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van Oosten, Jaap (NL-UTRE)

★ **Realizability: an introduction to its categorical side.**

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This book aims at beginning researchers in the field of realizability and so emphasizes technical tools rather than any overview of methods or results. The central object here which created the categorical approach to realizability is Martin Hyland's effective topos called *Eff*. The author advises that readers interested in getting directly to that topos can skip Chapter 1 and will only need "some parts of Chapter 2" (p. xii). However, that opening material will be needed for any research career on this and other realizability toposes. The reader is assumed to know some amount of general category theory as well as to have an "acquaintance with the notion of a topos" (p. vi).

The tools are presented very clearly and this is especially advantageous for the idea of a tripos. The standard reference on triposes has been Andrew Pitts's 1982 Ph.D. dissertation ["The theory of triposes", Univ. Cambridge, Cambridge], not reviewed in MR as it was never published. Considerable simplification has been possible since that pioneering work. This book gives a very clear exposition and should become the reference.

Chapter 1 deals with the idea of a partial combinatory algebra or *pca* and gives a dozen kinds of examples. The paradigm *pca* is the set of natural numbers, regarded as codes of partial recursive functions, so that one natural number n may or may not have a defined value when applied as a partial recursive function to another number m . The most obvious variants use partial recursive functions relative to some oracle. Other variants use other models of computation or of data types. This chapter gives clear proofs of various standard theorems showing to what extent certain kinds of *pcas* share classical features of the partial recursive functions.

Chapter 2 explains the idea of tripos and shows a canonical construction on any tripos which produces a topos, with the internal logic of that topos closely related to the tripos. Then Chapter 2 shows how each *pca* gives a tripos, with the corresponding topos called a realizability topos. That term comes from the way an element a of a *pca* may realize a formula φ in the logic of the tripos or the topos, in the rough sense that a encodes calculations or data which show φ is true.

Chapter 3 on the effective topos *Eff* is half of the book. It recapitulates the necessary material from Chapter 2 so it is relatively self-contained. Indeed, this chapter briefly describes *Eff* directly in terms of assemblies but treats this only as a theorem and does not mention that it suffices as a rigorous definition of the topos independent of the tripos machinery. Assemblies and this construction were introduced by A. Carboni, P. J. Freyd and A. Scedrov [in *Mathematical foundations of programming language semantics (New Orleans, LA, 1987)*, 23–42, Lecture Notes in Comput. Sci., 298, Springer, Berlin, 1988; [MR0948482 \(89g:68043\)](#)]; and some corrections are in the reviewer's book *Elementary categories, elementary toposes* [Oxford Univ. Press, New York, 1992; [MR1182992 \(94c:18001\)](#) (pp. 229–239)].

The chapter develops first-, second-, and third-order arithmetic in *Eff*. The first-order part

coincides with classical Kleene realizability and notably it verifies a strong version of Church's thesis saying *every* function from the natural numbers to themselves is recursive. Thus it refutes classical logic and the axiom of choice—although in fact it verifies countable choice. The result is allied to intuitionistic arithmetic and analysis. Later the chapter describes interpretations of intuitionistic or constructive set theory in Eff due to McCarty and to Lubarsky, Streicher, and van den Berg.

None of the results is exactly what any given intuitionist or constructivist might want. This is a crucial feature of the approach: it gives a canonical meaning to recursive realizability of any first- or higher-order statement of arithmetic independent of any philosophical conception of what 'ought' to count as intuitionistic or constructive. Other pcas, as described in Chapters 1 and then 4, give other notions of realizability. In effect one pca is one notion of first-order realizability, and the topos context gives each one of those a canonical higher-order extension.

Almost at the beginning of work on Eff, Hyland saw it contains striking new kinds of sets and categories: if we do set theory inside of Eff then a nontrivial 'set' A may be the same size as the 'set' A^A of all functions from it to itself. In classical set theory this happens only when A is a singleton. As a more refined variant, a small category, that is, a category with a 'set' of arrows, may have arbitrary 'set' sized limits without just being a pre-order. Van Oosten gives an extensive but intuitive account of this in terms of fibrations without attempting the fully rigorous "mathematical core of the argument" (p. 179).

These things suggested computationally plausible kinds of data structures which exist in the most naive sense in Eff while these naive forms are impossible in classical set theory. Classically, a model of untyped λ -calculus cannot represent every function from terms to terms by a term—unless there is only one term and so only one function from terms to terms! But each non-singleton 'set' A in Eff which is the same size as the function set A^A gives a nontrivial model of untyped λ -calculus with that naively desirable property. Chapter 3 discusses synthetic domain theory, which is more or less an axiomatic theory of categories of data types and shows that many kinds of extremely powerful, simple, naive, and classically impossible constructions work in the internal logic of Eff. Similar results hold for an axiomatic computability theory developed by Andrej Bauer. Here an observation on fixed points by William Lawvere (p. 233) leads to naively simple, classically impossible, arguments which lead to classically valid results on computability.

Chapter 3 closes with general comments on open questions about Eff. Some are related to André Joyal's characterization of Eff by a universal property. Van Oosten lays out several versions of this. Then he shows how little is yet known about the related question of geometric morphism to or from Eff. This is particularly frustrating to the reviewer, since one advantage the tripos construction seems to have over constructing Eff directly from assemblies is that triposes themselves admit a natural notion of geometric morphism (p. 86ff.) and one could hope it would give a good theory of geometric morphisms between realizability toposes and others. But it has not yet.

Chapter 4 describes a great many variant notions of realizability. This is particularly close to van Oosten's 1991 Ph.D. dissertation ["Exercises in realizability", Univ. Amsterdam, Amsterdam] collecting several articles he published around that time. It covers a great deal done since then by himself and others.

Altogether, the book gives an extensive survey of research on realizability toposes with a 12-page

bibliography.

Reviewed by *Colin McLarty*

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