Supersingular abelian varietes and foliations

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In this lecture I analyze some definitions of the concept "Supersingular Abelian Variety". Moreover we see that supersingular abelian varieties on the one hand have properties quite different from non-supersingular abelian varieties on the other hand.
My talk was a survey lecture: this note does not contain any new material.

Introduction

Over a field of characterisitc zero one can study the Lie algebra of a Lie group. However, if we want to study the "p-structure" of an abelian variety A over a field K of characteristic p, the Lie algebra does not give enough information; e.g. multiplication by p annihilates that Lie algebra. Moreover the module $\lim_{\leftarrow} A(k)[p^i]$, where k is an algebraic closure of K, is not the right concept: it can very well be zero for a non-zero A, and it does not vary well in families.

For an abelian variety A and a given prime number ℓ different from the characteristic of the base field one studies the Galois module $T_\ell(A)$, the Tale-ℓ-module of A.

One studies in this case the *p-divisible group* of A

$$X = A[p^\infty] \quad := \quad \cup_i \; A[p^i] \quad = \quad \lim_{\rightarrow} \; A[p^i],$$

which can be seen as the p-adic analog of the Tate-ℓ group. Here $A[n]$ stands for the scheme-theoretical kernel of the multiplication-by-n morphism $\times n : A \to A$.

Various invariants attached to $X = A[p^\infty]$ can be used to study properties of A, and to define strata and leaves in $\mathcal{A}_g := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, the moduli space of principally polarized abelian varieties in characteristic p.

We will start to define an abelian variety to be supersingular if all slopes in the Newton polygon of A are equal to $1/2$. We will show this is equivalent to another definition. And then we will study some properties of (non)-supersingular abelian varieties.

(0.1) Notation. For an algebraic scheme T over a field K we write $\Pi_0(T)$ for the set of irreducible components of T_k, where $k \supset K$ is an algebraically closed field.

We write $K_{p,\infty}$ for the quaternion algebra with center $\mathbb{Q}$ exactly ramified at the prime number p and at the infinite place (notation of Deuring).

All base fields and base schemes considered will be in characteristic p. We write k for an algebraically closed field. We write g for the dimension of an abelian variety under consideration.

1 Newton polygons

(1.1) Definition, p-divisible groups. Suppose given $h \in \mathbb{Z}_{>0}$. Suppose given a base scheme S. Suppose given for every $i \in \mathbb{Z}_{>0}$ a finite, flat group scheme $G_i \to S$ of rank p^{ih}, and inclusions $G_i \subset G_{i+1}$ for every i such that $G_{i+1}[p^i] = G_i$. The inductive system $X = \{G_i \mid i\} \to S$ is called a p-divisible group of height h over S.

The notion of a Barsotti-Tate group, or BT-group, is the same as that of a p-divisible group; both terminologies will be used. For more information see [14], Section 1.

Remark. Note that for every j and every $s \geq 0$ the map $\times p^s$ induces a surjection:

$$G_{j+s} \twoheadrightarrow G_j = G_{j+s}[p^j] \subset G_{j+s}.$$

For every i and $s \geq 0$ we have an exact sequence of finite flat group schemes

$$0 \to G_i \longrightarrow G_{i+s} \longrightarrow G_s \to 0.$$

Example. Let $A \to S$ be an abelian scheme. For every i we write $G_i = A[p^i]$. The inductive system $G_i \subset G_{i+s} \subset A$ defines a p-divisible group of height $2g$. We shall denote this by $X = A[p^\infty]$ (although of course "p^∞" strictly speaking is not defined).

For p-divisible groups, inductive systems, we define homomorphisms by

$$\mathrm{Hom}(\{G_i\}, \{H_j\}) \quad = \quad \lim_{\leftarrow i}\ \lim_{j \to}\ \mathrm{Hom}(G_i, H_j).$$

Note that a homomorphism $A \to B$ of abelian schemes defines a morphism $A[p^\infty] \to B[p^\infty]$ of p-divisible groups.

(1.2) Discussion. Over any base scheme S (in any characteristic) for an abelian scheme $A \to S$ and for a prime number ℓ invertible on S one can define $T_\ell(A/S)$ as follows. For $i \in \mathbb{Z}_{>0}$ one chooses $N_i := A[\ell^i]$, and we define $\times\ell : N_{i+1} \to N_i$. This gives a *projective system*, and we write

$$T_\ell(A/S) \quad = \quad \{A[\ell^i] \mid i \in \mathbb{Z}_{>0}\} \quad = \quad \lim_{\leftarrow i}\ A[\ell^i].$$

This is called the ℓ-Tate-group of A/S. Any geometric fiber is $T_\ell(A/S)_s \cong (\underline{\mathbb{Z}_\ell}_s)^{2g}$.

One should like to have an analogous concept for this notion in case p is not invertible on S. This is precisely the role of $A[p^\infty]$ defined above. Historically a Tate-ℓ-group is defined as a projective system, and the p-divisible group as an inductive system; it turns out that these are the best ways of handling these concepts (but the way in which direction to choose the limit is not very important). We see that the p-divisible group of an abelian variety should be considered as the natural substitute for the Tate-ℓ-group.

Note that a finite étale group scheme is the same as a finite group with a continuous action of the fundamental group of the base. Suppose A is an abelian variety over a field K, and $n \in \mathbb{Z}_{>0}$ is not divisible by the characteristic of K. The finite group scheme $A[n]$ is an étale group scheme over K. This group scheme determines and is determined by the module $A[n](\overline{K})$ with continuous action under the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Hence the Tate-$\ell$-group can be considered as the structure of a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module on the group $\mathbb{Z}_\ell^{2g}$.

In order to carry this analogy further we investigate aspects of $T_\ell(A)$ and wonder whether these can be carried over to $A[p^\infty]$. The first is a twist of a pro-group scheme. What can be said in analogy about $A[p^\infty]$? We will see that *up to isogeny* $A[p^\infty]$ is a "twist" of an ind-group scheme over $\mathbb{F}_p$; however "twist" here should be understood not only in the sense of separable Galois theory, but also using inseparable aspects: the main idea of Serre-Tate parameters.

(1.3) Theorem. (Duality theorem for abelian schemes, see [21], Theorem 19.1) *Let $\varphi : B \to A$ be an isogeny of abelian schemes. We obtain an exact sequence*

$$0 \quad \to \quad \mathrm{Ker}(\varphi)^D \quad \longrightarrow \quad A^t \quad \xrightarrow{\varphi^t} \quad B^t \quad \to \quad 0.$$

(1.4) The Serre dual of a p-divisible group. Consider a p-divisible group $X = \{G_i\}$. The exact sequence $G_{j+s}/G_j = G_s$ defines by Cartier duality, see [21], I.2, an exact sequence

$$0 \to G_s^D \longrightarrow G_{j+s}^D \longrightarrow G_j^D \to 0.$$

The inclusions

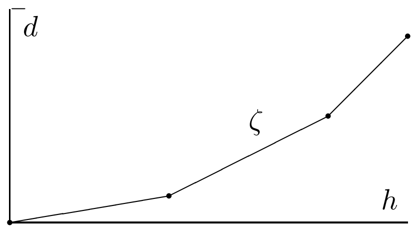
$$G_s^D \hookrightarrow G_{s+1}^D \quad = \quad (G_{s+1} \twoheadrightarrow G_s)^D,$$

define the p-divisible group $X^t = \{G_s^D\}$, called *the Serre dual* of X. Using the duality theorem we conclude: $(A[p^\infty])^t = A^t[p^\infty]$ (which is less trivial than notation suggests...).

In order to being able to handle the isogeny class of $A[p^\infty]$ we need the notion of Newton polygons.

(1.5) Newton polygons. Suppose given integers $h, d \in \mathbb{Z}_{\geq 0}$; here $h =$ "height", $d =$ "dimension". In case of abelian varieties we will choose $h = 2g$, and $d = g$. A Newton polygon γ (related to h and d) is a polygon $\gamma \subset \mathbb{Q} \times \mathbb{Q}$ (or, if you wish, in $\mathbb{R} \times \mathbb{R}$), such that:

- γ starts at $(0, 0)$ and ends at (h, d);
- γ is lower convex;
- any slope β of γ has the property $0 \leq \beta \leq 1$;
- the breakpoints of γ are in $\mathbb{Z} \times \mathbb{Z}$; hence $\beta \in \mathbb{Q}$.



Note that a Newton polygon determines (and is determined by)

$$\beta_1, \cdots, \beta_h \in \mathbb{Q} \text{ with } 0 \leq \beta_1 \leq \cdots \leq \beta_h \leq 1 \quad \leftrightarrow \quad \zeta.$$

Sometimes we will give a Newton polygon by data $\sum_i (m_i, n_i)$; here $m_i, n_i \in \mathbb{Z}_{\geq 0}$, with $\gcd(m_i, n_i) = 1$, and $m_i/(m_i + n_i) \leq m_j/(m_j + n_j)$ for $i \leq j$, and $h = \sum_i (m_i + n_i)$, $d = \sum_i m_i$. From these data we construct the related Newton polygon by choosing the slopes $m_i/(m_i + n_i)$ with multiplicities $h_i = m_i + n_i$. Conversely clearly any Newton polygon can be encoded in a unique way in such a form.

Remark. The Newton polygon of a polynomial. Let $g \in \mathbb{Q}_p[T]$ be a monic polynomial of degree h. We are interested in the p-adic values of its zeroes (in an algebraic closure of $\mathbb{Q}_p$). These can be computed by the Newton polygon of this polynomial. Write $g = \sum_j \gamma_j T^{h-j}$. Plot the pairs $(j, v_p(\gamma_j))$ for $0 \leq j \leq h$. Consider the lower convex hull of $\{(j, v_p(\gamma_j)) \mid j\}$. This is a Newton polygon (without the slope condition as in the definition above). The slopes of the sides of this polygon are precisely the p-adic values of the zeroes of g, ordered in non-decreasing order. (Suggestion: prove this as an exercise.)

(1.6) Incorrect. Here we indicate what the Newton polygon of a p-divisible group is (in a slightly incorrect way ...). Consider "the Frobenius endomorphism" of X. This has a "characteristic polynomial". This polynomial determines a Newton polygon, which we write as $\mathcal{N}(X)$, the Newton polygon of X. For an abelian variety A we write $\mathcal{N}(A)$ instead of $\mathcal{N}(A[p^\infty])$.

Well, this "definition" is correct over $\mathbb{F}_p$ as ground field. However over any other field $F : X \to X^{(p)}$ is not an endomorphism, and the above "construction" fails. Over a finite field there is a method which repairs this. However we need the Newton polygon of an abelian variety over an arbitrary field. As an intuitive concept, please accept the "explanation" given above: $\mathcal{N}(X)$ is "the Newton polygon of the Frobenius on X". Let us give a precise definition.

(1.7) *Here is the correct definition* of the Newton polygon of a p-divisible group X over a field K (of characteristic p).

For coprime $m, n \in \mathbb{Z}_{\geq 0}$ one defines a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$, see [19], page 38, or see [4], Section 4: this is a p-divisible group of dimension m, and its Serre dual is $(G_{m,n})^t = G_{n,m}$ is of dimension n. We attach to $G_{m,n}$ the Newton polygon consisting of $m+n$ slopes equal to $m/(m+n)$.

Exercise. *Show that this is indeed the Newton polygon of Frobenius in this case.*

In case $X_k \sim \Pi_i \ (G_{m_i.n_i})$ we take for $\mathcal{N}(X)$ the Newton polygons of the summands, with all slopes ordered in non-decreasing order. Over an algebraically closed field this classifies isogeny classes of p-divisible groups:

Theorem (Dieudonné and Manin), see [19], "Classification theorem " on page 35,

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}.$$

This means: for every p-divisible group X over a field we define its Newton polygon $\mathcal{N}(X)$; over en algebraically closed field, every Newton polygon comes from a p-divisible group and

$$X \quad \sim_k \quad Y \quad \Longleftrightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y).$$

Hence we see: *a p-divisible group X over a field of characteristic p determines uniquely a Newton polygon, and the isogeny class of X is uniquely determined by its Newton polygon $\mathcal{N}(X)$.*

We will write $\mathcal{N}(X) = \sum_i \ (m_i.n_i)$ in case $X_k \sim \Pi_i \ (G_{m_i.n_i})$. We write $\mathcal{N}(A)$ instead of $\mathcal{N}(A[p^\infty])$.

(1.8) Definition. *Let ξ be a Newton polygon with slopes $0 \leq \beta_1 \leq \cdots \leq \beta_h$. We say that ξ is* symmetric *if $\beta_i = 1 - \beta_{h+1-i}$ for $1 \leq i \leq h$.*

Fact. *The Newton polygon $\mathcal{N}(A)$ of an abelian variety A is symmetric.*

For an abelian variety over a finite field this was proved in [19], page 71. The general case follows from the duality theorem, which implies that $A^t[p^\infty] = (A[p^\infty])^t$.

(1.9) We say that A has p-rank equal to f if $A[p](k) \cong (\mathbb{Z}/p)^f$. Equivalently:

$$\mathcal{N}(A) = f\cdot(1,0) + \left(\sum_{m_i>0, n_i>0} (m_i, n_i) \right) + f\cdot(0,1).$$

(1.10) For elliptic curves there are two Newton polygons possible:
(ord) $\mathcal{N}(E) = (1,0)+(0,1)$; the Newton polygon has slopes 0 and 1; equivalently: $E[p](k) \cong \mathbb{Z}/p$; equivalently $f(E) = 1$; in this case we say that E is an *ordinary* elliptic curve;
(ss) $\mathcal{N}E = (1,1)$; the Newton polygon has two slopes equal to $1/2$; equivalently: $E[p](k) = 0$; equivalently $f(E) = 0$; in this case we say that E is a *supersingular* elliptic curve.

A remark on terminology. In characteristic zero the j-invariant of an elliptic curve with CM is called a *singular* j-invariant (but note that the curve is not singular at all). Deuring remarked in 1940 that there are elliptic curves in characteristic p where the endomorphism algebra has rank 4 over $\mathbb{Q}$. Hence Deuring called these "*supersingular*"; a more correct terminology would be: "an elliptic curve with a supersingular j-invariant", but that terminology is too long.

2 Supersingular

(2.1) Definition 1. *An abelian variety A over a field K* (of characteristic p) *is supersingular if all slopes in its Newton polygon are equal to $1/2$; i.e.*

$$A \text{ is supersingular} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathcal{N}(A) = g{\cdot}(1,1).$$

Remark. Equivalently: $A[p^\infty] \otimes k \sim (G_{1,1})^g$.

(2.2) Notation. Let G be a group scheme over a field $K \supset \mathbb{F}_p$. Consider

$$\alpha_p := \mathrm{Ker}(F : \mathbb{G}_a \to \mathbb{G}_a).$$

Choose a perfect field L containing K. Note that $\mathrm{Hom}(\alpha_p, G_L)$ is a right-module over $\mathrm{End}(\alpha_p \otimes_{\mathbb{F}_p} L) = L$. We define

$$a(G) = \dim_L \left(\mathrm{Hom}(\alpha_p, G_L)\right).$$

Remarks. For any field L we write α_p instead of $\alpha_p \otimes_{\mathbb{F}_p} L$ if no confusion is possible.

The group scheme $\alpha_{p,K}$ over a field K corresponds under by Diedonné theory (in case K is perfect) to the module K^+ with operators $\mathcal{F} = 0$ and $\mathcal{V} = 0$.

If K is not perfect it might happen that $\dim_K (\mathrm{Hom}(\alpha_p, G)) < \dim_L (\mathrm{Hom}(\alpha_p, G_L))$.

However if L is perfect, and $L \subset L'$ is any field extension then $\dim_L (\mathrm{Hom}(\alpha_p, G_L)) = \dim_{L'} (\mathrm{Hom}(\alpha_p, G_{L'}))$; hence the definition of $a(G)$ is independent of the chosen perfect extension L.

Theorem (See [24], Section 4; also see [34], Th. 2 on page 140). Definition 1 *is equivalent to:*

Definition 2. *An abelian variety A over a field K* (of characteristic p) *is supersingular iff*

there exists an isogeny $A_k \sim_k E^g$, where E is a supersingular elliptic curve.

Sketch of a proof. Clearly $\mathcal{N}(E^g) = g{\cdot}(1,1)$ if E is a supersingular elliptic curve. Hence (2) $\Rightarrow$ (1).

Conversely suppose that $\mathcal{N}(A) = g{\cdot}(1,1)$. Then there exists an isogeny $A_k[p^\infty] \sim (G_{1,1})^g$. Hence there exists an isogeny $A_k \sim_k B$ with $a(B) = g$. Let $[(B,\mu)] = y \in \mathcal{A}_g \otimes \mathbb{F}_p$. Let T be an irreducible component of $\mathcal{A}_g \otimes \mathbb{F}_p$ containing y. Deformation theory shows that any infinitesimal deformation of B keeping $a(-) = \dim(B)$ constant is trivial. Hence the set of points $y' \in T$ with $y' = [(B', \mu')]$ and $a(B') = g$ is zero-dimensional and closed. This proves that B is defined over a finite field.

Next we show: *let B be a simple abelian variety defined over $\mathbb{F}_q$, with $q = p^n$ and B supersingular. Then there exists an isogeny $B_{\mathbb{F}} \sim_{\mathbb{F}} E^g$, where E is a supersingular elliptic curve.*

Suppose n is even (or perform a quadratic extension, and replace B by a simple factor if necessary). Let $\pi = \pi_B$ be the Weil q-number associated with B; i.e. π is the algebraic number defined as the endomorphism $\pi = F^n$, the "geometric Frobenius" on $B/\mathbb{F}_q$. Then $\mid \psi(\pi) \mid = \sqrt{q}$ for every $\psi : \mathbb{Q}(\pi) \to \mathbb{C}$ by the Weil conjecture. Note that $\beta := \pi/\sqrt{q}$ is a unit at all finite places not dividing p. The condition that all Frobenius slopes are equal to $1/2$ shows that β is also a unit at every finite place dividing p. A well-known theorem in number theory, see [1], page 105, Th.2, shows that under these conditions β is a root of unity, say $\beta^s = 1$ for some $s \in \mathbb{Z}_{>0}$. Hence $\pi^s \in \mathbb{Q}$, which shows that $B \otimes \mathbb{F}_r \sim E^g$, with $r = q^s$, where E a supersingular elliptic curve over $\mathbb{F}_r$, see [35], page 97. $\square$

(2.3) Remark. In characteristic zero an abelian variety of CM-type is defined over a number field, see [32], Proposition 26 on page 109. However in positive characteristic there are examples that an abelian variety admitting *sufficiently many complex multiplications* which are not defined over a finite field (hence the first part of the proof is not superfluous). However, as Grothendieck showed, see [22], the isogeny class of a smCM abelian variety is defined over a finite field (hence, in retrospect, it is not surprising what we see in the proof).

(2.4) The number of isomorphism classes of elliptic curves over $\mathbb{F} := \overline{\mathbb{F}_p}$ was computed by Hasse, Deuring and Igusa. See [8], [13]. It can be given as a class number. Also see [15] page 118. It can also be given by:

$$\sum_{j(E)} \frac{1}{\#(\mathrm{Aut}(E)} \quad = \quad \frac{p-1}{24},$$

the sum taken over all isomorphism classes of supersingular elliptic curves over $\mathbb{F}$.

Deligne and Shioda proved:
Suppose $g \geq 2$. Let $E_1, \cdots, E_{2g}$ be supersingular elliptic curves over $\mathbb{F}$. Then

$$E_1 \times \cdots \times E_g \quad \cong \quad E_{g+1} \times \cdots \times E_{2g}.$$

See [33], Th. 3.5.

Write $\Lambda_{g,j}$ for the set of isomorphism classes (E^g, μ), where E is a supersingular elliptic curve, and $\mu : E^g \to (E^g)^t$ is a polarization with $\mathrm{Ker}(\mu) = E^g[F^j]$. This number $\Lambda_{g,j}$ can be expressed as a class number; e.g. see [12], Section 2; see [18], Chapter 4, Chapter 8.

(2.5) Theorem/Definition, see [23], Th. 2. *Let A be an abelian variety over a field K. Write k for an algebraic closure of K. The following are equivalent:*
(1) $a(A) = \dim(A) = g$.
(2) *There exists an isomorphism $A_k \cong E^g$, where E is a supersingular elliptic curve.*
(3) Definition *A is called* superspecial. $\square$

Remark. Note the following curious fact.

For $g = 1$ a polarization on an elliptic curve is uniquely determined by its degree, e.g. $\#(\Lambda_{1,0}) = 1$, and there are many isomorphism classes of supersingular elliptic curves (many, if $p \gg 0$).

However,

for $g \geq 2$ there is only one isomorphism class of a superspecial abelian variety of dimension g, but $\#(\Lambda_{g,0})$ is large for $p \gg 0$.

Note: if $g \leq 2$ and $f(A) = 0$ then A is supersingular. However for $g \geq 3$ there are p-rank zero abelian varieties which are not supersingular.

3 Supersingular versus non-supersingular

We discuss some properties, and we will see that supersingular abelian varieties have properties in many cases very different from non-supersingular abelian varieties. In this section no proofs are given, and perhaps some definitions are lacking. We freely use the theory of foliations, to be explained in the next section. In considerations below, unless specified otherwise, we will work over an algebraically closed field k. We will consider an abelian variety A, which is supersingular, or non-supersingular as indicated. We write $X = A[p^\infty]$. We write σ for the supersingular Newton polygon and in this section we use ξ for a symmetric, non-supersingular Newton polygon. Some references are lacking, because material is still under construction, but you might find on my home page relevant information.

(3.1) Note that *the Manin conjecture* has been proved. One can use the Honda-Tate theory, but also a pure characteristic p proof is available:
For every symmetric Newton polygon ξ and every prime number p there exists an abelian variety A in characteristic p with $\mathcal{N}(A) = \xi$;

see [19], page 76, Conjecture 2; [35], page 98; [27], Section 4.

(3.2)
(ss) Any homomorphism $G_{1,1} \to X = A[p^\infty]$ can be p-adically approximated by a homomorphism $E \to A$. In particular *any A of dimension at least two is not absolutely simple.*
(non-ss) For any ξ there exists an abelian variety A over $\mathbb{F}$ with $\mathcal{N}(A) = \xi$ which is absolutely simple.

(3.3)
(ss) For any k and any A we have $\mathrm{End}^0(A) \cong \mathrm{Mat}_g(K_{p,\infty})$.
(non-ss) For any ξ there exists A over k such that $\mathrm{End}(A) = \mathbb{Z}$.

(3.4)
(ss) For any A over $\mathbb{F}$, its endomorphism ring is not commutative.
(non-ss) For any ξ there exists an abelian variety A over $\mathbb{F}$ such that $\mathrm{End}^0(A)$ is a field (necessarily a CM-field of degree $2 \cdot \dim(A)$ over $\mathbb{Q}$). See [17].

(3.5)
(ss) Let $W_\sigma \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ be the supersingular locus. The set $\Pi_0(W_\sigma))$ equals $\Lambda_{g,g-1}$. This uses [20], and [18] or [29]. For fixed g and $p \gg 0$ we have $\#(\Pi_0(W_\sigma) > 1$, i.e. W_σ is geometricaly reducible in these cases. Some examples:

for $g = 1$ we have W_σ is geometricaly irreducible $\Leftrightarrow p \leq 11$;

for $g = 2$ we have W_σ is geometricaly irreducible $\Leftrightarrow p \leq 11$;
for $g = 3$ we have W_σ is geometricaly irreducible $\Leftrightarrow p = 2$.
(non-ss) The stratum W_ξ is geometrically irreducible. See [5].

(3.6) We discuss $\Pi_0(W_\sigma)$. For (A, λ), a supersingular abelian variety with a principal polarization there exists a unique Polarized Flag Type Quotient, see [20]. In [18] and in [29] we show the (quite difficult) theorem that $W_\sigma(a = 1)$ is everywhere dense in W_σ. Combination of these facts shows:

$$\Pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}.$$

From this it follows:

$$\#(\Pi_0(W_\sigma)) = H_g(p, 1) \text{ if } g \text{ is odd}, \qquad \#(\Pi_0(W_\sigma)) = H_g(1, p) \text{ if } g \text{ is even}.$$

This is one of the main results of [18]. For notation e.g. see [18], 4.5 and Chapter 8.

(3.7) The Hecke orbit of a moduli point $x = (A, \lambda)$ is the set of moduli points of polarized abelian varieties isogenous with $A, \lambda)$. For the definition of the Hecke orbit of a moduli point, see [4], Section 1 (actually there are two definitions which are slightly different). For the following results see [2], Prop. 1 on page 448.
(ss) *For $x \in W_\sigma$ and a prime number ℓ different from p the Hecke orbit $\mathcal{H}_\ell(x) \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is finite.*
(non-ss) *For $x \notin W_\sigma$ and a prime number ℓ different from p the Hecke orbit $\mathcal{H}_\ell(x) \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is non-finite.*

(3.8) The following can be deduced from the previous subsection. Also see [31].
(ss) Any central leaf is finite.
(non-ss) Any central leaf has positive dimension.

(3.9)
(ss) An EO stratum which is contained in W_σ is geometrically reducible. Conjectured by FO, proven by S. Harashita, see [10].
(non-ss) An EO stratum which is not contained in W_σ is geometrically irreducible. This was conjectured by FO, see [28], 14.1, and is proven to be true in [9].

(3.10) Let A be an abelian variety simple over a finite field $K = \mathbb{F}_q$. We write π_A for its geometric Frobenius.
(ss) The abelian variety A is supersingular iff $(\pi_A)^r$ is totally real for some $r \in \mathbb{Z}_{>0}$.
(non-ss) The abelian variety A is non-supersingular iff $(\pi_A)^r$ is totally complex for all $r \in \mathbb{Z}_{>0}$.

The various differences alluded to above can be understood better in the theory of foliations.

4 Understanding via foliations

Basic reference [30].

(4.1) Let $[(A,\mu)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$. We defined the (central) leaf passing through this point by:

$$C(x) \quad = \quad \{[(B,\zeta)] \mid \exists \Omega : \quad (B,\zeta)[p^\infty] \otimes \Omega \cong (A,\mu)[p^\infty] \otimes \Omega\},$$

i.e. the moduli points are given by the geometric isomorphisms class of $(A,\mu)[p^\infty]$. In the definition Ω is an algebraically closed field.

In [30] also isogeny leaves are defined (the characteristic p version of Rapoport-Zink spaces). Moreover in Section 5 of that paper it is shown that any component of an open Newton polygon stratum up to finite morphisms is the product of a finite cover of irreducible components of a central leaf, respectively an isogeny leaf. Hecke correspondences are finite-to-finite above central leaves (and may blow up and down isogeny leaves in a fashion difficult to describe explicitly). Hence we obtain the feeling that central leaves behave very much like moduli spaces in characteristic zero, while isogeny leaves describe the characteristic p pathologies of Hecke correspondences.

(4.2) Note that for any $x \in \mathcal{A}_g \otimes \mathbb{F}_p$ for the Hecke orbit $\mathcal{H}(x)$ we have $\mathcal{H}(x) \subset C(x)$ (this follows immediately from the definition). Several years ago I made the conjecture that this Hecke-ℓ orbit is dense in $C(x)$, and the full Hecke orbit is dense in the related Newton polygon stratum, see [30], 6.1. This "Hecke orbit conjecture" has been proved and we hope a proof will appear not too long from now in [6]; this implies (using the product structure) that the full Hecke orbit of a point is dense in the related Newton polygon stratum.

(4.3) For any non-supersingular Newton polygon a central leaf is positive-dimensional. This follows from the fact that $\mathcal{H}_\ell(x)$ in this case is non-finite. At one extreme, for an ordinary x the isogeny leaf passing through x is just one point, and the central leaf is the whole ordinary locus. The same for the case of p-rank $= g-1$. At the other extreme, for a supersingular point its central leaf is finite, and any component of W_σ is contained in an isogeny leaf; this explains why considerations about W_σ are more algebraic than geometric in nature. For non-supersingular Newton polygon strata geometric properties have to be taken into account. For all non-supersingular cases the dimension of a central leaf is positive; for the exact values see [31].

It can be proved that a non-supersingular leaf is geometrically irreducible (see [5] for the exact formulation of this statement and a proof).

5 Remarks and questions

(5.1) One reason why supersingular abelian varieties are interesting is because of the following facts.
(1) In many cases a stratum in the moduli space has supersingular points in its boundary. We can see those points the cases where "the p-structure is the most degenerated".
(2) Moduli spaces have geometric aspects. However many properties (such as the number of components) of the supersingular locus can be studied via algebraic methods.
For example these are the basic tools to prove the irreducibility of non-supersingular Newton polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$, see [5].

(5.2) An actual construction of isogeny classes of abelian varieties over finite field can be carried out explicitly using the Honda-Tate theory, see [35], [11]. This is a wonderful tool to use (often easy algebraic) number theory in order to construct such isogeny classes with certain properties. This tool can be used for more explicit constructions and questions, e.g. see [36], [17], [3].

(5.3) The supersingular locus is well-understood for low values of p and of g. However already for $g = 3$ interesting questions remain to be studied. Does this problem connect to a conjecture stated by Ibukiyama at this conference?

(5.4) Long ago I conjectured the irreducibility of non-supersingular Newton polygon strata, see [26], 8B. Note one the one hand that every supersingular abelian variety of dimension g over k has endomorphism algebra isomorphic with $\mathrm{Mat}_g(K_{p,\infty})$; we have seen that this algebra plays a crucial role in computing $\#(\Pi_0(W_\sigma))$. On the other hand for every irreducible comonent of W_ξ, and for the generic point η of that component the related abelian variety A over $\overline{\mathbb{F}_p(\eta)}$ has endomorphism ring equal to $\mathbb{Z}$; can this be used to prove irreducibility ? I never managed to carry our this idea. It seems that the geometry of a central leaf has to come into play. This phenomenon is non-present in the supersingular case (there any central leaf is finite), but has to be taken into account in the non-supersingular case. Hence it might be that this idea does not lead directly to a proof of the irreducibility of W_ξ.

(5.5) What is the closure of a central leaf? More precisely, let $x \in W^0(\xi)$, let $\xi' \prec \xi$. describe $W^0(\xi') \cap (C(x))^{\mathrm{Zar}}$. This seems an important question to settle in order to have a better understanding of the theory of foliations in this case.

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