNMB, Sep–Nov 2019

Program for this week and the next

Dealing with NP-hard problems: Approximation

Basic definitions
Ad-hoc approaches
LP-based approaches
Approximation Schemes
In-approximability

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We leave decision problems, and return to optimization problems

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Let X be a minimization problem.
The optimal objective value of instance I is denoted $\mathrm{opt}(I)$.
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We aim for polynomial time algorithms with good approximation ratios
still possible for problems whose decision versions are NP-complete!

Ad-hoc approaches

Vertex cover (VC)

Instance: undirected graph $G = (V, E)$
Goal: find a vertex cover of smallest possible size
(vertex cover = subset of vertices that touches every edge)

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Theorem

This poly-time approximation algorithm has approximation ratio 2.

Makespan minimization

Instance: m machines; n jobs with processing times $p_1, \ldots, p_n$
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Is this bound tight?

Can sharpen above analysis to guarantee $2 - 1/m$ (exercise).

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- 7 bridges of Köningsberg (Kalingrad): can a tour cross all bridges once?

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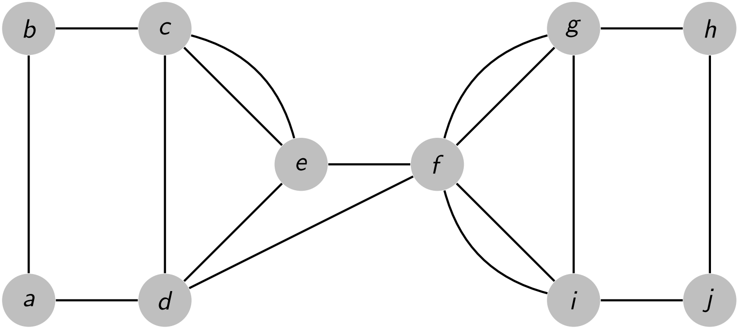
  e.g. some edge $\{u, v\}$ may occur multiple times
- A Eulerian tour of a graph is a tour visiting all *edges* exactly once.



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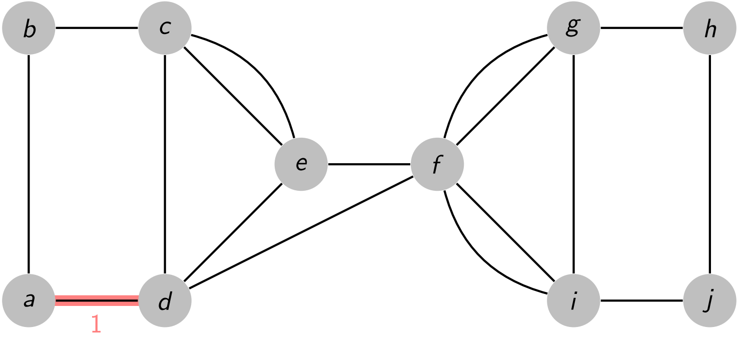

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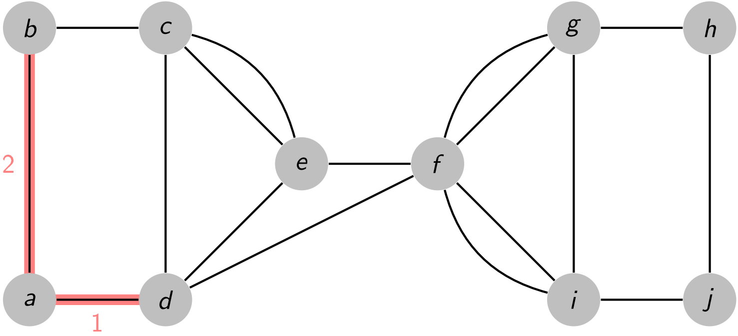

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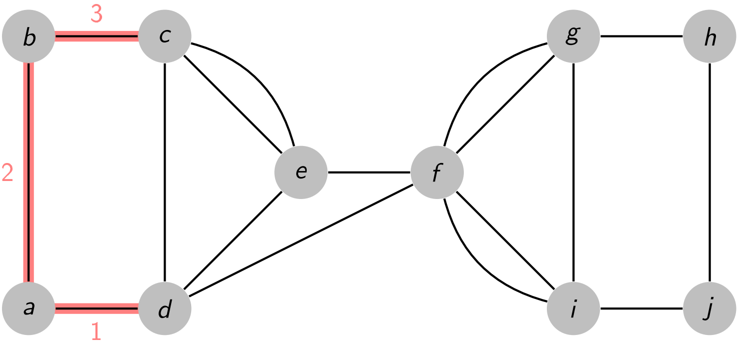

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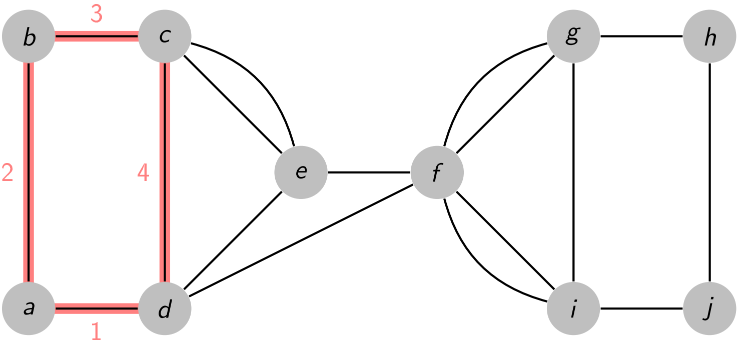

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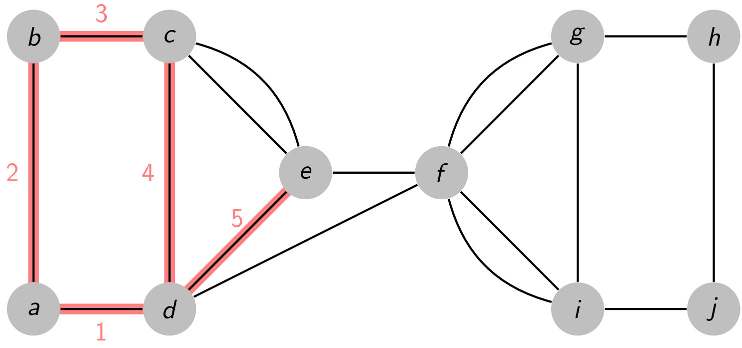

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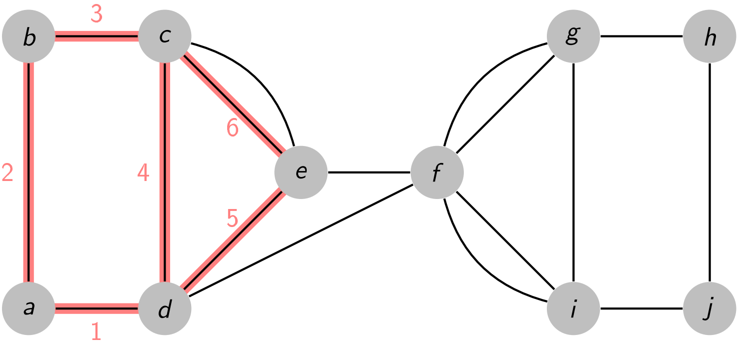

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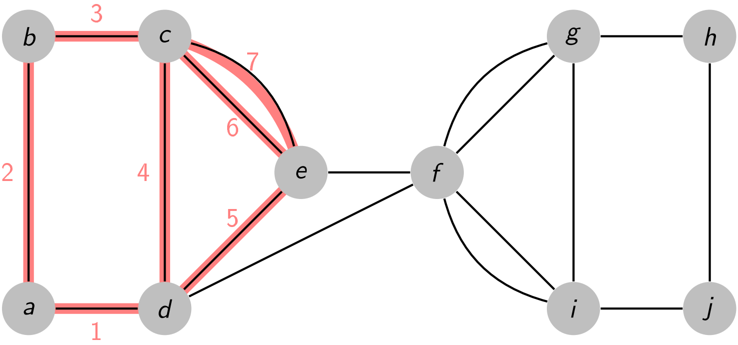

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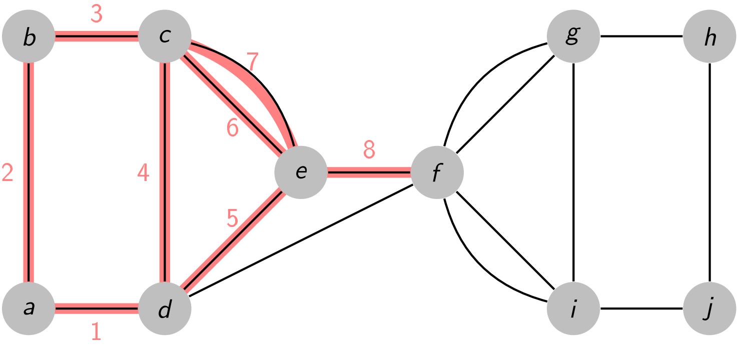

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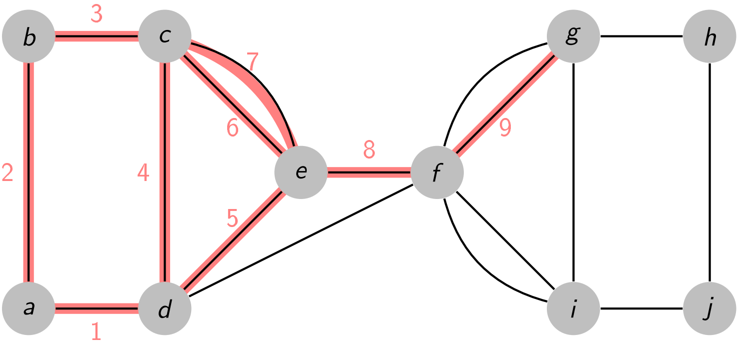

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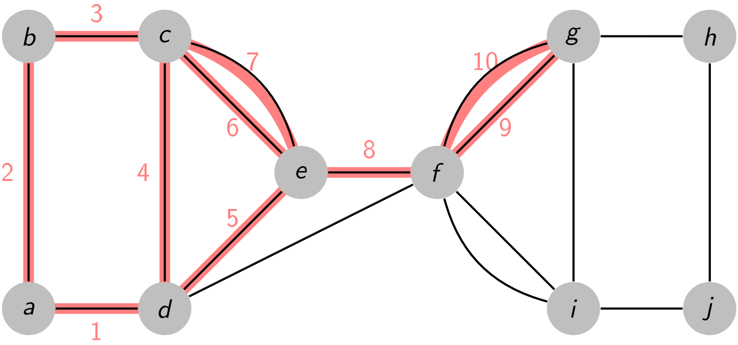

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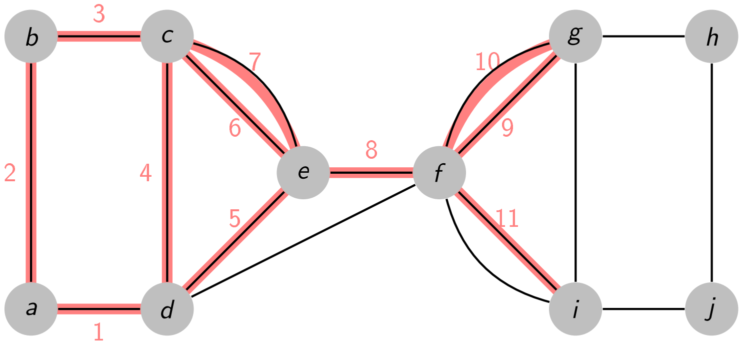

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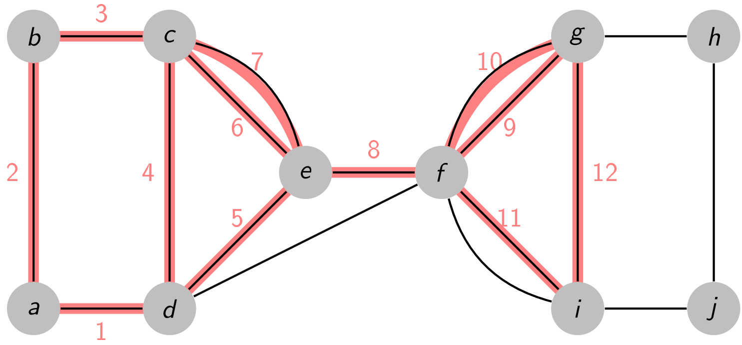

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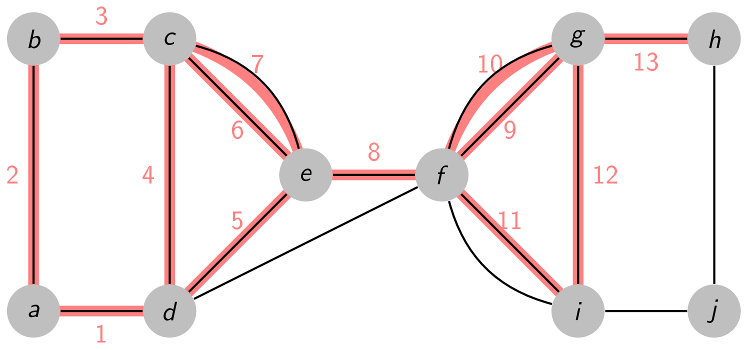

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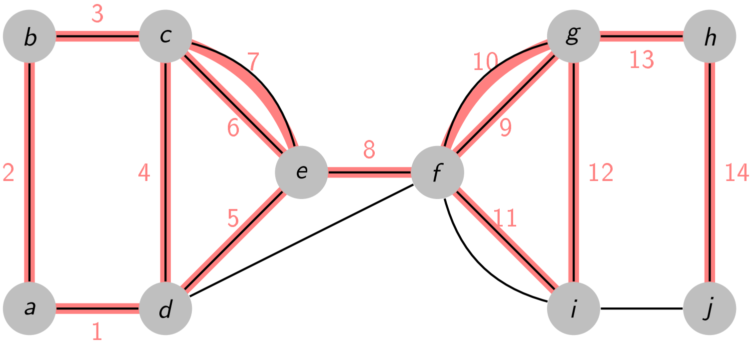

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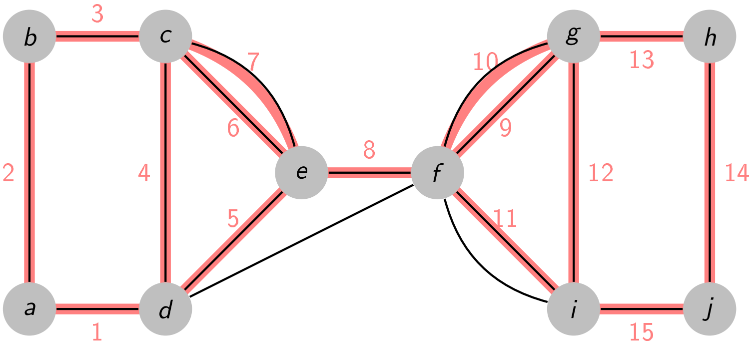

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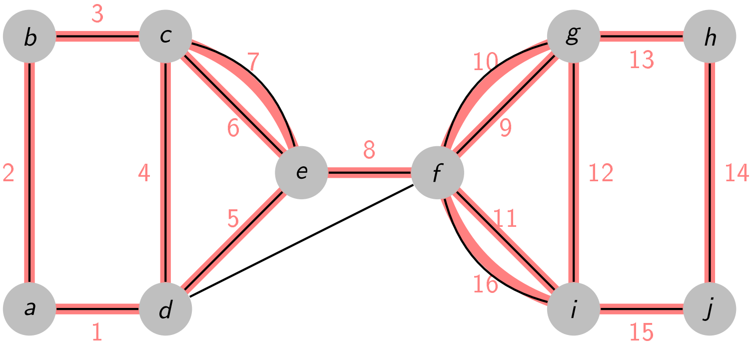

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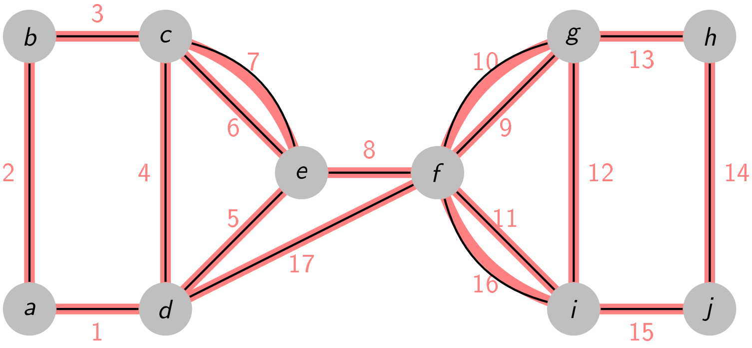

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(This is a constructive polynomial-time algorithm)

Travelling Salesman Problem

TSP (Optimization version)

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Goal: find roundtrip of smallest possible length

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Assumption: *metric* TSP

We now assume that the distances satisfy the triangle inequality
$d(x, y) + d(y, z) \geq d(x, z)$ for all cities x, y, z

Important example: Points in the plane (triangle ineq. by Pythagoras).
Even here, still NP-complete (harder reduction, beyond scope for us)

Travelling Salesman Problem

TSP (Optimization version)

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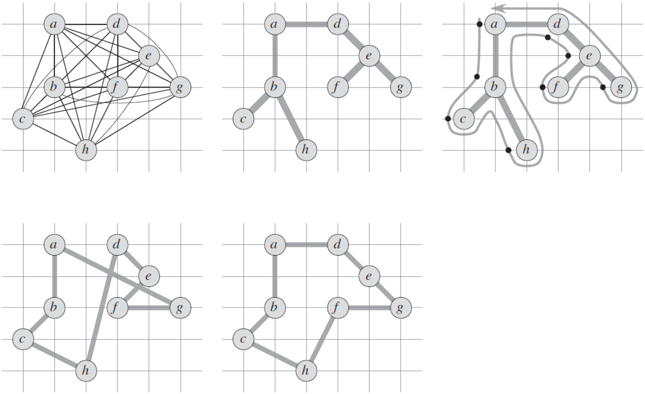
- $\mathrm{opt}(I) \geq$ twice the length of min. weight perfect matching, if n even

Double-tree algorithm



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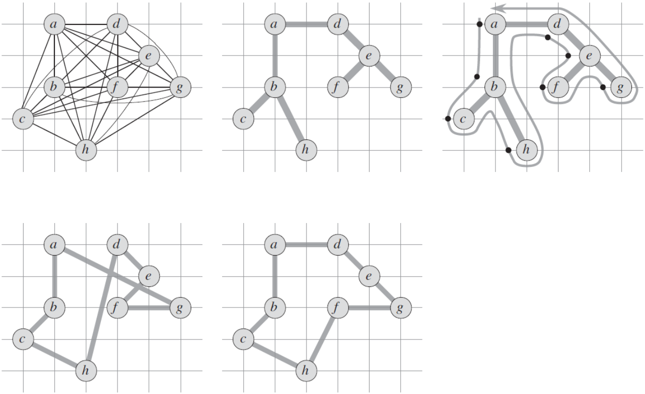

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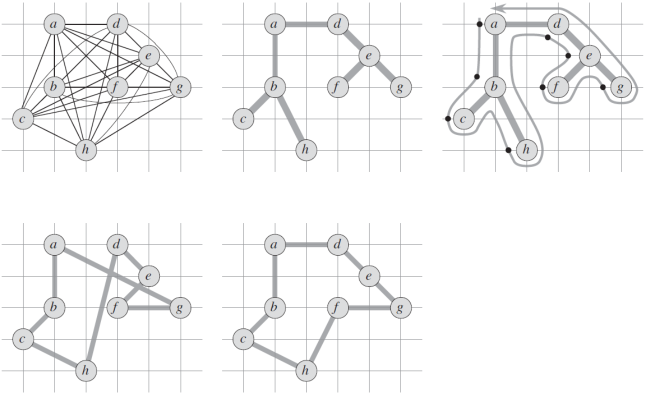
1. Compute a minimum spanning tree MST
2. Double every edge in MST to get a Eulerian graph



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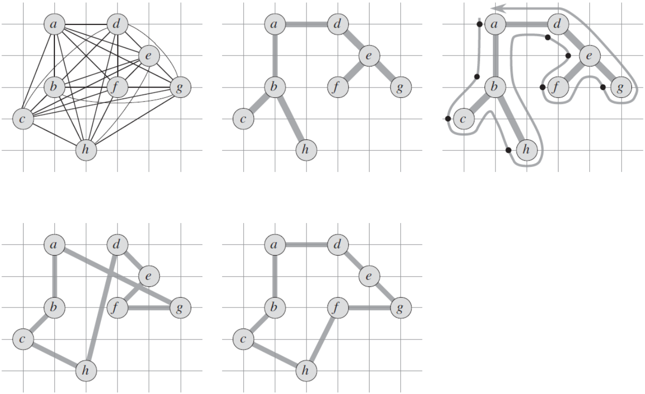

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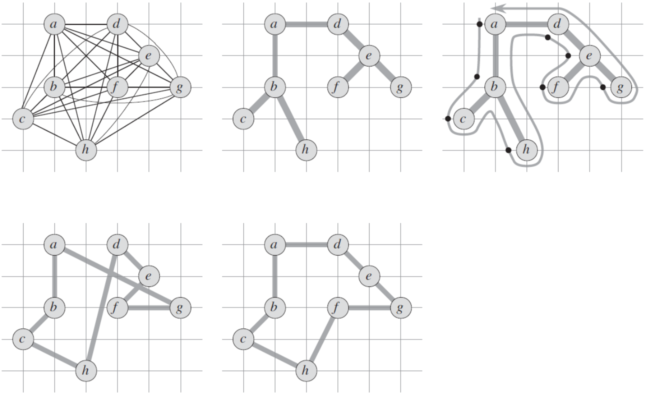

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$A(I) \leq$ twice length of MST $\leq 2 \cdot \mathrm{opt}(I)$

Christofides-Serdyukov algorithm

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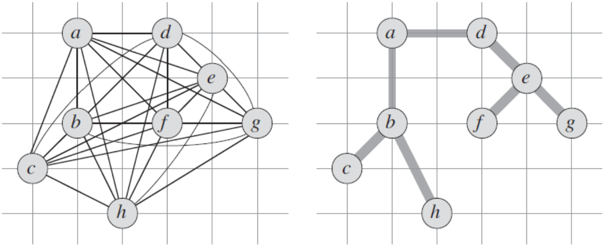

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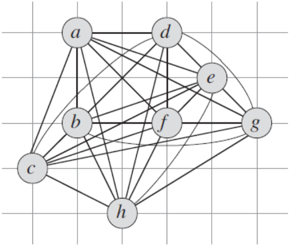

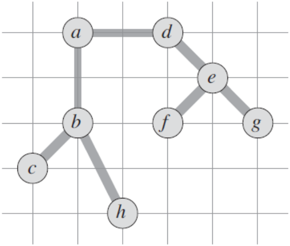

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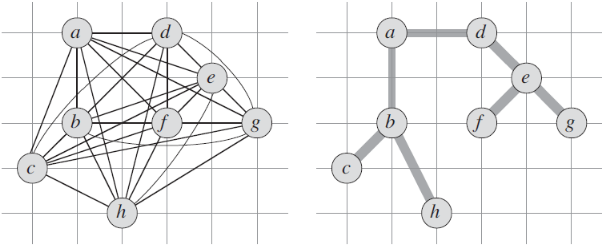

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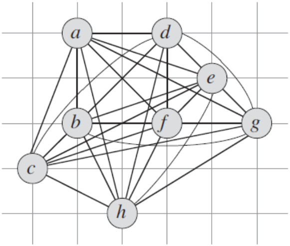

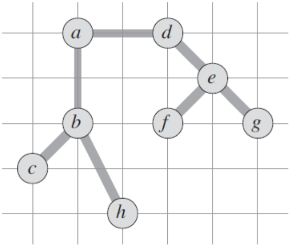

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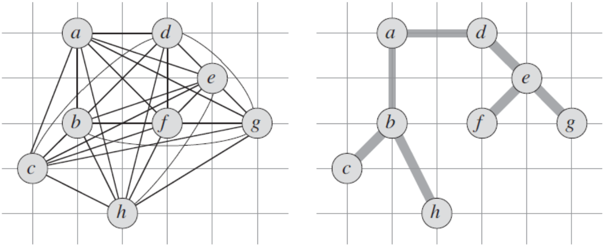

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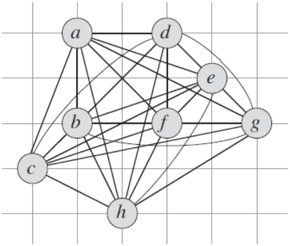

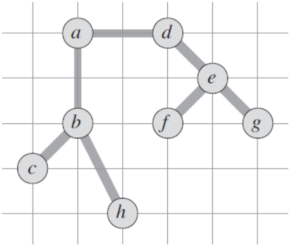

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Where did we use the triangle inequality?

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4. Round the optimal LP solution to approximate ILP solution (preserving feasibility!)

Weighted vertex cover (VC)

Instance: a graph $G = (V, E)$; weights $w : V \to \mathbb{R}^+$
Goal: find a vertex cover of smallest possible weight
(e.g. find a vertex cover $X \subseteq V$ minimizing $\sum_{v \in V} w(v)$)

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ILP formulation

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\text{minimize} & \sum_{v \in V} w(v) \cdot x_v & \\
\text{subject to} & x_u + x_v \geq 1 & \text{for every edge } \{u, v\} \in E \\
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This poly-time approximation algorithm has approximation ratio 2.

Proof:

General approach is centered around three values: opt_{ILP}, opt_{LP}, app (result of the rounding)

Observation

$\mathrm{opt}_{LP} \;\leq\; \mathrm{opt}_{ILP} \;\leq\; \mathrm{app} \;\leq\; 2\mathrm{opt}_{LP}$

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Two examples (with unit weights)

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Therefore the integrality gap of the LP relaxation is $\mathrm{opt}_{ILP}/\mathrm{opt}_{LP} = 2$