Some questions in algebraic geometry

(preliminary version, June 1995)

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"Een zot kan meer vragen
dan tien wijzen kunnen beantwoorden."
Een oud Nederlands spreekwoord.

"One fool can ask more questions
than ten wise men can answer."
An old Dutch saying.

Introduction

In June 1995 several mathematicians will gather in Utrecht for a conference on

"Arithmetic and geometry of abelian varieties,"

and this seems a good occasion to share with them some of the questions that have occupied my mind over the years.

These pages contain some of these problems. Their choice, approach and presentation do, I am afraid, bear a distinctly personal stamp - my own. Very likely they will bring a smile to the lips of some of the experts present at the conference, and there may even be some raisings of eyebrows. Yet I hope that among you there may be some that are willing to spend some time with these musings, even though there are, as we all realize, more important questions to be studied in the mathematics of today.

Most of these ideas presented here I have over the past thirty years shared with colleagues, but the part which each of them has played in shaping my thoughts is in most cases no longer traceable. Only such ideas that I believe to have originated in my own mind and which, moreover, I consider likely to be true, are here labelled as "conjectures".

Very likely some of these questions might have been formulated more precisely. I may also not be aware of the fact that some of them have already been solved, or were found to be connected to other problems: references to that effect I may easily have missed.

If you have any comments on the problems as they are presented here, please let me know.

NB This is the 1995 version, NOT UPDATED.

Contents

1. Finite group schemes.

Suppose S is a scheme, which we suppose to be noetherian and connected. Let $N \to S$ be a finite, flat group scheme over S. For every point $s \in S$ the fiber N_s is a finite group scheme; its rank does not depend on the point s chosen; we call this number the rank of $N \to S$.

1A. Question: *Is a finite group scheme $N \to S$ annihilated by its rank?*
I.e. let n be the rank of $N \to S$, and let $m_n : N \to N$ be "exponentiation by n", does this map factor through the unity section $e : S \to N$:

$$(m_n : N \to N) \quad \stackrel{?}{=} \quad (N \to S \stackrel{e}{\to} N).$$

1.1. I do not know whether Cartier duality (leading from a non-commutative group scheme to a group scheme in non-commutative geometry) could be of any help in this question. One could try to phrase and answer the analogous question in non-commutative geometry.

1.2. In case the group scheme is commutative the answer is affirmative, as was proved by Deligne, see the theorem on page 4 of [65].

In case S is the spectrum of a field K, the answer is affirmative: it suffices to show the statement over an extension of K, so take the algebraic closure $\overline{K}$; for étale group schemes the result follows from plain group theory; in characteristic zero every group scheme is reduced

(Cartier), hence in that case we are done; in positive characteristic, over a perfect field, a finite group scheme is the direct sum of a local and an étale group scheme; the result follows for local group schemes by the structure theorem [66], 14.4, theorem.

Note that it suffices to answer the question for all cases where S is the spectrum of a local artin ring.

I expect that in case of a "small extension" $R \to R'$ with a finite flat group scheme N over R such that $N' = N \otimes R'$ is annihilated by its rank, the liftability problem of N' to R can be linked with the annihilation of N by its rank, and possibly an answer to Question (1A) can be obtained in this way.

2. Endomorphisms of abelian varieties.

Let K be a field, and let X be an abelian variety over K. We denote by $m(X)$ the rational number

$$m(X) = \frac{2g}{[D:\mathbb{Q}]}$$

(standard notation introduced by Shimura, see [62], page 156), where $g = \dim(X)$, and $D = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, the endomorphism algebra of X.

2A. Question: *Suppose given $m \in \mathbb{Q}_{>0}$. Does there exist a simple abelian variety over a field such that $m = m(X)$?*

2.1. In case $\mathrm{char}(K) = 0$, the number m is the rank of the lattice Λ (as $D-$module) which can be used to define the complex torus $\mathbb{C}^g/\Lambda \cong X(\mathbb{C})$; hence in case D is a division algebra it follows that this rank in an integer.

It is easy to see that any integer $m \in \mathbb{Z}_{>1}$ can be realized in this way (consider Hilbert type real multiplications). For $m = 1$ we can take a simple CM abelian variety.

If X is simple, and $m(X)$ is not an integer, then $\mathrm{char}(K) > 0$. If X is simple, and $2{\cdot}m(X)$ is not an integer, then $\mathrm{char}(K) > 0$ and $\mathrm{End}^0(X)$ is an algebra of Type IV.

Note that for any $d \in \mathbb{Z}$ the number $m := \frac{1}{d}$ can be realized: for $d = 2$ choose a supersingular elliptic curve; for $d > 2$, choose an integer n with $0 < n < d$, and define a p^d Weil number as a zero of $T^2 + p^n{\cdot}T + p^d$; this constructs an isogeny class of a simple abelian variety X of dimension d over a finite field, where its endomorphism algebra is an algebra of rank d^2 over an imaginary quadratic field, see [64], "problème de Manin" and page 253-04.

Possibly the question can be answered in case $m = \frac{a}{d}$ by using the previous example by Tate, letting D act diagonally on $Y_0 = X^a$ and trying to produce a deformation in characteristic p of Y_0 such that this action is exactly all that survives (could methods as in [54] be used?).

3. The Torelli mapping.

Let $\mathcal{M}_g$ be the coarse moduli scheme over Spec($\mathbb{Z}$) for complete, irreducible, nonsingular curves of genus g. Let $\mathcal{A}_{g,1}$ be the coarse moduli scheme over Spec($\mathbb{Z}$) of principally polarized abelian varieties. The Torelli morphism is denoted by

$$j : \mathcal{M}_g \ \rightarrow \ \mathcal{A}_{g,1},$$

it maps the moduli point of C to the moduli point of the canonically polarized $Jac(C)$. This morphism is injective on the set of geometric points, cf. [3]. It is known that

$$j_{\mathbb{Q}} : \mathcal{M}_g \otimes \mathbb{Q} \ \rightarrow \ \mathcal{A}_{g,1} \otimes \mathbb{Q}$$

is an immersion, see [55], Corollary 3.2. More generally, if $[C] \in \mathcal{M}_g(k)$, with k algebraically closed and char(k) not dividing the order of Aut(C), then j is an immersion at this point.

3A. *Determine all points in $\mathcal{M}_g$ at which the Torelli mapping j is not an immersion.*

4. Lifting automorphisms of algebraic curves.

Let k be a field of characteristic p, let C_0 be an algebraic curve over k (absolutely irreducible, reduced, non-singular, complete). Let H_0 be a subgroup of the automorphism group:

$$H_0 \subset \mathrm{Aut}(C_0).$$

4A$_?$ Conjecture: *In case the group H_0 is* cyclic, *the pair (C_0, H_0) can be lifted to characteristic zero.*
I.e. there should exist an integral domain R of characteristic zero, a homomorphism $R \rightarrow k$, a curve $\mathcal{C} \rightarrow \mathrm{Spec}(R)$ (flat, proper, smooth) with $\mathcal{C} \otimes k \cong C_0$ and a subgroup $H \subset \mathrm{Aut}_R(\mathcal{C})$ such that the two subgroups are identified: $H \xrightarrow{\sim} H_0$.

4.1. In case we drop the condition "cyclic", counterexamples are easy to give. Commutative counterexamples are easy to give. Here is a non-commutative example. Consider the normalization C_0 of the plane projective curve defined by the affine equation $Y^p - Y = X^2$. This curve has genus $g = (p-1)/2$ if the characteristic of the base field is not equal to 2. As Roquette showed: choose $p \geq 5$; if the base field is $\overline{\mathbb{F}_p}$, this curve has an automorphism group of order $8g(g+1)(2g+1) = 2(p-1)p(p+1)$, see [59], Satz 1. Any lifting of this curve to characteristic zero has, by a theorem of Hurwitz, an automorphism group of order at most $84(g-1)$. Hence this curve C_0 with its full group of automorphisms cannot be lifted to characteristic zero.

Note that in the previous example for any automorphism β_0 of C_0 the pair (C_0, β_0) can be lifted to characteristic zero (see (4.2) below). However in [47], page 166 we see a non-commutative subgroup $H_0 \subset \mathrm{Aut}(C_0)$ of order 20 such that (C_0, H_0) cannot be lifted to characteristic zero.

4.2. In [60] we find: for a curve C_0 and an automorphism $\sigma_0 \in \mathrm{Aut}(C_0)$ such that p^2 does not divide the order of σ_0, the pair (C_0, σ_0) can be lifted to characteristic zero.

4.3. In [13] we find a lifting of a curve plus a group of automorphisms in characteristic p to a possibly singular curve in characteristic zero.

5. Algebraic curves with sufficiently many complex multiplications.

Let X be an abelian variety over a field K; we say that X admits *sufficiently many complex multiplications* (abbreviated smCM) if the endomorphism algebra $\mathrm{End}^0(X)$ contains a commutative, semisimple algebra of rank $2g$ over $\mathbb{Q}$, where $g = \dim(X)$ (several other definitions can be given). For example, if X is a simple abelian variety over a field of characteristic zero which admits smCM, then $\mathrm{End}^0(X)$ is a CM-field of degree $2{\cdot}\dim(X)$ over $\mathbb{Q}$. We say that an algebraic curve C is a CM-curve if $\mathrm{Jac}(C)$ admits smCM. A point $[(X,\lambda)] \in \mathcal{A}_g(\mathbb{C})$ is called a CM-point if X admits smCM.

5A. Question (Coleman, see [6], page 238, Conjecture 6): *Let*

$$\mathcal{C}_g := \{[C] \in \mathcal{M}_g(\mathbb{C}) \mid C \quad \text{is a CM-curve}\}.$$

Is it true that for a fixed $g \in \mathbb{Z}_{\geq 4}$ this number is finite:

$$\#(\mathcal{C}_g) \quad < \infty \ \ \mathbf{?}$$

5.1. It is not so difficult to show that for every $g \leq 3$ the set $\mathcal{C}_g$ is infinite: for these values

$$j(\mathcal{M}_g) \subset \mathcal{A}_{g,1}$$

is Zariski-dense, and the set of CM-points in $\mathcal{A}_{g,1}(\mathbb{C})$ is dense (in the Zariski topology, but also in the classical topology), and the conclusion follows. Note however that this approach does not construct explicitly CM-curves of genus 2 or 3. And this is the difficulty of the problem: on the one hand it is "easy" to construct algebraic curves, but how can we read off from properties of C what are the endomorphisms of $\mathrm{Jac}(C)$? It is easy to construct principally polarized CM abelian varieties. For $g \leq 3$ a principally polarized abelian variety over an algebraically closed field is a Jacobian (possibly of a reducible curve), see [57]. However for $g > 3$ it seems difficult to decide for a given polarized abelian variety whether it is the Jacobian of an algebraic curve. Typically: being a Jacobian ánd properties of $\mathrm{End}(X)$ are not easy to compare.

5.2. In [20] Johan de Jong and Rutger Noot prove that $\mathcal{C}_4$ and $\mathcal{C}_6$ are infinite sets. They get round the difficulty "CM versus being a Jacobian" in the following way. Choose a prime number ℓ. They construct a family of curves of genus $g = \ell - 1$ over an open set $U = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ in $\mathbb{P}^1_{\mathbb{C}}$, where every fiber has an automorphism of order ℓ, by considering $Y^\ell = X(X-1)(X-\lambda)$. Note that $[\mathbb{Q}(\zeta_\ell) : \mathbb{Q}] = \ell - 1 = g$ hence every fiber has something like "half CM".

$\boldsymbol{\ell}$**=2.** For $\ell = 2$ this is the Legendre family of elliptic curves, and we know that for infinitely may values of λ the corresponding fiber is an elliptic curve with CM.

$\boldsymbol{\ell}$**=3,** $\boldsymbol{\ell}$**=5 and** $\boldsymbol{\ell}$**=7.** For a prime number $\ell > 2$ one can try to show the analogous fact that for infinitely many values of λ the corresponding fiber has a Jacobian with smCM. For $\ell \leq 7$ one easily sees that the image of U in $\mathcal{A}_g$ is (an open set in) a one-dimensional Shimura variety (of PEL type); for such varieties we know the set of CM-points is dense, and we are done.

$\boldsymbol{\ell}$ **>7.** For a prime number $\ell > 7$, this gives a one-dimensional family in $j(\mathcal{M}_g) \subset \mathcal{A}_{g,1}$ but the smallest Shimura variety containing it has dimension at least two, and we expect (see (6) and (7) below) that the number of values for λ where the corresponding fiber has a Jacobian with smCM is finite.

5.3. By the way, note that for a prime number p, the set $\mathcal{M}_g(\overline{\mathbb{F}_p})$ is infinite, and every point corresponds with a Jacobian which admits smCM (by a theorem by Tate, see [63]). We see that the characteristic p analog of the question by Coleman gives an infinite set for every p and every $g > 0$.

5.4. One could formulate the *"local Coleman problem"*: fix an algebraic curve C_0 over $\overline{\mathbb{F}_p}$, and consider the set $\mathcal{L}(C_0)$ of isomorphism classes of CM-curves defined over $\mathbb{C}$ which modulo p reduce to C_0 (one has to give proper and precise definitions). Suppose that $\mathrm{Jac}(C_0)$ is an ordinary abelian variety. Is the set $\mathcal{L}(C_0)$ finite? For $g \leq 4$ and for $g = 6$ this set is infinite, for other values we do not know the answer in general.

Note that the completion of $\mathcal{A}_{g,1}$ at an ordinary point $[(X_0, \lambda_0)]$ has a group structure (by a theorem by Serre and Tate), and that in this group structure the torsion points are exactly the CM-liftings (called quasi-canonical liftings).

Consider the Manin-Mumford conjecture: if a (possibly singular) Riemann surface S contained in a complex torus T has the property that $\#(S \cap Tors(T)) = \infty$, then the genus of S is one; the Manin-Mumford conjecture was proved by Raynaud, see [58]. We see a striking analogy between the local Coleman conjecture on the one hand, and the Manin-Mumford conjecture on the other hand. This analogy stimulated me with respect to (6A) below.

6. Dense sets of CM-points.

We have already seen that Shimura varieties contain a dense set of CM-points. We expect that the converse is also true:

6A$_?$ Conjecture: *Let $V \subset \mathcal{A}_g \otimes \mathbb{C}$ be a subvariety which contains a Zariski-dense set of CM-points. Then V is a subvariety of Hodge type* (terminology as in [34], I.3.8 and see [34], IV.1.2.)

6.1. Also see [52], (2). After I formulated this conjecture I found out that a special case ($\dim(V) = 1$) already had been formulated by André, see [1], page 215, Problem 1.

6.2. As for the local problem: let $[(X_0, \lambda_0)] = x_0 \in \mathcal{A}_{g,1}(\overline{\mathbb{F}_p})$ be an ordinary point, and consider the completion

$$\mathcal{G} = (\mathcal{A}_{g,1})^{\wedge}_{x_0}$$

in that point; then $\mathcal{G}$ is a formal group scheme, in fact this is a formal torus (over some extension of the base ring) it is isomorphic with $(\mathbb{G}_m^{\wedge})^{g(g+1)/2}$. Consider an algebraic variety $V_K \subset \mathcal{A}_g \otimes K$ over a field K in characteristic zero, whose extension $\mathcal{V}$ contains x_0; write $\mathcal{V}^{\wedge}_{x_0} \subset \mathcal{G}$ for its completion. Rutger Noot showed: V is a subvariety of Hodge type (in the terminology of [34]) if and only if $\mathcal{V}^{\wedge}_{x_0}$ is a formal subtorus (up to translation over a torsion point, up to taking irreducible components), for precise formulations see: [38], Prop. 2.2.3, see [39], Th. 3.7; for the converse see [34], III.5.2

6.3. In his PhD-thesis [34] Ben Moonen answers a particular case of Conjecture 6A. Sets of CM-points which have enough points of good, ordinary reduction can be studied via Serre-Tate parameters, for a precise formulation see [34], IV.1.4 and IV.1.8.

6.4. However, here is already a case of Conjecture (6A) which I am unable to solve: Let

$$W = \mathcal{A}_{1,1} \times \mathcal{A}_{1,1} = \mathbb{A}^1 \times \mathbb{A}^1,$$

and consider a set $S \subset W(\mathbb{C})$ of CM-points. Suppose that S is not contained in a finite union of curves of the folowing type: horizontal fibers, vertical fibers and modular curves. In this case the conjecture would imply that the Zariski-closure of S is $W \otimes \mathbb{C}$.

7. Shimura varieties contained in the Torelli locus.

This section is largely taken from [52]. Consider the Torelli morphism

$$j : \mathcal{M}_g \longrightarrow \mathcal{A}_{g,1}$$

see (3). The image of this mapping will be denoted by

$$j(\mathcal{M}_g) = \mathcal{J}_g^o \subset \mathcal{A}_{g,1}$$

and will be called the *open Torelli locus*, its Zariski-closure is called the *Torelli locus*, denoted by $\mathcal{J}_g \subset \mathcal{A}_{g,1}$. Note that for $g \leq 3$ we have $\mathcal{J}_g = \mathcal{A}_{g,1}$, and for $g \geq 4$ the set $\mathcal{J}_g$, which is of relative dimension $3g-3$ over $\mathrm{Spec}(\mathbb{Z})$, is a proper closed subset of $\mathcal{A}_{g,1}$.

7A. *For a fixed integer $g \in \mathbb{Z}_{\geq 4}$ determine all varieties of Hodge type of positive dimension in $\mathcal{A}_{g,1} \otimes \mathbb{C}$ which are contained in the Torelli locus $\mathcal{J}_g$, and which meet the open Torelli locus $\mathcal{J}_g^o$.*

7B. Here is an example of the previous question for which I do not know the answer. In [36] Mumford constructed Shimura varieties of dimension one in $\mathcal{A}_{4,1} \otimes \mathbb{C}$ (which are not Shimura varieties of PEL type). Let us call these curves "Mumford curves."
Question: *Is any of the Mumford curves contained in the Torelli locus $\mathcal{J}_4$?*

7.1. Note that a complete answer to (7A), and a positive answer to Conjecture (6A) would settle completely Coleman's Question (5A):
If (6A) *holds, and $\#(\mathcal{C}_g) = \infty$, then there exists a variety of Hodge type of positive dimension as in* (7A). Indeed, the Zariski closure of this infinite set is a finite union of varieties of Hodge type, at least one of these is as in (7A).
Conversely, for a variety of Hodge type V as in (6A) we have that $V \cap \mathcal{J}_g^0$ is open in V, and the set of CM-points is Zariski-dense in V, hence the existence of a variety as in (6A) implies that $\#(\mathcal{C}_g) = \infty$.

Consider $\mathcal{L}(C_0)$ as in (5.4), with g =genus(C_0). Suppose this is an infinte set. Methods of [34] can be used to show that the Zariski closure of this set in $\mathcal{A}_g$ is a finite union of subvarieties of Hodge type, and we would have a variety as in (7A) for this value of g.

7.2. An obvious observation: there are many varieties of Hodge type of positive dimension in $\mathcal{J}_g$ (for $g \geq 2$): take $g = g_1 + \cdots + g_r$ with all $g_i \leq 3$, take varieties of Hodge type $V_i \subset \mathcal{A}_{g_i}$, at least one of positive dimension (see 5.1), the image of the product of these gives a variety of Hodge type in $\mathcal{J}_g$ which however does not meet $\mathcal{J}_g^o$ if $r > 1$.

7.3. In [34], II.3 we find a characterization of varieties of Hodge type over $\mathbb{C}$ as totally geodesic subvarieties of $\mathcal{A}_g \otimes \mathbb{C}$. It might be that an approach along these lines could shed some light on Question (7A).

Suppose $g \geq 4$. We can see that $\mathcal{J}_g \otimes \mathbb{C}$ is not a subvariety of Hodge type of $\mathcal{A}_{g,1} \otimes \mathbb{C}$. It would be interesting to have a better understanding of the analogy between methods of differential geometry, and say methods as developed in [9].

8. Strata given by Newton polygons.

For an abelian variety X in positive characteristic we can define its *Newton polygon* $\mathcal{N}(X)$; we write NP for "Newton Polygon". For an abelian variety over a finite field a definition can be given using properties of the geometric Frobenius. A definition can be given by comparing the kernels $X[p^i]$ and $X[F^j]$ for large values of i and j. A definition can be given by decomposing the p-divisible of X over an algebraically closed field up to isogeny into isosimple factors, see [32], page 35; just to fix soms notations: the isosimple p-divisible group as denoted by Manin by $G_{m,n}$ (with $m, n \in \mathbb{Z}_{>0}$ relatively prime) has dimension m, its Serre-dual has dimension n, and the part of the NP given by this isogeny factor has slope $n/(m+n)$ over an interval of length $m+n$. In this way a formal isogeny type

$$X[p^\infty] \otimes k \sim \sum \; G_{m_i,n_i} \otimes k$$

defines the (lower convex) NP of X (by ordering the slopes $n_i/(m_i+n_i)$ in a non-decreasing order).

For a given value g all Newton polygons belonging to g form a partially ordered set (where the ordering is given by "lying above"). For a given NP, say β, we consider the set

$$W_\beta \subset \mathcal{A}_g \otimes \mathbb{F}_p$$

consisting of all polarized abelian varieties X such that no part of the NP $\mathcal{N}(X)$ lies below β. By theorems by Grothendieck and Katz, see [23], page 143, Corollary 2.3.2, it is known that W_β is a *closed* subset of $\mathcal{A}_g \otimes \mathbb{F}_p$. We write $W_{\beta,1} = W_\beta \cap \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for the locus in W_β of principally polarized abelian varieties.

8A. Question: *Above we have defined W_β as a closed subset. What is a good "functorial definition", what is a good definition of this stratum as a scheme?*

8.1. Stratification by p-rank. For an abelian variety X in characteristic p we define $f(X)$, its p-rank, by:

$$f(X) = f \iff X(\overline{K})[p] \cong (\mathbb{Z}/p)^f.$$

Note that $0 \leq f \leq \dim(X)$. For a given value f we define

$$V_f := \{(X, \lambda) \mid f(X) \leq f\}.$$

It is easy to see that this is a closed set in $\mathcal{A}_g \otimes \mathbb{F}_p$, and it is easy to see that every component of V_f has dimension at least $(g(g+1)/2) - g + f$. In fact:

Theorem: *Every component of V_f has exactly this dimension:*

$$\dim(V_f) = (g(g+1)/2) - g + f.$$

In case of principal polarizations this was proved by Koblitz, see [25], Theorem 7 on page 163, for the general case see [42], Theorem 4.1.

8.2. We remark that the p-rank strata are particular cases of the NP-strata. For any g and any f with $0 \le f \le g$ there is a unique NP β such that every NP lying above β has p-rank at most f, in particular

$$W_\beta = V_f.$$

In fact this is clear for $f = g$, for $f = g - 1$ take $f(1,0) + (1,1) + f(0,1)$ and for $f \le g - 2$, take the NP given by $f(1,0) + (g - f - 1, 1) + (1, g - f - 1) + f(0,1)$.

8.3. As usual, we define the *ordinary* NP ρ by $g(1,0) + g(0,1)$, i.e. the NP belonging to $g = f$ (i.e. only slopes 0 and 1 appear). We define the *supersingular* NP σ by $g(1,1)$ (i.e. the NP is a line, all slopes are equal to $\frac{1}{2}$). The locus $\mathcal{S}_{g,1}$ of principally polarized supersingular abelian varieties

$$W_\sigma \supset \mathcal{S}_{g,1} \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$$

has dimension equal to

$$\dim(\mathcal{S}_{g,1}) = [\frac{g^2}{4}]$$

(as conjectured in [43],page 616, [ ]: integral part). The number of components of this supersingular locus can be expressed as a class number (and this number is large for p large):

$$\#(\text{irreducible components of} \quad \mathcal{S}_{g,1} \otimes \overline{\mathbb{F}_p}) = \begin{cases} H_g(p,1) & \text{if} \quad g \quad \text{is odd,} \\ & \\ H_g(1,p) & \text{if} \quad g \quad \text{is even.} \end{cases}$$

For $g = 1$ this is due to Deuring (using a class-number computation by Eichler), and to Igusa, for $g = 2$ see [21], Theorem 5.7, for $g = 3$ see [22], Theorem 6.7, and for general g, see [28], Theorem 4.9. In particular we see that this number is large for p large (e.g. for $g = 1$ this number equals 1 iff $p \in \{2, 3, 5, 7, 13\}$, for $g = 1$ it is 1 iff $p \le 11$, for $g = 3$ it is 1 iff $p = 2$). To summarize: *in most cases the supersingular locus is reducible.*

8B? Conjecture: *Let β be a NP, with $\beta \ne \sigma$, i.e. W_β is not the supersingular locus. Then for principally polarized abelian varieties:*

$$W_{\beta,1} \quad \textit{is irreducible.}$$

In some cases this has been proved. Faltings (for all p) and Chai (for $p > 2$) showed that $W_{\rho,1} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is irreducible. In [53] it is shown that $V_{g-1,1} \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is irreducible (and in that paper the previous result by Faltings and Chai is proved again). I have proved that $V_{0,1} \subset \mathcal{A}_{3,1} \otimes \mathbb{F}_p$ is irreducible (unpublished). For the general case I have glimpses of a possible proof of (8B).

9. Strata given by NP in the moduli space of curves.

Consider the strata

$$W_\beta \subset \mathcal{A}_g \otimes \mathbb{F}_p$$

given by Newton Polygons, and consider the open Torelli locus

$$j(\mathcal{M}_g \otimes \mathbb{F}_p) = \mathcal{J}_g^o \otimes \mathbb{F}_p \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$$

and its closure, the Torelli locus $\mathcal{J}_g \otimes \mathbb{F}_p$.

9A. Question: *For a given $g \in \mathbb{Z}_{\geq 4}$ and a given NP β describe the intersection*

$$W_\beta \cap (\mathcal{J}_g \otimes \mathbb{F}_p).$$

9.1. At the moment I have no reasonable guess what kind of answer could be expected. Does there exist g and β such that this intersection is empty? Note that $\dim(W_\beta)$ is independent of p; however, if the intersection in the question is not empty, does the dimension (of a component of) this intersection depend on p? Note that $\dim(\mathcal{S}_{g,1}) = [g^2/4]$ and $\dim(\mathcal{J}_g \otimes \mathbb{F}_p) = 3g-3$, hence if

$$g \geq 9 \quad \text{then} \quad \dim(\mathcal{S}_{g,1}) + \dim(\mathcal{J}_g \otimes \mathbb{F}_p) < \dim(\mathcal{A}_g \otimes \mathbb{F}_p).$$

However the intersections of the various W_β with the Torelli locus are not “as transversal as possible”: G. van der Geer and M. van der Vlugt construct for arbitrary p and large g supersingular curves in characteristic p of genus g, see [14], [15].

It might very well be that intersections of the p-rank strata V_f with the Torelli locus are transversal in the sense that I expect that (for $g \geq 2$):

$$\dim(V_f \cap (\mathcal{J}_g \otimes \mathbb{F}_p)) \stackrel{?}{=} 3g-3-g+f.$$

One could hope that intersections as above give effective cycles in $\mathcal{M}_g \otimes \mathbb{F}_p$ of which the Chow classes can be computed. This might give insight in the Chow ring of $\mathcal{M}_g$.

9.2. We see a kind of question which in general is difficult. Consider two subsets of a moduli space, each characterized by certain properties of the objects we classify. Try to determine properties of the intersection of these sets. If the properties are difficult to compare, such a question seems difficult in general. Example are the following: consider CM-Jacobians (see 5), or Jacobians in positive characteristic with a given Newton polygon as above, or Jacobians with a given endomorphism ring [5], intersecting Mumford curves with the Torelli locus as in (7B), or study moduli spaces of hyperelliptic supersingular curves [48].

10. Algebraic computation of fundamental groups.

For an algebraic variety Grothendieck has defined the (algebraic) fundamental group. For an algebraic curve in characteristic zero the structure of its fundamental group is determined with the help of comparison with the topological fundamental group (and the same for the prime-to-p part of the fundamental group of an algebraic curve in characteristic p).

10A. Question: *Can we determine the structure of the fundamental group of an algebraic curve by purely algebraic methods?*

10.1. Already for $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ it seems that this question remained unanswered up to now. Ofer Gabber communicated to me that this is the algebraic computation of the fundamental group in characteristic zero can be reduced to this case.

11. Abelian varieties over number fields, and the type of their reductions modulo a prime.

Let X be an abelian variety over a number field K, and let Σ_K be the set of finite primes of K. For every $v \in \Sigma_K$ at which X has good reduction we can ask what is the type of the abelian variety X_v, e.g. its p-rank or its NP $\mathcal{N}(X_v)$. We say that an abelian variety Y in positive characteristic is *ordinary* if its p-rank is maximal, i.e. $\mathcal{N}(Y) = \rho$. For a NP β we write

$$\mathcal{R}_\beta(X) := \{v \in \Sigma_K \mid \mathcal{N}(X_v) = \beta\}$$

for the set of finite places of K where X has good reduction with NP equal to β. We mention in this section the question:
Given an abelian variety X over a number field K and a NP β; what can be said about $\mathcal{R}_\beta(X) \subset \Sigma_K$?

For the *ordinary* locus Serre made the following conjecture (see [61]):
11A$_?$: *For every X and K as above, there exists a finite extension* $[L : K] < \infty$ *such that*

$$\mathcal{R}_\rho(X_L) \subset \Sigma_L$$

has Dirchlet density equal to one. In fact the conjecture is more precise: L should be chosen in such a way that the image of the Galois representation on each of the the Tate groups of X_L is connected.

11.1. Every time you think about this conjecture , it seems reasonable but difficult. For elliptic curves this was proved by Serre; for the case $\dim(X) \leq 2$ it was shown to be true by Ogus, see [45], Corollary 2.9. For CM abelian varieties it holds. For certain other special cases it has been proved. In general (11.A) seems unknown, even it seems unknown whether every abelian variety over a number field has at least one place of good, ordinary reduction (something which would be very nice to know, that would enable us to use Serre-Tate parameters for every such abelian variety).

11.2. Example: Consider an Albert algebra D of Type II(1), i.e. a quaternion algebra with centre equal to $\mathbb{Q}$ such that D is indefinite, i.e. $D \otimes \mathbb{R} \cong \mathrm{Mat}(2, \mathbb{R})$. Consider (a component of) a Shimura variety of PEL type associated with this algebra, i.e. a complete curve in $V = V_{\mathbb{C}} \subset \mathcal{A}_2 \otimes \mathbb{C}$ such that every geometric point of V corresponds with an abelian variety which has multiplication by an order in D.

Let X be an abelian surface defined over a number field K such that an order in an algebra of Type II(1) acts on X. Then any reduction X_v at a prime of K does not have p-rank equal to one: $f(X_v) \neq 1$. This provides an example of an abelian variety in characteristic zero and a Newton Polygon such that no reduction modulo any p has the given Newton polygon:

$$\mathcal{R}_\beta(X) = \emptyset$$

if $\beta = (1, 0) + (1, 1) + (0, 1)$ and X has multiplication by an algebra of Type II(1).

Using the "Raynaud trick" (see [53]) we show that any such V modulo p contains at least one supersingular point. The analogy with the case of elliptic curves is striking, and one can expect:

11B? Conjecture: *Suppose X is an abelian surface over a number field K such that $X \otimes \overline{K}$ has multiplication by an order in a* Type II(1) *algebra; for the supersingular NP σ we expect:*

$$\#(\mathcal{R}_\sigma(X)) \quad = \quad \infty \qquad ?$$

We have seen that for a given abelian variety X over a number field and a given NP β the set $\mathcal{R}_\beta(X)$ can be empty. However for the ordinary locus, $\beta = \rho$, see (11A), and for the supersingular locus $\beta = \sigma$ such reductions might exist.

11C. Question: *Given an abelian variety X over a number field K, and the supersingular NP σ; what can be said of the set $\mathcal{R}_\sigma(X)$ of places of good, supersingular reduction? Is it infinite for every X? Can it be finite or empty for some choice of X?*

Note that Elkies showed that for every elliptic curve E defined over a number field K which has a real embedding (is this essential?) the set of supersingular reductions is infinite, see [10], [11]. For abelian varieties of higher dimension this question seems interesting and difficult. Even for abelian surfaces in general the answer seems unknown.

11.3. Here are some rather non-founded ideas and questions. Suppose $[(X, \lambda)] \in \mathcal{A}_g(\overline{\mathbb{Q}})$, and let $V \subset \mathcal{A}_g \otimes K$ be the smallest variety of Hodge type containing this point, defined over a number field K. Consider the set of prime places of K where reduction of V gives a subset of the moduli space in positive characteristic which intersects the supersingular locus in a set of codimension at most one; if this set is finite one could expect that $\mathcal{R}_\sigma(X)$ is finite, otherwise one could try to show that $\mathcal{R}_\sigma(X)$ is infinite. It seems worthwhile (and difficult) to pursue this idea.

12. CM-liftings.

Suppose X_0 is an abelian variety over a finite field k. By a theorem of Tate [63] we know that X_0 has smCM. We say that X is a CM-lifting of X_0 if there exists a domain R, with $K \supset R \to k$, an abelian scheme $\mathcal{X} \to \mathrm{Spec}(R)$ such that K has characteristic zero, such that $X_0 \cong \mathcal{X} \otimes k$, and such that $X := \mathcal{X} \otimes K$ admits smCM.

Note that if the p-rank of X_0 is at least $\dim(X) - 1$, then a CM-lifting exist, see [50], Theorem A.

However, for every $g \in \mathbb{Z}_{\geq 3}$ and $f \leq g - 2$ there exists an abelian variety X_0 over a finite field with $\dim(X_0) = g$ with p-rank $f(X) = f$ which does not admit a CM-lifting, see [50]. The proof of this fact in [50] probably can be improved.

Note that if $\mathcal{X} \to \mathrm{Spec}(R)$ is an abelian scheme, $R \to k$ is a residue class map, $X_0 \cong \mathcal{X} \otimes k$ with $\mathrm{char}(k) = p > 0$, then the natural map $\mathrm{End}(X) \to \mathrm{End}(X_0)$ is injective, and the index

$$[\mathrm{End}^0(X) \cap \mathrm{End}(X_0) : \mathrm{End}(X)] \quad \text{is a power of} \quad p.$$

12A. Question: *Describe how the Tate-p-groups scheme of an abelian variety X in characteristic zero reduces to a submodule of the Dieudonné module of X_0; use this to show that in certain cases CM-liftings do not exist.* In particular it might be used to answer the following questions:

12B. Question: *Let E be a supersingular curve over $\overline{\mathbb{F}_p}$. Does there exist an abelian surface X_0 isogenous with $E \times E$ which does not admit a CM-lifting, or, does every supersingular abelian surface over $\overline{\mathbb{F}_p}$ admit a CM-lifting?*

In [50] examples were constructed using [27], where we started with a simple abelian variety of low p-rank over a finite field with a commutative endomorphism algebra. However the following case we could not decide:

12C. Question: *Let Y be a simple abelian variety over $\overline{\mathbb{F}_p}$ such that* $\mathrm{End}(Y)$ *is not commutative. Is Y isogenous with an abelian variety X_0 over $\overline{\mathbb{F}_p}$ which does not admit a CM-lifting?* See [50], Question C. Note that different CM-liftings of Y or of X_0 may belong to different CM-fields.

13. Serre-Tate parameters in the non-ordinary case.

Let X_0 be an *ordinary* abelian variety in positive characteristic, and λ_0 a principal polarization, we write $[(X_0, \lambda_0)] = x \in \mathcal{A}_{g,1}(\mathbb{F}_p)$. Serre and Tate showed that the formal scheme $(\mathcal{A}_{g,1})^{\wedge}_{x_0}$ has "canonical coordinates", see [31], see [24], Chapter 5, see [33]. Note that the torsion points in the formal group

$$(\mathcal{A}_{g,1})^{\wedge}_{x_0} \quad \cong \quad ((\mathbb{G}_m)^{\wedge})^{g(g+1)/2}$$

correspond with the "quasi-canonical liftings" of X_0, i.e. the CM-liftings (but $\mathrm{End}(X)$ need not be a maximal order in $\mathrm{End}^0(X) = \mathrm{End}^0(X_0)$).

We pose the question whether such a "canonical" parametrization is possible in the non-ordinary case.

13A. Question: *Suppose given an abelian variety X_0 with a polarization λ_0 over a finite field and a CM-lifting (X, λ). Can we define "canonical coordinates" on $(\mathcal{A}_{g,1})^{\wedge}_{x_0}$?*
Various attempts have been made in the past, see [30], [16], [68], [2], [67].

Note that different choices of a CM-lifting of X_0 belonging to different CM subalgebras of $\mathrm{End}^0(X_0)$ may give quite different coordinate systems.

This question should be made much more precise before it can be taken seriously.

13.1. Remark: The terminology "canonical lifting" might cause confusion. I intend to use this phrase only in case X_0 is an ordinary abelian variety. Some authors use this concept for an arbitrary abelian variety over a finite field requiring that the geometric Frobenius can also be lifted; for an ordinary X_0 we do get the right concept, but for non-ordinary abelian varieties there may be many liftings such that the geometric Frobenius lifts along (e.g. a supersingular elliptic curve E over $\mathbb{F}_{p^n}$ such that "$\pi_{E/\mathbb{F}_{p^n}} = F^n$" is multiplication by an integer, e.g. $\pi_{E/\mathbb{F}_{p^n}} = p^{n/2} \cdot 1_E$). Also for abelian varieties over a non-finite field in positive characteristic we can define a canonical lifting for an ordinary X_0 in the Serre-Tate theory, however in that case there is no geometric Frobenius.

14. Complete subvarieties.

This material is partly taken from [51]. We consider moduli spaces (of curves, of abelian varieties), which are in general non-complete (non-compact when over $\mathbb{C}$), and we study the following type of question:

Let W be a **complete subvariety** *of a certain moduli space;*
can we give a sharp bound for the dimension of W?

We think that the answer to this question may depend on the characteristic of the base field in consideration.

14.1. Example: *For every prime number p and for every $g \in \mathbb{Z}_{>0}$ the set $V_0 \subset \mathcal{A}_g \otimes \mathbb{F}_p$ is a closed subset of dimension $g(g-1)/2$ which is* **complete.**
See [46], Th. (1.1a), and [42], Th. 4.1. For the definition of V_0, see (8.1).
Furthermore: *if K is a field, and $W \subset \mathcal{A}_{g,1} \otimes K$ is a complete subvariety then* $\dim(W) \leq g(g-1)/2$ (G. van der Geer, unpublished). We see that in positive characteristic this bound is attained.

14A? Conjecture: *Let $W \subset \mathcal{A}_3 \otimes \mathbb{C}$ be a complete subvariety then* $\dim(W) < 3 = g(g-1)/2$.
In fact, we expect that for $g \geq 3$ and $W \subset \mathcal{A}_3 \otimes \mathbb{C}$, a complete subvariety, then $\dim(W) < g(g-1)/2$. It seems interesting to know what the maximum is of the dimension of such complete subvarieties for a fixed g.

14B? Conjecture: *Let $W \subset \mathcal{A}_{3,1} \otimes \mathbb{F}_p$ be a complete subvariety of dimension 3. Then we expect that this implies that $W = V_{0,1}$.* Note: $V_{0,1}$ is irreducible for $g = 3$, see (8B).
If (14B) turns out to be correct, then (14A) follows (using a calculation of certain Chow classes, as was done by G. van der Geer, unpublished). Actually, this line of thought was stimulated by computations by Carel Faber [12], Ann. Math. 132, page 413, and a conjecture made in 1988 by Manin (unpublished), that the Chow classes of the $V_{0,1} \subset \mathcal{A}_{3,1} \otimes \mathbb{F}_p$ for various p should be proportional to each other with rational factors.

14.2. Example: *For $g \geq 2$, and any field K a complete subvariety $W \subset \mathcal{M}_g$ has* $\dim(W) \leq g-2$.
This was proved by Diaz in characteristic zero, see [7], Theorem 4. For an arbitrary base field we find this in [29], corollary in Section 1, where Looijenga proved part of a conjecture by C. Faber.

14.3. It could be true that a complete subvariety W in $\mathcal{M}_g \otimes \mathbb{C}$ for $g \geq 3$ has $\dim(W) < g-2$, and it might be true that there does exist a complete subvariety of dimension equal to $g-2$ in $\mathcal{M}_g \otimes \mathbb{F}_p$. However we have little evidence for this. Already the case $g = 4$ this seems unsolved: does there exist a complete surface $W \subset \mathcal{M}_4 \otimes K$ for some field K?

14.4. For the moduli space $\mathcal{M}_g^{\sim}$ of "nice curves", i.e. curves of "compact type", one can phrase analogous results (dimension bounded by $2g-3$) and analogous expectations.

15. Hecke orbits: dense sets of points.

Consider the moduli space $\mathcal{A} = \mathcal{A}_g \otimes \mathbb{F}_p$ of polarized abelian varieties in characteristic p (or a component of this). Consider a point $[(X, \lambda)] = x \in \mathcal{A}$, and consider the set $\mathcal{G}(x)$ consisting of all points corresponding with isogenous polarized abelian varieties, i.e. if (X, λ) is a polarized abelian variety over some field K with moduli point $[(X, \lambda)] = x$ then $\mathcal{G}(x)$ consists of all moduli points of pairs $[(Y, \mu)] \in \mathcal{A}$ such that there exists an isogeny $\varphi : Y \to X \otimes k$ over some field containing K, and an integer $m \in \mathbb{Z}_{>0}$ such that $\varphi^*(\lambda) = m{\cdot}\mu$. This set $\mathcal{G}(x)$ is called the *Hecke orbit* of x in $\mathcal{A}$. If we consider only isogenies with degree prime to p we write

$$\mathcal{G}^{(p)}(x) = \{[(Y,\mu)] \mid \exists \varphi : Y \to X \otimes k, \quad m \in \mathbb{Z}_{>0}, \quad p \not| m, \quad \varphi^*(\lambda) = m{\cdot}\mu\},$$

the Hecke-prime-to-p orbit. The ℓ-power Hecke orbit $\mathcal{G}_\ell(x)$ is the set where we consider only isogenies where the $\deg(\varphi)$ is a power of the prime number ℓ. Clearly

$$\mathcal{G}_\ell(x) \subset \mathcal{G}^{(p)}(x) \subset \mathcal{G}(x)$$

if $\ell \neq p$. These definitions can be found in [4], and in that paper by Chai we find (Th. 2):
Theorem: *Suppose ℓ is a prime number different from p, and $x = [(X,\lambda)]$ such that X is an* ordinary *abelian variety; then $\mathcal{G}_\ell(x)$ is dense in $\mathcal{A}$.*

15A? Conjecture: *Suppose $[(X,\lambda)] = x \in \mathcal{A}$, we write $\beta = \mathcal{N}(X)$ for the Newton polygon of X. We expect that the Hecke orbit $\mathcal{G}(x)$ is dense in the Newton-polygon stratum defined by β.*

15.1. Note that in general $\mathcal{G}_\ell(x)$ or $\mathcal{G}^{(p)}(x)$ has no chance to be dense in a Newton-polygon stratum. For example consider the locus of 3-dimensional abelian varieties with Newton polygon given by $(2,1)+(1,2)$, this is an open subset of the p-rank-zero locus $V_0 \subset \mathcal{A}_3$. For such an abelian variety either $a(X) = 2$ (this is in the 2-dimensional locally closed subset $V_0(a = 2) \subset V_0$) or $a(X) = 1$ (this gives a set dense in $V_{0,1}$). Clearly $\mathcal{G}^{(p)}$ does not move a point with $a = 2$ to a point with $a = 1$, so in the first case we see that $\mathcal{G}^{(p)}(x) \subset V_0(a = 2)$; if the conjecture is correct this should be a dense subset. In case $a(X) = 1$ we can show that $\mathcal{G}^{(p)}(x)$ is not dense in $V_{0,1}$, but if the conjecture holds, the Zariski closure of this set has dimension 2.

Such examples can be given in great generality. We do not see a proof of the conjecture using methods as in [4]. It might be that this conjecture can be approached via the intriguing Question (Q 2) in [4].

16. Special subsets of moduli spaces.

In [26], I.3 we find the definition of the "special subset" $\mathrm{Sp}(V)$ of a variety V defined over a field (say of characteristic zero). This subset, which is geometrically defined, has applications in arithmetic (this is the subset where we expect "most of the rational points" when working over a number field).

16A. Question: *What is* $\mathrm{Sp}(\mathcal{M}_g \otimes \mathbb{Q})$, *what is* $\mathrm{Sp}(\mathcal{A}_g \otimes \mathbb{Q})$?

16.1. This seems a natural, and difficult question. In [35] we find some suggestions about this.

Note that the moduli space $\mathcal{A}_{g,1,n} \otimes \overline{\mathbb{F}_p}$ contains many rational curves where $n \in \mathbb{Z}_{\geq 3}$ prime to p and $g \geq 3$.

17. Special subsets in open surfaces.

Consider a scheme over an order in a number field with generic fiber a variety U. Is there a way to predict where "most of the integral points" should be found? For a open set $U \subset \mathbb{P}^2$

obtained by deleting a curve it seems that a good definition is available: take the Zariski closure of the union of all rational curves in $\mathbb{P}^2$ which have at most two places outside U. Do we know what the special subset of such an open variety U is, when we have enough information about the deleted curve? Can this be generalized to arbitrary (non-complete) varieties?

17A. Question: *Is there a good definition of the "open special subset" of a (possibly non-complete) variety? What is this in case of an open set* $U = \mathbb{P}^2$-(a curve)?

18. Intersections of components of moduli spaces of polarized abelian varieties in positive characteristic.

In [37] Mumford raised the question whether components of the moduli space of (non-principally) polarized abelian varieties intersect in positive characteristic. Norman gave examples that indeed this happens for abelian surfaces, see [40], [41]. In his PhD-thesis Johan de Jong analyzed this situation, describing intersections of components of $\mathcal{A}_{g,d} \otimes \mathbb{F}_p$, see [18], Chapter I, and [19] (the index d: polarizations of degree d^2). A sequence of elementary divisors is a sequence $\delta = \{\delta_1, \cdots, \delta_g\}$ such that $\delta_1 | \delta_2 \cdots | \delta_g$; we write $d = \Pi_{j=1}^{g}\ \delta_j$.

18A. Question: *Determine for arbitrary sequences δ, $\overline{\delta}$ of elementary sequences the intersection of the moduli spaces $(\mathcal{A}_{g,\delta} \otimes \mathbb{F}_p)^c$ and $(\mathcal{A}_{g,\overline{\delta}} \otimes \mathbb{F}_p)^c$.* Notation of [19], the exponent "c": take the Zariski-closure.

19. Minimal modular parametrizations.

19.1. We write (STW)$_?$ for the Shimura-Taniyama-Weil conjecture, which says that *every elliptic curve over $\mathbb{Q}$ is modular* (proved by A. Wiles for semistable elliptic curves over $\mathbb{Q}$).

19.2. Given a real number $\alpha \in \mathbb{R}$ we write (MO;α) for the following conjecture: Suppose $A, B \in \mathbb{Z}_{>0}$ are positive integers which are relative prime, and write $C := A + B$; we write

$$cond(ABC) := \Pi\ p, \quad \text{the product taken over all prime numbers that divide} \quad ABC.$$

(MO;α)$_?$: *If $A, B \in \mathbb{Z}_{>0}$ are relatively prime, then*

$$C \quad < \quad (cond(ABC))^{\alpha}.$$

This conjecture is called the Masser-Oesterlé conjecture, also called the A, B, C-conjecture; there are much better, sharper formulations of this conjecture, e.g. see [44], Section 3.

19.3. Suppose (STW) does hold. We can study the following boundedness condition (where $n \in \mathbb{Z}_{>0}$):
$(B_n)_?$: *For every elliptic curve E over $\mathbb{Q}$ with conductor $N := conductor(E)$ there exists a parametrization (non-constant morphism) over $\mathbb{Q}$:*

$$\varphi : X_0(N) \to E \quad with \quad \deg(\varphi) < N^n.$$

Certainly it is not easy to give upper bounds for the degree of the minimal modular parametrization see [69], Section 6. I have no idea whether a polynomial expectation/question like $(B_n)_?$ is reasonable.

19A$_?$ *Suppose Conjecture* (STW) *is correct, and suppose that there exists a positive integer n such that the boundedness B_n holds. Then there exists $\alpha \in \mathbb{R}$ such that* (MO;α) *holds.*
This seems to be reasonable, once deep results like (STW)$_?$ and $(B_n)_?$ are settled.

Acknowledgments: During many years I have profited on a large scale from the generous way in which teachers, colleagues, friends, and students have helped to shape my mathematical ideas. But however much I would like to thank them all for their friendship, and their personal and mathematical influence upon me, it would be impossible to list all their names.

Many of the ideas put forward in the preceding pages that I might be inclined to consider as my own, may in fact be the fruit of cooperation, or may even have originated from somebody else. In several instances, too, my references to other people's contributions might have been more explicit. Yet such an effort would certainly have resulted in a very incomplete picture of the state of affairs. Therefore I decided only to mention, in gratitude, the names of those of whom I am distinctly aware that they left their imprint on the questions listed above:

Yves André, Aldo Andreotti, Frits Beukers, Mikhail Borovoi, Irene Bouw, Lucia Caporaso, Ching-Li Chai, Ciro Ciliberto, Robert Coleman, Elisabetta Colombo, Bas Edixhoven, David Eisenbud, Torsten Ekedahl, Noam Elkies, Carel Faber, Ofer Gabber, Bert van Geemen, Gerard van der Geer, Dick Gross, William Haboush, David Harbater, Joe Harris, Johan Huisman, Tomoyoshi Ibukiyama, Elham Izadi, Johan de Jong, Toshiyuki Katsura, Neal Koblitz, Michael Larsen, Hendrik Lenstra jr., Ke-Zheng Li, Eduard Looijenga, Yuri Manin, Ben Moonen, Jaap Murre, M. Ram Murthy, Rutger Noot, Peter Norman, Tadao Oda, Takayuki Oda, Martin Pikaart, Richard Pink, Pietro Pirola, Marius van der Put, Masa-Hiko Saito, René Schoof, Tsutomu Sekiguchi, J-P. Serre, Alice Silverberg, Jozef Steenbrink, Kate Stevenson, John Tate, Jaap Top, Kenji Ueno, Noriko Yui, Don Zagier, Yuri Zarhin, Thomas Zink

(but, of course the responsability for the contents of the above is completely mine).

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