The inefficiency of equilibria

Chapters 17,18,19 of AGT

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Outline

Formalization

- Like usual, we consider a game with
  - k players $\{1, \ldots, k\}$,
  - sets of strategies $\mathcal{S}_i$ for each player,
  - a utility function $u_i : \mathcal{S} \rightarrow \mathbb{Z}$ for each player, where $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_k$ is the set of all strategy vectors.
- unlike before, we also introduce a social function $\sigma : \mathcal{S} \rightarrow \mathbb{Z}$.
- Denote $E \subseteq \mathcal{S}$ for the set of all equilibria, and $S^* \in \mathcal{S}$ for the social optimum (the strategy vector maximizing σ).

Definition

The price of anarchy is $\min_{S \in E} \frac{\sigma(S)}{\sigma(S^*)}$.

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Definition

The price of stability is $\max_{S \in E} \frac{\sigma(S)}{\sigma(S^*)}$.

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Our tool: Potential function method

In general, the potential function method is the following:

- Suppose we want prove some property of some implicitly given subset E of a set S (for example, it is nonempty).
- Define a potential function $\phi : \mathcal{S} \rightarrow \mathbb{Z}$ such that E are exactly the (local) optima of ϕ.
- Since ϕ has a local optimum, E is non-empty.
- Algorithmically, this is also useful since an element of E can be found by optimizing ϕ (but in chapters 17,18 and 19 of the book people don't care).

Congestion games

Definition

A congestion game is a game with

- k players,
- a ground set of resources R,
- a cost function $c_r : \{1, \dots, k\} \to \mathbb{Z}$ for every $r \in R$, and
- a strategy set $\mathcal{S}_i \subseteq R$ for every player i.

In a strategy profile $S = (S_1, \dots, S_k)$, the cost of a player is defined as $c^i(S) = \sum_{r \in S_i} c_r(n_r)$, where $n_r(S)$ is the number of strategies in S containing r.

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Theorem (Rosenthal (1973))

Every congestion game has at least one pure equilibrium.

For a strategy profile $S = (S_1, \ldots, S_k)$ and an alternative strategy $S_i' \in \mathcal{S}_i$, denote (S_{-i}, S_i') for the strategy vector $(S_1, \ldots, S_{i-1}, S', S_{i+1}, \ldots, S_k)$.

Proof.

$$\text{Define}\phi_r(S) = \sum_{i=1}^{n_r(S)} c_r(i) \quad \text{and} \quad \phi(S) = \sum_{r \in R} \phi_r(S)$$

$$\begin{aligned} c^i((S_{-i}, S'_i)) - c^i(S) &= \sum_{r \in S'_i \setminus S_i} c_r(n_r(S)+1) - \sum_{r \in S_i \setminus S'_i} c_r(n_r(S)) \\ &= \sum_{r \in S'_i \setminus S_i} \phi_r((S_{-i}, S'_i)) - \phi_r(S) \\ &- (\sum_{r \in S_i \setminus S'_i} \phi_r(S) - \phi_r((S_{-i}, S'_i))) \\ &= \phi((S_{-i}, S'_i)) - \phi(S) \end{aligned}$$

□

Potential games

Definition

An exact potential function is a function $\phi : \mathcal{S} \to \mathbb{N}$ such that for every strategy vector S, player i and $S'_i \in \mathcal{S}_i$:

$$c^i((S^{-i}, S'_i)) - c^i(S) = c^i((S^{-i}, S'_i)) - \phi(S)$$

More general, ϕ is said to be ordinal if

$$sgn(c^i((S^{-i}, S'_i)) - c^i(S)) = sgn(\phi((S^{-i}, S'_i))\phi(S))$$

Definition

A game is an exact/ordinal potential game if it admits an exact/ordinal potential function.

Theorem (Rosenthal (1973))

Every congestion game is an exact potential game.

But, are exact potential function also useful in other settings? No:

Theorem (Monderer and Shapley (1996))

Every exact potential game is isomorphic to a congestion game.

- Potential game P with n players, k strategies each, potential ϕ.
- Create congestion game C with n players, k strategies and resource set $(\{0,1\}^k)^n$.
- Player i plays strategy q in P: uses all resources where player i chooses q in his subset in C.

Proof idea.

- Given a strategy vector S of P, define:
  - $b_{ij}(S) = 1$ if $j = \{q\}$ and q is used by player i in S, and 0 otherwise.
  - $b^p_{ij}(S) = 0$ if player $p \neq i$ plays a strategy in S that is contained in j, and 1 otherwise.
- Every resource $r = b_{ij}(S)$ is used by every player in S. Define $q_r(n) = \phi(r)$ and 0 otherwise.
- Every resource $r' = b^p_{ij}(S)$ is used only by player p. Define $q'_r(1) = c^i(r') - \phi(r')$ and 0 otherwise.

□

Ordinal potential games

- But for finding equilibria, ordinal potential functions also suffice.
- So what exactly is the scope of the "ordinal potential function method"?
- This appears to be exactly the complexity class *PLS* (to be defined in a few minutes).

Computational complexity of Congestion games

Now we study the computational complexity of the CONGESTION problem:

Given A congestion game, where the strategy sets are given explicitly.

Asked Construct an equilibrium.

Computational complexity of Congestion games

- Let l be the maximum size of a strategy set and W the maximum cost a resource has.
- How fast can we find an equilibrium?
- Brute-force: l^k.
- Using potential function: $\mathcal{O}^*(W)$.
- Can we expect a polynomial algorithm, is it NP-hard to find one?
- We already know there is a solution but have to find one (= TFNP), so NP-hardness doesn't make sense, but maybe we can prove it to be hard for one of these kind of classes?

Polynomial Local Search

Definition

A local search problem *P* belongs *PLS* if:

- For every instance, a polytime algorithm. computes an initial feasible solution.
- the objective function is polytime computable
- there is a polytime algorithm that states that a solution is locally optimal or gives a better one in it's neighborhood

(Recall PPAD are all problems reducible to the "END-OF-THE-LINE" problem. Similarly, PLS can be defined as all problems reducible to the "FIND-SINK" problem.)

Ordinal potential functions

The promise of a few slides back:

Theorem (Fabrikant et al. (2004))

The class of ordinal potential games "essentially" comprises of all problems of PLS.

Congestion is PLS-complete

Definition

A local search problem P belongs to PLS if:

- For every instance, a polytime algo. computes an initial feasible solution.
- the objective function is polytime computable
- there is a polytime algo that states that a solution is locally optimal or gives a better one in it's neighborhood

- First note that by Rosenthal's proof, CONGESTION is in PLS.
- We prove that CONGESTION is PLS-hard by a reduction from the PLS-complete L-MAX-CUT.

Reduction from L-MAX-CUT

Given: A graph $G = (V, E)$ with weighted edges.

Asked: A local maximum cut. That is, a cut that can not be improved by changing side of one vertex.

- Create a player for each vertex, and resources r_e^L and r_e^R for each edge $e \in E$.
- Each player $v \in V$ has two strategies:
  - use all resources r_e^L for every edge $e = (v, w)$
  - use all resources r_e^R for every edge $e = (v, w)$
- If a resource r_e^L (r_e^R) is used by one player, the cost is zero. If used by 2, the cost is $w(e)$.
- Minimizing the cost is maximizing weight of edges crossing.

Network congestion

- Given a digraph $G = (V, E)$ with positive weights on the edges and a source-sink pair (s_i, t_i) for every player i.
- Resources are edges, the strategies of player i are all $s_i t_i$ paths (hence, given implicitly).
- Pseudo-polynomial algorithm still applies, using shortest path computations.
- Polynomial if all source-sink pairs are the same, using a min-cost flow algorithm (Fabrikant et al. (2004))
  - for every edge $e \in E$, create n parallel edges with costs $c_e(1), \ldots, c_e(n)$ and capacity 1.
  - A min-cost st-flow of value n is the global optimum of the potential function, hence an equilibrium.
- Also know to be PLS-complete.

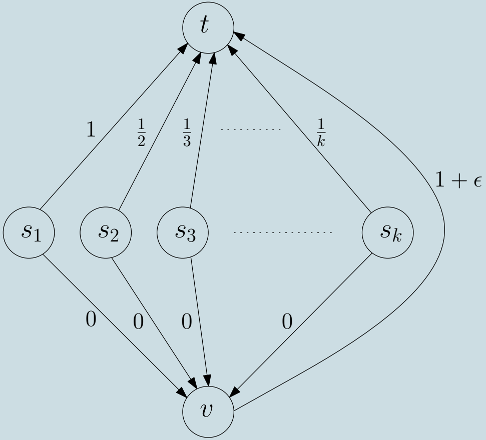
Shapley network design (aka Multicast)

- A special type of Network congestion where we are given
  - digraph $D = (V, E)$,
  - weight function $w : E \to \mathbb{Z}$, and
  - a source-sink pair (s_i, t_i) for every player i.
- The resource set is E, and the strategy set $\mathcal{S}_i$ for player i are all $s_i t_i$-paths.
- The cost of a resource $r \in E$ is given as

$$c_r(S) = \frac{w(r)}{n_r(S)}$$

- Define the social function $\sigma : \mathcal{S} \to \mathbb{N}$ as the sum of the costs of all players.

Shapley network design



Price of stability

Theorem

The price of stability in the MULTICAST *game is at most* $\ln(k)$.

Proof.

$$\phi_r(S) = \sum_{i=1}^{n_r(S)} c_r(i) = \sum_{i=1}^{n_r(S)} \frac{w(r)}{i} = w(r) \sum_{i=1}^{n_r(S)} \frac{1}{i} \leq w(r) \ln(n_r(S))$$

$$\sigma(S) \leq \phi(S) \leq \sigma(S) \ln k$$

□

FPT-ness of Multicast

What if we parameterize MULTICAST by the number of players? Is it FPT/XP?

Observation

There always exists an equilibrium without undirected cycles.

- Add weighted arcs for each shortest path
- Now look for an equilibrium with sources/ vertices of in-degree at least 2 / vertices of in-degree at least 2 / sinks.
- The number of vertices with in-degree / out-degree is at least 2 $2k$.
- $n^{2k} f(k)$ possibilities.

FPT-ness of Multicast

What if we parameterize MULTICAST by the number of players? Is it FPT/XP?

- It is even FPT. An FPT algorithm can be obtained by doing some dynamic programming similar to the dynamic programming algorithm for weighted Steiner Tree.

Conclusion

- Remember: Potential function method!!
- Used for all kinds of games to prove properties of equilibria.
- Exact potential games = congestion games; potential games = PLS.
- Potential function implies pseudo-polynomial algorithm for finding equilibrium.
- If you know there is a solution but want to construct one, and you want to prove your problem to be "hard". Look at subclasses of the complexity class TFNP, or define yet at another one.

Thanks for attending!!!

Any questions?