MORE CONSEQUENCES OF FALSIFYING SETH AND THE ORTHOGONAL VECTORS CONJECTURE

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Fine-Grained Complexity hypotheses and consequences

Problem k-SAT: Given formula in k-CNF with n variables, is it satisfiable?

Strong Exponential Time Hypothesis (SETH):
$\forall \varepsilon > 0\ \exists k$: k-SAT has no $O(2^{(1-\varepsilon)n})$-time algorithm

Hitting Set, Set Splitting, NAE-SAT: $O^*(2^n)$ but not $O^*((2-\varepsilon)^n)$ [CDLMNOPSW'16]
Independent Set: $O^*(2^{tw})$ but not $O^*((2-\varepsilon)^{tw})$ [LMS'11]
Subset Sum: $\tilde{O}(n+t)$ but not $t^{1-\varepsilon}2^{o(n)}$ [B'17, ABHS'18+]

Problem Orthogonal Vectors: Given sets $A, B \subseteq \{0,1\}^d$ of size n, are any $a \in A, b \in B$ orthogonal? $(\sum_i a_i \cdot b_i = 0)$
OV-Hypothesis (OVH) (moderate dimension):
$\forall \varepsilon, \delta > 0$: OV in $d = n^\delta$ has no $O(n^{2-\varepsilon})$-time algorithm

No $O(n^{2-\varepsilon})$ algorithm for: Edit Distance, LCS, Diameter-2, Frechet distance, NFA Acceptance, RegExp Matching, ... [BI'15,ABVW'15,BK'15,VWR'13,B'14,BI'16,BGL'17]
No $O(m \cdot n^{2-\varepsilon})$ algorithm for All Pairs Maxflow [KT'17]
Dynamic graph algorithms, Hardness of Approximation in P, ...

Reasons to believe in fine-grained hypotheses:
- **decades** of effort
- **restricted algorithms** cannot refute them
- falsifying them implies **circuit lower bounds**
- falsifying them implies fast algorithms for **more problems**

Our focus

Our Contributions

We find more consequences of falsifying SETH/OVH:

Thm1 If SETH fails then:
there are $O((2-\varepsilon)^n)$-time algorithms for sparse-TC^0-SAT

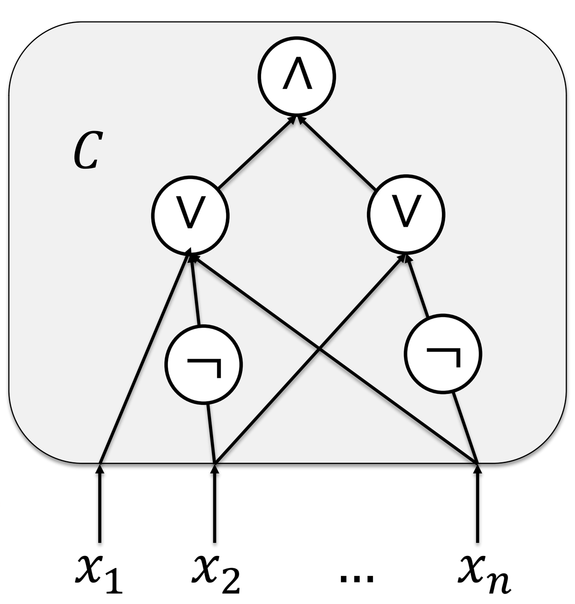
Thm2 If OVH fails then:
(i) there are $O((2-\varepsilon)^n)$-time algorithms for sparse-TC^1-SAT
(ii) there are $O(n^{(1-\varepsilon)k})$-time algorithms for weighted k-Clique

Consequences for Circuit Sat

$\mathcal{C}$ = class of circuits

$\mathcal{C}$-SAT: Given a circuit $C \in \mathcal{C}$ on n variables, is C satisfiable?

$\omega(\mathcal{C})$ = $\inf\{ w \mid \mathcal{C}\text{-SAT is in time } O^*(2^{w \cdot n}) \}$



SETH: $\lim_{k\to\infty} \omega(k\text{−CNF}) = 1$

$\Longleftrightarrow$ (sparsification lemma [IPZ'01]) $\lim_{k,c\to\infty} \omega(c\text{−sparse } k\text{−CNF}) = 1$

sparse TC0 circuits

$\Longleftrightarrow \lim_{c,d\to\infty} \omega(c\text{−sparse depth−}d\ \{\vee,\wedge,\neg,THR\}\text{−circuit}) = 1$

not known to be in time $2^n/n^{\omega(1)}$ for $c = 100, d = 4$

Thm 1

previously similar results known for sparse (formulas / AC^0 / VSP-circuits) [SS'12,DW'13,CDLMNOPSW'16]

OVH fails $\Longrightarrow \lim_{c,d\to\infty} \omega(c\text{−sparse depth−}(d \log n)\ \{\vee,\wedge,\neg,THR\}\text{−circuit}) < 1$

Thm 2(i)

sparse **TC1** circuits

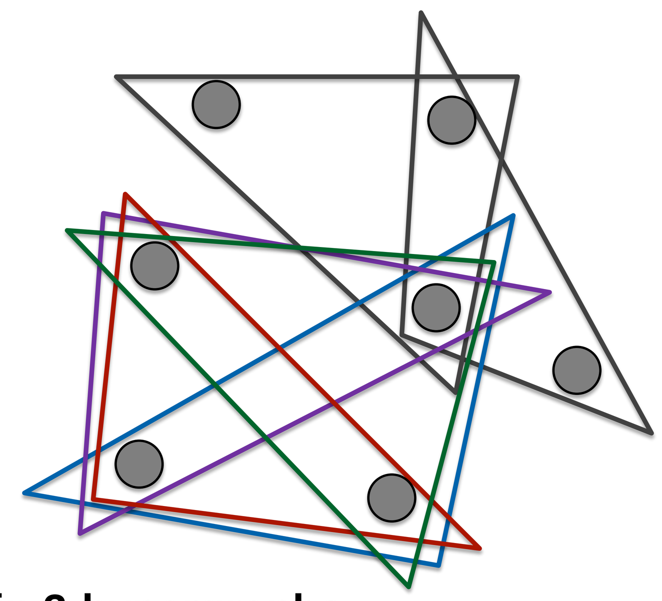
Proof techniques:
- Refinement of techniques from proof of Cook-Levin theorem
- Valiant's depth reduction

Consequences for Clique problems

(Weighted) k-Clique in Hypergraphs

r-hypergraph: $G = (V, E)$ with $E \subseteq \binom{V}{r}$ — 2-hypergraph = graph

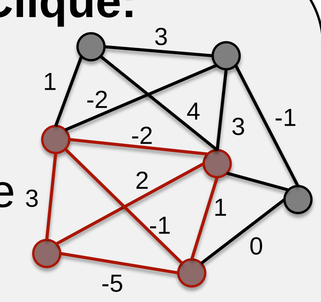
k-Clique in r-hypergraph:
Find vertices $v_1, \dots, v_k$ s.t.
for any $e \subseteq \{v_1, \dots, v_k\}$ of size r we have $e \in E$



$O(n^{0.79k})$ known for **k-Clique in graphs** [NP'85]

$O(n^{k-\varepsilon})$ not known for **Neg-k-Clique in graphs** or **k-Clique in 3-hypergraphs**

Problem Negative-k-Clique:
Given edge-weighted graph G is there a k-Clique with negative total edge-weight?



Neg-k-Clique-Hypothesis:
$\forall \varepsilon > 0, k \geq 3$: Neg-$k$-Clique has no $O(n^{k-\varepsilon})$ algorithm

Tree Edit Distance is not in $O(n^{3-\varepsilon})$ [BGMW'18]
Maximum Weight Rectangle is not in $O(n^{d-\varepsilon})$ [BDT'16]
Viterbi's Algorithm cannot be improved to time $O(t\, n^{2-\varepsilon})$ [BT'17]

OVH fails $\Longrightarrow$ $O(n^{k-\varepsilon})$ for **Neg-k-Clique in r-hypergraphs,**
for any $k \gg r$ and weights bounded by $n^{f(k)}$

Thm 2(ii)

Proof outline Thm 2(ii)

Chain of reductions

Neg-$2k$-Clique r-hypergraphs $\xrightarrow{1}$ ExactWeight-$2k$-Clique r-hypergraphs $\xrightarrow{2}$ $2k$-Clique $2r$-hypergraphs $\xrightarrow{3}$ OV

1 Reduce threshold to exact target by rounding [VW'09, NLZ'12]

If (G, w) has a positive k-clique, then:
(G, w) has a k-clique of weight t for some $t \in \{1, 2, \dots, k^r\}$
or
$\left(G, \left\lfloor \frac{w}{2} \right\rfloor\right)$ has a positive k-clique
$\longrightarrow$ solve $k^r \log W = O(\log n)$ instances of ExactWeight-k-Clique

2 Removing weights by increasing arity *inspired by [ALW'14]*

Fix $r = 2$ for simplicity (the same arguments extend for $r > 2$)

1. Use base $B \approx \log n$ expansion to replace ints $w(e)$ with vectors $w'_\ell(e)$
   - guess carries c_ℓ to embed addition into $\mathbb{Z}^d$
2. Assume target vector t equals $\vec{0}$
3. Define $w''(h)$ so that **(+)** holds

Now the weights are significantly reduced (i.e. at most $polylog(n)$)

$$\sum_{e \subseteq C} w'_\ell(e) = 0 \quad \forall \ell$$
$$\Leftrightarrow \sum_\ell \Big(\sum_{e \subseteq C} w'_\ell(e)\Big)^2 = 0$$
$$\Leftrightarrow \sum_{e_1, e_2 \subseteq C} \sum_\ell w'_\ell(e_1) \cdot w'_\ell(e_2) = 0$$
$$\overset{(+)}{\Leftrightarrow} \sum_{h \subseteq C, |h| = 4} w''(h) = 0$$

4. Guess the weight of every hyperedge in the solution

$polylog(n)^{O(k^4)}$ overhead

5. Keep only edges with the guessed weight.

3 Encode Constraint Satisfaction problem as Orthogonal Vectors

Fix $r = 2$ for simplicity (same arguments extend for $r > 2$)

A contains a vector $a = a(u_1, \dots, u_k)$ for every k-clique $\{u_1, \dots, u_k\}$ of G

B contains a vector $b = b(v_1, \dots, v_k)$ for every k-clique $\{v_1, \dots, v_k\}$ of G

We have a dimension for every tuple (i, j, u, v) with $i, j \in [k]$, $\{u, v\} \in V^2 \setminus E$:

$$a_{(i,j,u,v)} = \begin{cases} 1 & \text{if } u_i = u \\ 0 & \text{ow.} \end{cases} \qquad b_{(i,j,u,v)} = \begin{cases} 1 & \text{if } v_j = v \\ 0 & \text{ow.} \end{cases}$$

a, b are non-orthogonal $\Leftrightarrow \exists$ dimension (i, j, u, v) such that $u_i = u$ and $v_j = v$
$\Leftrightarrow (u_1, \dots, u_k, v_1, \dots, v_k)$ is no $2k$-clique

The OV instance is on $n = O(|V|^k)$ vectors of dimension $d = O(|V|^2)$