Lifting Galois covers of algebraic curves

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Introduction

We discuss the question: *Does a pair (C,H) of an algebraic curve C over a field of positive characteristic and a subgroup H of Aut(C) admit a lift to characteristic zero*?

- We give motivating (counter)examples and general theory.
- Formulate a conjecture.
- Discuss partial results and possible approaches.

This talk serves as an introduction to the talk
Andrew Obus: *Proof of the Oort Conjecture.*

1. A question. In this talk we say C is an algebraic curve over a field K if C is proper and smooth of dimension one over K and absolutely irreducible. We say $\mathcal{C} \to S$ is a (relative) curve if this morphism is proper and smooth and of relative dimension one with absolutely irreducible geometric fibers.

(1a) **Definition.** Suppose given a curve C_0 over $\kappa \supset \mathbb{F}_p$ and a subgroup $H \subset \mathrm{Aut}(C_0)$. A lifting of (C_0, H) (to characteristic zero) consists of

$$\kappa \longleftarrow R \subset \mathrm{Quot}(R) = K,$$

where R is an integral characteristic zero domain and a relative curve

$$\mathcal{C} \longrightarrow \mathrm{Spec}(R), \quad H \subset \mathrm{Aut}(\mathcal{C})$$

such that

$$(\mathcal{C}, H) \otimes_R \kappa \cong (C_0, H).$$

(1b) **Question.** *For which pairs* (C_0, H) *does a lift exist*?

Note that the lifting problem for C_0 is formally smooth. However we will see that in general the lifting problem for (C_0, H) can be obstructed; in some cases a lifting does not exist, in several cases ramification in R is needed to make a lifting possible.

In order to have a positive answer to this question it suffices to consider the case where $\kappa = k$, an algebraically closed field; however if we insist on the extra property R to be normal, the choice of κ might have an influence on the answer; for such aspects in a slightly different lifting problem, see [2].

2. (Counter-)examples. We give several (easy) examples of an action $H \subset \mathrm{Aut}(C_0)$ that cannot be lifted to characteristic zero.

Note: *for any curve C over any field we have*

$$\mathrm{genus}(C) \geq 2 \quad \Longrightarrow \quad \#(\mathrm{Aut}(C)) < \infty.$$

Note: *for any curve C with* $\mathrm{genus}(C) \geq 2$ *over any field* $K \supset \mathbb{Q}$ *we have*

$$\text{(Hurwitz)} \qquad \#(\mathrm{Aut}(C)) \leq 84(g-1).$$

(2a) **Example** (Roquette 1970, see [23]). There are many curves in positive characteristic having more automorphisms than admissible in characteristic zero. Here is one example. Consider over a field of characteristic $p = 5$ the affine curve defined by

$$Y^2 = X^p - X;$$

the curve C_0 is the normalization of any Zariski closure of this affine curve, considered over $\mathbb{F} = \overline{\mathbb{F}_p}$. This is a curve of genus 2. We see that

$$\#(\mathrm{Aut}(C_0)) = 120; \quad \text{note that} \quad 120 > 84(g-1) = 84.$$

Conclusion. The pair $(C_0, H = (\mathrm{Aut}(C_0))$ cannot be lifted to characteristic zero: *the group H is too large.*

We omit to mention the rich theory of bounds on automorphism groups in positive characteristic (Stichtenoth, Singh), and the study of many examples.

(2b) **Automorphisms of $\mathbb{P}^1$.**
Let $\sigma \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ be of order n. This automorphism has two fixed points. After normalizing these to 0 and ∞ this automorphism is given by $x \mapsto \zeta_n{\cdot}x$ in an affine coordinate. The cover $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}/ < \sigma >$ is called a *Kummer cover.*

We see:

$$m > 2 \quad \Longrightarrow \quad (\mathbb{Z}/2)^m \subsetneqq \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}),$$

and for $p > 2$ we have

$$m > 1 \quad \Longrightarrow \quad (\mathbb{Z}/p)^m \subsetneqq \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}).$$

Let $\kappa \supset \mathbb{F}_p$; let $m \in \mathbb{Z}_{>0}$ and assume $\dim_{\mathbb{F}_p}(\kappa) \geq m$. Then there exists

$$H = (\mathbb{Z}/p)^m \hookrightarrow \mathrm{Aut}(\mathbb{P}^1_\kappa)$$

such that every automorphism in H has no fixed points on $\mathbb{A}^1_\kappa = \mathbb{P}^1_\kappa - \{\infty\}$. Every element σ in H is given by $x \longmapsto x + a$ for some $a \in \kappa$. The cover $\mathbb{P}^1_\kappa \longrightarrow \mathbb{P}^1_\kappa/ < \sigma >$ is called an *Artin-Schreier cover.*

(2c) **Conclusion.** We see that

$$H = (\mathbb{Z}/2)^m \hookrightarrow \mathrm{Aut}(\mathbb{P}^1_\kappa) \text{ with } m > 2,$$
$$\text{respectively } H = (\mathbb{Z}/p)^m \hookrightarrow \mathrm{Aut}(\mathbb{P}^1_\kappa) \text{ with } p > 2 \text{ and } m > 1$$

cannot be lifted to characteristic zero: although $\#(\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})) = \infty$, *in this case the group H is too large to fit into this automorphism group.*

(2d) See [16], § 1. Consider the curve C_0 given in (2a). Define $\beta, \gamma \in \mathrm{Aut}(C_0)$ given by:

$$\beta(x) = x + 1, \quad \beta(y) = y : \quad \gamma(x) = -x, \quad \gamma(y) = 2y.$$

Note that "$2 = \sqrt{-1} \in \mathbb{F}_5$". We see

$$\beta \neq 1, \ \beta^5 = 1, \ \gamma^2 \neq 1, \ \gamma^4 = 1, \quad \gamma^{-1}\beta\gamma = \beta^4,$$

$$N := <\beta> \quad \subset \quad H := <\beta, \gamma> \quad \subset \quad \mathrm{Aut}(C_0), \qquad \#(H) = 20$$

and

$$\mathbb{Z}/5 \cong N \subset H$$

is a normal subgroup, with $\mathbb{Z}/4 \cong Q := N/H$ generated by (the residue class of) γ.
Claim. *The pair (C_0, H) cannot be lifted to characteristic zero.*
As $20 < 84$ the argument of (2b) does not give the result in this case.

Suppose a lifting $C_0 \subset \mathcal{C} \supset C = \mathcal{C}_\eta$ would exist with $H \subset \mathcal{C}$ over some integral domain in characteristic zero, and C defined over K; note that genus$(C/N) = 0$. In that case we study

$$C \longrightarrow C/N = D \longrightarrow C/H = P.$$

We see that the wildly ramified cover $C_0 \longrightarrow C_0/N = D_0$ would lift to a cover in characteristic zero. By the Zeuthen-Hurwitz-Hasse theorem we see that the different of $C_0 - D_0$ equals 12:

$$2g(C_0) - 2 = (2g(D_0) - 2).5 + \deg(\delta); \quad \deg(\delta) = 2.2 - 2 + 2.5 = 12.$$

The branch locus of $C \to D$ (perhaps after extending K) consists of three points $\{P_1, P_2, P_3\}$: the lifting of $C_0 \to D_0$ splits the branch locus in this way. Moreover the group

$$<\gamma> \cong Q \cong \mathbb{Z}/4 \quad \text{would act faithfully on} \quad (D, \{P_1, P_2, P_3\}).$$

However the curve D_0 is rational; an action of the cyclic group Q on D has two fixed points; a union of orbits of this faithful actins has cardinality $1, 2, 4, 5, 6, \cdots$. We see $\{P_1, P_2, P_3\} \subset D$ is not the union of orbits of an action $\mathbb{Z}/4 \subset \mathrm{Aut}(D)$ on $D \cong \mathbb{P}^1$. This contradiction shows the claim. □
The group H does not have the correct structure for a subgroup of automorphisms of a curve of genus 2 in characteristic zero.

(2e) It looks like that a global argument is necessary in (2d). Hoqwever, This is a local problem. It is amusing to study more closely the different of the cover $C_0 \to D_0$ in (2c); ramification is only at "$y = \infty$". We can define C_0 by pasting the affine curve given as a neighborhood of 0

$$U^{(0)} : \quad Y^2 = X^5 - X \quad \text{under} \quad \xi = \frac{1}{X}, \quad \eta = \frac{Y}{X^3}$$

to a neighborhood of ∞

$$U^{(\infty)}: \quad \eta^2 = \xi - \xi^5.$$

The cover $C_0 \to D_0$ sends $Z =$ "∞/C_0'' $= (\eta = 0, \xi = 0)$ to $P =$ "∞/D_0''

$$Z \in C_0, \quad Z \longmapsto P \in D_0.$$

The completion of the local rings

$$Z \longmapsto P, \quad \mathcal{O}_{C_0,Z} \supset \mathcal{O}_{D_0,P},$$

gives, choosing local parameters

$$k[[z]] \hookleftarrow k[[t]], \quad z = \eta, \quad t = \frac{\xi^3}{\eta}.$$

Note that t is invariant under the action of β. In order to compute the different of the cover $C_0 \to C_0/ < \beta >= D_0$ at $Z \longmapsto P$ we compute dt/dz: under $\eta^2 = \xi - \xi^5$ we have

$$t = \frac{\xi^3}{\eta} = \eta^5 + 3\eta^{13} + (\text{terms of order at least 51}) \text{ in } \quad k[[z]] = k[[\eta]],$$

because $\xi^3/\eta = (\eta^2 + \xi^5)^3/\eta = (\eta^2 + (\eta^2 + \xi^5)^5)^3/\eta$, etc.

We see, as expected, that the different has degree 12 at P; this was already predicted by the fact that the different of a deformation would give 3 branch points with different equal to $3 \times (p-1) = 12$. This proves that the *local lifting problem* given by $\mathbb{Z}/5 = N \hookrightarrow \mathrm{Aut}(k[[z]])$ above does not admit a llifting. See (4d).

(2f) Once there was a question whether an ordinary CM Jacobian in characteristic p would have a canonical lift which again is a CM Jacobian. In [20] we see this is not the case. This provides us with examles of groups of automorphisms which cannot be lifted. The more general case of CM liftings of ordinary Jacobians was settled in [4]. Also see [8], IV Theorem 2.6, [10], Theorem 6.6.

Here is (a special case of) an example mentioned in [20]. Choose $p = 5$, and $H = X^3 - X$ and $G \in \kappa[X]$ of degree 3 and relatively prime with H. Study the (normalization of a Zariski closure of the) curve C_0 given by $Y^p - H^{p-1}Y = H^{p-1}G$. It is shown: $\mathrm{genus}(C_0) = 8$, the curve is not hyperelliptic, and it Jacobian $J_0 = \mathrm{Jac}(C_0)$ is an ordinary abelian variety; the canonically lift of this principally polarized abelian variety is not a Jacobian.

(2g) **Deforming (lifting) an Artin-Schreier cover to a Kummer cover,** see [25], Introduction and § 1. In positive characteristic p we have an Artin-Schreier exact sequence

$$0 \to \underline{\mathbb{Z}/p} \longrightarrow \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0$$

on the one hand, and in characteristic zero we have the Kummer exact sequence

$$0 \to \mu_p \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \to 0$$

(apologies for writing even in the multiplicative case 0 for the trivial group) on the other hand. Is there a mixed characteristic lifting combining these two sequences? I do not know "abstract, general theory" answering this question. However we can write down explicit formulas for an exact sequence over a mixed characteristic ring (sufficiently ramified) which does combine these two sequences. See [25], pp. 347-353 for explicit formulas and for “an explanation”. See (4e) and (4f).

3. A conjecture. Lifting questions have been on my mind for more than 40 years; e.g. see [15]. A survey of some results available in 1985 can be found in [16]. After 1986 I was expecting the following to be true.

We have seen several “reasons” why lifting of (C_0, H) (for a“large” group H) is not possible. However we expect to be true:

Conjecture (see [16] and [18]). *Let C_0 be a curve in positive characteristic and $<\beta>= H \subset$* $\mathrm{Aut}(C_0)$ *be a* cyclic *group of automorphisms. In this case we conjecture the pair (C_0, H) can be lifted to characteristic zero.*

There are (at least) two approaches to this conjecture. Ons is to study lifting/deformations of Artin-Schreier covers of Witt groups to characteristic zero; it seems that one should understand the geometry of compactifications of these groups. Another method could be deforming first in characteristic p the problem at hand into one which is more manageable; this method was applied successfully to he lifting problem of abelian varieties, see [12] and to a proof of a conjecture by Grothendieck, see [19] § 8 for a description and for references. Anyhow, one should first reduce to the local cae, see (4d). Let us see whether a final answer to conjecture above uses such methods.

4. Theory / some results After deformation theory became available theoretic results proving lifting properties were proved (curves, principally polarized abelian varieties). In some obstructed cases counter examples became available. Remaining cases are sometimes proved by a combination of theory and “tricks”.

Lifting groups of automorphisms of curves is the topic today.

CM liftings of abelian varieties were studied in [17], and complete answers to those problems are now available and will be published in [2].

(4a) **In characteristic zero** we have the **Riemann existence and uniqueness theorem** which can be formulated as:

Theorem. *Let D be a curve over $\mathbb{C}$ and $\{P_1, \cdots P_m\} \subset D$ and H a finite group; there is a one-to-one correspondence between on the one hand the set of all $(C, H \subset \mathrm{Aut}(C))$*

$$C \longrightarrow C/H \cong D \quad \text{unramified outside} \quad \{P_1, \cdots P_m\}$$

and on the other hand the set of all continuous group homomorphisms

$$\pi_1(D - \{P_1, \cdots P_m\}) \quad \twoheadrightarrow H$$

(we omit the choice of a base point in the notation).

My interest in studying liftings (deformations) of curves with groups of automorphisms is centered around the (too vague) question how much we can retrieve the essence of this theorem in mixed or in pure positive characteristic. As long as wildly ramified covers are avoided this is part of general theory:

(4b) **Tame covers.**
Theorem (see SGA 1, Exp. XIII by Mme Michele Raynaud). *Let $S = \mathrm{Spec}(R)$ be a scheme, with R a complete local* DVR *as above, $\mathcal{D} \longrightarrow S$ proper and smooth, and $\mathcal{B} \subset \mathcal{D} \to S$ a divisor with normal crossings relative to S; the case of interest to us: $\mathcal{D}/S$ is a curve, and $\mathcal{B}$ is the union of a finite set of mutually disjoint sections $\mathcal{P}_i : S \to \mathcal{D}$. Write $\mathcal{U} = \mathcal{D} - \mathcal{B}$. Then*

$$\pi_1^t(\mathcal{U}_\eta) \xrightarrow{\sim} \pi_1^t(\mathcal{U}_0)$$

is an isomorphism (again, we omit the choice of a base point in the notation). Here "t" stands for the requirement that coverings are at most tamely ramified above $\mathcal{B}$.

(4c) As a **corollary** we see that in order to prove the conjecture (3) (for abelian groups ...) it suffices to prove the conjecture for finite cyclic groups of order a power of p. Indeed, for a commutative group H we can write $H = H^{(p)} \times H(p)$ (the prime-to-p part, and the p-power part). If any $H(p)$-cover lifts, then any H cover lifts by (4b).

(4c) **Wildly ramified covers.** A covering of curves $C \to D$ is wildly ramified at $Q \mapsto P$ if the characteristic p of the base field divides the order of the ramification group at Q.

Any *abelian* cover $C \to D$ of algebraic curves can be obtained by pulling back (and completing) and isogeny of their generalized Jacobians, see [26], VI.13, Corol. on page 128. In practice this means that wildly ramified abelian covers (in the local case) can be obtained by studying Artin-Schreier type coverings of Witt groups.

(4d) **The local conjecture.**
Conjecture. *Let k be an algebraically closed field of characteristic p, let H he a finite* cyclic *group acting on $k[[z]]$. The conjecture says that there exists a characteristic zero domain $R \twoheadrightarrow k$ and an action of H on $R[[z]]$ lifting the action of H on $k[[z]]$.*

(4e) **Local-global.** *Suppose a finite group H acts on C, and suppose that for every $Q \in C$ the action of H on the completion of the local ring $\mathcal{O}_{C,Q}$ lifts, then the action of H on C admits a lift.* See [5], [1], [3].

(4f) **The order not divisible by p^2.** The main result of [25], the beginning of the story, says that the conjecture (3) indeed is true in case p^2 does not divide the order of the cyclic group H. In the proof, as recorded in (2g), the local problem is solved for the cyclic group of order p. the local-global problem for the abelian case was settled, and the result follows.

(4g) **The order not divisible by p^3.** The case of a cyclic group H, where p^3 does not divide $\#(H)$ was proved in [5].

(4h) We have seen in (2) examples of (non-cyclic) groups with an action on a curve that does not lift. However for some non-cyclic groups any action lifts; for this problem see e.g. [3], [13], [28], [27] and several other publications. In [3], Conjecture 2 we find a statement which groups (besides cyclic groups) should have the property that any action of that group lifts.

5. Recent results.

(5a) The case of the conjecture (3) that p^4 does not divide $\#(H)$ was announced in [14].

(5b) In [14] a large class of cyclic covers which can be lifted is determined. In [22] an arbitrary cyclic cover in characterstic p is deformed into a a cyclic cover for which [14] proves the conjecture. Hence we see that the papers [14] and [22] together claim a proof of Conjecture (3) for every cyclic cover of a curve in characteristic p.

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