Abstract

This article presents a generalisation of the two main methods for obtaining class models of constructive set theory. Heyting models are a generalisation of the Boolean models for classical set theory which are a kind of forcing, while realizability is a decidedly constructive method that has first been develloped for number theory by Kleene and was later very fruitfully adapted to constructive set theory. In order to achieve the generalisation, a new kind of structure (applicative topologies) is introduced, which contains both elements of formal topology and applicative structures. The generalisation not only deepens the understanding of class models and leads to more efficiency in proofs about these kind of models, but also makes it possible to prove new results about the special cases which were not known before and to construct new models.

Generalising Realizability and Heyting Models for Constructive Set Theory

Albert Ziegler*

January 13, 2008

1 Introduction

1.1 Models of Set Theory

Models of set theory have played an extremely important role and shaped this discipline for a long time. They greatly enhance our understanding of set theoretic universes and furnish inspiration about many set theoretic principles as well as yielding concrete independance and proof theoretic results.

Models which can be defined as class models within set theory itself proved to be of particular interest and for classical set theory these are mainly the inner models and forcing constructions [8] which have turned out not only to obtain the vast number of noted independance results like the independance of the Continuum Hypothesis and the Axiom of Choice but also to be very worthwhile subjects of study in their own right.

It is reasonable to assume that for the same reasons why studying models of classical set theory remains so fruitful, research about models of constructive set theory can be of great potential benefit to this discipline.

Probably the most important constructions for class models of constructive set theory are realizability models and Heyting-algebra valued models (which are similar to forcing with Boolean Algebras). The previous have been explored by Rathjen for CZF [10] (and before that by McCarty for IZF [9]) while the latter have been detailed by Gambino [6]. In both cases, special attention needed to be payed due to the predicativity of the background theory.

Aczel [4] noticed how realizability models and Heyting models for higher order logic have a common generalisation in an impredicative context and suggested that this might also work for set theories and also in a predicative context (i.e. CZF rather than IZF).

This suggestion lead to the herein presented work and although the presented solution may at first blush look quite different from Aczel's generalisation for higher order logic (for example, we use a new structure called applicative topology instead

*Ludwig-Maximilian University of Munich, Department of Mathematics, aziegler@math.lmu.de

of Heyting algebras augmented by an application operation), many important ingrediences are actually the same.

This publication mainly draws from by Diploma thesis [13], supervised by Wilfried Buchholz.

1.2 CZF

The constructive set theory this article is concerned about (although its results extend to various others) is CZF [5, 3], which has acquired a very prominent role among constructive set theories. It uses intuitionistic logic with predicatively and constructively acceptable variants of the axioms of Zermelo-Fraenkel set theory. The theory will be briefly recapitulated during the following lines.

The basic axioms of Extensionality, Pairing, Union and Emptyset remain unchanged as does Infinity.

Separation is only demanded to hold for bounded formulas[1] to achieve of predicativeness. So for a bounded formula $\Phi(x)$,

$$\forall x \exists y.\ y = \{z \in x | \Phi(z)\}$$

This restricted scheme is called Δ_0-Separation.

Replacement is replaced by the constructively stronger Strong Collection, an axiom positing not only images of sets under functions (like Replacement) but under total relations (multi-valued functions). It means that

$$\forall x \in a \exists y\ \theta(x, y)$$

implies the existence of a set b with

$$\forall x \in a \exists y \in b\ \theta(x, y) \wedge \forall y \in b \exists x \in a\ \theta(x, y)$$

Foundation is reduced to the scheme of Set Induction which is just the induction principle for the $\in$-relation.

Instead of the Powerset Axiom, CZF includes the weaker axiom of Subset Collection. It demands for every formula θ and for all A, B the existence of a C such that for any u:

$$\forall x \in A \exists y \in B\ \theta(x, y, u) \to \exists z \in C.(\forall x \in a \exists y \in z\ \theta(x, y, u) \wedge \forall y \in z \exists x \in a\ \theta(x, y, u))$$

One of the most important applications of Subset Collection is that it entails the class of all functions between two given sets to be a set.

So the full list of Axioms is: Extensionality, Emptyset, Pairing, Union, Δ_0-Separation, Strong Collection, Subset Collection, Set Induction.

2 Applicative Topologies

Generalising Heyting and realizability models compels to first generalise the algebraic structures with which these models work, i.e. formal topologies[2] and applicative

[1]Recall that a formula is called *bounded* in case it only contains quantifiers that are restricted, i.e. $\forall a \in b$ or $\exists a \in b$

[2]Indeed, these are better suited in a predicative contexts than Heyting algebras themselves [6].

structures, also known as partial combinatory algebras or pca's..

Recall that a formal topology $(S, \leq, \lhd)$ consists of a poset $(S, \leq)$ together with a covering relation $\lhd \subset S \times \wp(S)$ which normally is a proper class. It needs to fulfill the following axioms:

$$a \in p \rightarrow a \lhd p$$
$$a \leq b \in p \lhd q \rightarrow b \lhd q$$
$$a \lhd p, q \rightarrow a \lhd \{b | \exists c \in p, d \in q. b \leq c, d\}$$

Here $p \lhd q$ is to be understood as $\forall x \in p. x \lhd q$.

Formal topologies are said to be set-presented if the relation $\lhd$ is given by a function $R : S \rightarrow \wp(\wp(S))$ such that

$$a \lhd p \Leftrightarrow \exists u \in R(a)\ u \subseteq p$$

While formal topologies have the complexity of a class (due to $\lhd$), set-presented formal topologies only have the complexity of a set, as they can be seen to consist only of $S, \leq$ and R with $\lhd$ only as a defined notion.

It is interesting to study a definition weaker than the following where the formal topology need not be set-presented. This yields relative class models of CZF without Subset Collection — but as this article intends to study full CZF, these things are better done elsewhere [13].

Definition 1. *A* ***applicative topology*** *is a set-presented formal topology* $(S, \leq, \lhd, R)$ *equipped with a subset* $\nabla \subseteq S$*, two special elements* $k \in \nabla$ *and* $s \in \nabla$ *as well as a partial binary operation* $\circ$ *called application. The following axioms need to be satisfied:*

1. $\forall x \in p, y \in q\ xy\downarrow \rightarrow (a \lhd p \wedge b \lhd q) \rightarrow ab\downarrow \wedge ab \lhd \{xy | x \in p, y \in q\}$
2. $xy\downarrow\ \wedge x, y \in \nabla \rightarrow xy \in \nabla$
3. $\forall x, y \in S.\ kxy\downarrow \wedge kxy \lhd \{x\}$
4. $\forall x, y \in S\ sxy\downarrow$
5. $\forall x, y, z \in S.\ ((xz)(yz)\downarrow\ \vee sxyz\downarrow) \rightarrow (sxyz\downarrow\ \wedge (xz)(yz)\downarrow\ \wedge sxyz \lhd \{(xz)(yz)\})$
6. $\nexists e \in \nabla\ e \lhd \{\}$

Frames with a set presentation g where g is closed by meets and contains $\top$[3] yield applicative topologis where the application operation is just the meet, ∇ the top element and the $\lhd$-relation is defined as

$$a \lhd p :\leftrightarrow a \leq \bigvee p$$

Applicative structures with discrete topology and $\nabla = S$ also yield applicative topologies. An example for applicative topologies which come neither from applicative structures nor from formal topologies are Oosten's ordered pca's [7] with the

[3] If there is any set presentation, there is also one with these properties

minimal topology[4] on them. The herein presented applicative topologies can be seen as a generalisation of ordered pcas.

The applicative structures can be thought of containing both computational information (as realizability structures do) and information about different cases which might happen (as Heyting algebras do). Those cases which are sure to happen resp. those computations which we can trust are stored in ∇.

Having fixed such an S, we can talk about applicative terms t (consisting of free variables and constants in S) and their inductively values t^S as is usually done for applicative structures.

Definition 2. *1. A closed term t denotes ($t\downarrow$) if $\exists a\in S\ t^S = a$.*

2. A term t convinces ($t!$) if for all substitutions σ of elements of S for the free variables, we have: $t[\sigma]\downarrow \rightarrow\ t[\sigma]^S \in \nabla$

3. Let $t \trianglelefteq t'$ mean that if one of the terms denotes, then both do and $t^S \lhd \{t'^S\}$.

The $\trianglelefteq$-relation plays a similar role as the $\simeq$-relation does for pcas, although it requires more subtle a handling since it is only a partial order, not an equivalence relation.

The most important feature of applicative structures is their combinatorical completeness, which applicative topologies cannot hope to match. But they come sufficiently close for our purposes:

Lemma 1. *Let t be an applicative term, v a variable. Then there is a denotating applicative term $\lambda v.t$ for with the same free variables except v such that*

$$(\lambda v_n.t)v_n \trianglelefteq t$$

If all constants from t convince, then so does $\lambda v.t$. If v was free in t or $t'\downarrow$, it follows

$$(\lambda v.t)t' \trianglelefteq t[v := t']$$

Proof. Use the definition of $\lambda v_n.t$ by induction on t which is usually used for pca's [12]. It works for ordered pca's [7] and it works here. Q.E.D.

This enables us to prove a weaker form of the fixed point lemma:

Lemma 2. *There is a convincing applicative term τ^{fix}, such that*

$$\forall a\in S.\ \tau^{fix}[v_1 := a] \trianglelefteq a(\tau^{fix}[v_1 := a])$$

Proof. The term $\tau^{fix} = ((\lambda v_2.v_1(v_2v_2))(\lambda v_2.v_1(v_2v_2)))$ works. Q.E.D.

Also, applicative topologies contain elements p, l, r, D which serve as pairing, left and right projections and case distinctions (between l and r) in the way one would expect, e.g. $Dlab \trianglelefteq a$ instead of $Dlab \simeq a$ [13].

When talking about applicative terms as if they were elements of S, we use the convention of really talking about their values and implicitely add the demand that they denote to every statement about them. For example, the statement $ab!$ (for a, b elements of S) would be taken to mean that $\circ$ is defined for (a, b) and yields an element of ∇.

[4] $a \lhd p \leftrightarrow \exists b \in p\ a \leq b$

3 Defining the Model

The underlying idea for the definition of the relative model is that we equip every set not only with the information which its elements are, but also which computational and factual information this elementhood entails. So each set in the class model $V(S)$ will have its elements valued by members of S. This only resulting in a change at the surface, we demand that these valuations should be $\lhd$-closed for technical reasons. For $p \subseteq S$ we shall write $\lhd p$ for the set of $b \in S$ which are covered by p.

Definition 3. *$V(S)$ is the smallest class such that*

$$\forall a \subseteq S \times V(S).\, \forall (e,b) \in a\; a^{-1}b \;=\; \lhd(a^{-1}b) \;\rightarrow\; a \in V(S)$$

The proof in [13] that this definition works is just a simple alteration of Rathjen's proof in [10] and uses general facts about inductive definitions in CZF [11, 2].

The idea of the pairing is that the first component tells the reason for or computational content of the fact that the second component is in the set (as in realizability), combined with the case in which it is in the set (as in Heyting models). These reasons need be closed under $\lhd$ by our technical definition, yet when defining a set in $V(S)$, it often gets tiresome to manually close these sets. This makes the following convention valuable:

Definition 4. *For a set $a \subseteq S \times V(S)$, $S(a)$ is defined to be the smallest superset of a which is in $V(S)$*

This is predicative and unique, as for such sets a

$$S(a) = \{(e,y) | e \lhd \{f | (f,y) \in a\}, y \in range(a)\}$$

Note that this requires the separability[5] of $\lhd$.

Now the realizability relation can be defined by induction over formulae with parameters in $V(S)$:

Definition 5. *For the elements of S, define $e \Vdash \phi$ (read as: e realizes ϕ) inductively:*

1. $e \Vdash \bot$ *if* $e \lhd \emptyset$
2. $e \Vdash x \dot{\in} y$ *if* $e \lhd y^{-1}x$
3. $e \Vdash x \in y$ *if* $e \lhd \{f \in S | \exists z \in Bi(y). lf \Vdash z \dot{\in} y \wedge rf \Vdash x = y\}$
4. $e \Vdash x = y$ *if* $\forall z \in Bi(x) \forall f \Vdash z \dot{\in} x.\ lef \Vdash z \in y$ *und* $\forall z \in Bi(y) \forall f \Vdash z \dot{\in} y.\ ref \Vdash z \in x$
5. $e \Vdash \phi \wedge \psi$ *if* $le \Vdash \phi \wedge re \Vdash \psi$
6. $e \Vdash \phi \vee \psi$ *if* $e \lhd \{f \in S | (lf \trianglelefteq l \wedge rf \Vdash \phi) \vee (lf \trianglelefteq r \wedge rf \Vdash \psi)\}$
7. $e \Vdash \phi \rightarrow \psi$ *if* $\forall f \in S. f \Vdash \phi \rightarrow ef \Vdash \psi$
8. $e \Vdash \forall x \phi(x)$ *if* $\forall a \in V(S)\ e \Vdash \phi[a]$

[5] i.e. the fact that $\lhd p$ is always a set, which is true for set-presented formal topologies

9. $e \Vdash \exists x \phi(x)$ *if* $e \lhd \{f \in S | \exists a \in V(S)\ f \Vdash \phi[a]\}$

Note that for the atomic formulas, another inductive definition as in [10] is necessary. The atomic formulas of type $x \dot{\in} y$ can be seen as only a technical convenience to define the other cases more easily.

The above definition can relatively easily be seen as equivalent[6] to a definition where subtly altered clauses are used[13]. These alterations are often more convenient and the herein used are

1. $e \Vdash \exists x \in y\ \phi(x)$ if $e \lhd \{f \in S | \exists (lf, a) \in y\ rf \Vdash \phi[a]\}$
2. $e \Vdash \forall x \in y\ \phi(x)$ if $\forall (f, a) \in y\ ef \Vdash \phi[a]$

Both [10] and [6] similarily use extra clauses for bounded quantification, which makes it much easier to prove Δ_0-Collection.

We define the truth value $[\![\phi]\!]$ of a formula as the class of its realizers and call the formula realized ($\Vdash \phi$) if it has a convincing realizer, i.e. a realizer in ∇. These truth values are saturated with respect to the $\lhd$-relation.

Theorem 1. *Let ϕ be a closed formula of set theory with parameters in $V(S)$. Then*

$$^{\lhd}[\![\phi]\!] = [\![\phi]\!]$$

Proof. By induction over ϕ. The cases $\bot, \dot{\in}, \in, \vee$ and $\exists$ are trivial by definition. For edification, the $\wedge$ case will be done here. The details for the rest can be found in [13] (the atomic $a = b$ case seems the most complex).

Let ϕ be $\psi \wedge \theta$ and $^{\lhd}[\![\psi]\!] = [\![\psi]\!]$ as well as $^{\lhd}[\![\theta]\!] = [\![\theta]\!]$. Consider an arbitrary $e \lhd [\![\phi]\!]$, i.e.

$$e \lhd \{f \in S | lf \Vdash \psi \wedge rf \Vdash \theta\}$$

We have $lf\downarrow$ and $rf\downarrow$ for elements f of the right side[7], thus

$$le \lhd \{lf \in S | lf \Vdash \psi \wedge rf \Vdash \theta\} \subseteq \{f \in S | f \Vdash \psi\}$$

$$re \lhd \{rf \in S | lf \Vdash \psi \wedge rf \Vdash \theta\} \subseteq \{f \in S | f \Vdash \theta\}$$

As the truth values of ψ and ϕ are saturated with respect to $\lhd$, we get

$$le \Vdash \psi \wedge re \Vdash \theta$$

And hereby $e \in [\![\psi \wedge \theta]\!]$. q.e.d.

[6] In the sense that the realized formulas are the same

[7] This is part of the convention above: When terms are treated like elements of S in some statement, then this statement really talkes about their values and includes the statement that they denote.

4 The Consistency Results

To show that we get a relative model, we need consistency results: We need to know that the realized formulae are closed under deduction in an intuitionistic predicate calculus. Then we need to know that all axioms of the theory under consideration (i.e. CZF) are realized. That the relized formulae don't form a trivial theory is then clear by the definition of $\Vdash \bot$ (assuming CZF itself is correct).

Theorem 2. *If Γ is a set of formulae with $\Vdash \phi$ for all $\phi \in \Gamma$ and $\Gamma \vdash \psi$ in intuitionistic predicate logic, then $\Vdash \psi$. In particular, all intuitionistic tautologies are realized.*

Proof. See [13], Satz 57. The proof is lenghty but not overly illuminating and mostly follows [9]. q.e.d.

Main Theorem 1. *All axioms of CZF are realized.*

This is done by showing realizedness for each axiom, possibily using equivalent formulations for CZF in consideration of economy of time. By treating these new models using applicative topologies in the same way as Rathjen's [10] treats standard realizability (e.g. recursively defining the $\Vdash$-relation instead of the truth values $[\![\cdot]\!]$), it is made possible to incorporate many ideas from [10] in our approach and a substantial part of each proof for the consistency of the axioms is alike to the corresponding proof of [10].

Some axioms are quite straightforward:

Lemma 3.

1. *$\lambda v.v$ realizes $\forall xy.\forall z \in x\ z \in y \wedge \forall z \in y\ z \in x \ \rightarrow\ z = x$. This is equivalent to extensionality.*
2. *$p(pke_r)(pke_r)$ realizes $\forall xy \exists z.\ x \in z \wedge y \in z$. This is equivalent to Pairing (given Δ_0-Collection).*
3. *$k(kpke_r)$ realizes $\forall x \exists y \forall z \in x \forall w \in z.\ w \in y$. This is equivalent to the Union Axiom (given Δ_0-Collection).*
4. *$e := \tau^{fix}[v_1 := \lambda vx.x(k(vx))]$ realizes $\forall x((\forall y \in x \phi(y)) \rightarrow \phi(x)) \rightarrow \forall x \phi(x)$, This is an arbitrary instance of ϵ-induction.*

Proof.

1. Let $x, y \in V(S)$ with $e \Vdash \forall z \in x\ z \in y \wedge \forall z \in y\ z \in x$. This implies $le \Vdash \forall z \in xz \in y$, and consequently $\forall (f, z) \in x.\ lef \Vdash z \in y$. In the same way we get $\forall (f, z) \in x.\ ref \Vdash z \in x$.
2. Sei $x, y \in V(S)$. Sei $z = S(\{(k, x), (k, y))\})$. Then we have $pke_r \Vdash x \in z$, $pke_r \Vdash y \in z$ and hereby

$$p(pke_r)(pke_r) \in \{e | \exists ze \Vdash x \in z \wedge y \in z\}$$

Being an element, it is covered by this set which is what was to show.

3. Let $x \in V(S)$ and let $y = S(\{(k,w) | \exists (e,(f,w)) \in y\})$. This entails

$$k(kpke_r) \Vdash \forall z \in x \forall w \in z\; w \in y$$

as for $(e,(f,w)) \in x$ we have $k(kpke_r)ef \trianglelefteq pke_r \Vdash w \in y$ because $(k,w) \in y$. Consequently

$$k(kpke_r) \in \{e | \exists y. e \Vdash \forall z \in x \forall w \in z w \in y\}$$

Being an element, it is covered by this set.

4. Sei $f \Vdash \forall x. \forall y \in x \phi(y) \rightarrow \phi(x)$.

Let $V(S)_\gamma$ mean the class of those elements of $V(S)$ whose rank is an element of γ.

We show $ef \Vdash \phi(x)$ for all $x \in V(S)$ by set-induction.

Assume this is known for all $x \in V(S)_\beta$ with $\beta \in \alpha$ and let $x \in V(S)_\alpha$. Then

$$\begin{aligned} ef &\trianglelefteq (\lambda x. x(k(ex)))f \\ &\trianglelefteq f(k(ef)) \\ &\Vdash \phi(x) \end{aligned}$$

Note that $k(ef) \Vdash \forall y \in x \phi(y)$ by the induction hypothesis, as $ef \Vdash \phi(y)$ for all $y \in Bi(x)$.

Q.E.D.

Δ_0-Collection is also not difficult, but requires a special consideration: Both in realizability and in Heyting model theory, truth values of bounded formulae form sets also in a predicative context. This is also true for this new kind of model if bounded quantifiers are realized in the alternative way discussed above.

Lemma 4. *Let ϕ be a formula. $e := p(\lambda xy.p(p(xy))e_r)(\lambda x.p(p(lx)e_r)(rx))$ realizes $\forall x \exists y.\, \forall z \in x \phi(z) \rightarrow z \in y \wedge \forall z \in y.\, z \in x \wedge \phi(z)$. This is Δ_0-Collection.*

Proof. Let $x \in V(S)$. We need to show that

$$e \lhd \{e' | \exists y. e' \Vdash \forall z \in x \phi(z) \rightarrow z \in y \wedge \forall z \in y. z \in x \wedge \phi(z)\}$$

Actually, it will even be proved that e is an element of this set. The witnessing y will be

$$y := S(\{(pfg) | (f,z) \in x, g \Vdash \phi(z)\})$$

This is indeed as set, as $[\![\phi(z)]\!]$ is a set for each $z \in x[S]$ and $x[S]$ is again a set.

For $f \Vdash z \dot{\in} x$ and $g \Vdash \phi(z)$ we have:

$$\begin{aligned} lefg &\trianglelefteq p(p(fg)e_r) \\ &\in \{h | lh \Vdash z \dot{\in} y, rh \Vdash z = z\} \\ &\subseteq \{h | \exists z'. lh \Vdash z \dot{\in} y, rh \Vdash z' = z\} \\ &\subseteq [\![z \in y]\!] \end{aligned}$$

On the other hand, for $f \Vdash z\dot{\in}y$, i.e. $f = pgh$ with $g \Vdash z\dot{\in}x$ and $h \Vdash \phi(z)$ we get

$$l(rlf) \trianglelefteq p(lx)e_r \trianglelefteq pge_r \Vdash z \in x$$
$$r(rlf) \trianglelefteq rx \trianglelefteq h \Vdash \phi(z)$$

Q.E.D.

Infinity is a bit lengthy in detail [13], but in essence the same as in [10]. It requires first implementing natural numbers n as $\underline{n}$ in the applicative topology and then defining their interpretation $\bar{n}$ in $V(S)$ as $S(\{(\underline{i}, \bar{i}) | i \in n\})$. The interpretation of the natural numbers can then be defined as $S(\{(\underline{i}, \bar{i}) | i \in \omega\})$.

More difficulties occur with the higher axioms of CZF, namely Strong Collection and especially Subset Collection. The technical aspects of their proof are greatly facilitated by allowing pairing into the syntax of the interpreted set theory, interpreting (a, b) as the closure of $\{(l, a), (r, \{(l, a), (r, b)\})\}$. With this, we get

Lemma 5. *Let ϕ be a formula. Then $e := \lambda x.p(\lambda y.py(xy))(\lambda y.y)$ realizes*

$$\forall a.\forall x \in a \exists y \phi(x, y) \to \exists b.\forall x \in a \exists y \in b\ \phi(x, y)\ \wedge\ \forall y \in b \exists x \in a\ \phi(x, y)$$

This is Strong Collection.

Proof. Let $a \in V(S)$ and $f \Vdash \forall x \in a \exists y \phi(x, y)$ which means

$$\forall (g, x) \in a fg \lhd \{h | \exists y h \Vdash \phi(x, y)\}$$

The class on the right hand side needn't be a set and remembering the convention that covered by a class means covered by a subset of the class, we get

$$\forall (g, x) \in a \exists c. fg \lhd c \wedge \forall h \in c \exists y\ h \Vdash \phi(x, y)$$

We want to collect these y in a set but would lose the information where the y come from if we don't include the information about g in the set. A good technical way to store the information is collecting the $(pg(fg), y)$ in some set Y dependant of g, x and c:

$$\forall (g, x) \in a \exists c. fg \lhd c \wedge \exists Y. \forall h \in c \exists (pg(fg), y) \in Y\ h \Vdash \phi(x, y) \wedge \forall (g', y) \in Y. g = pg(fg) \wedge \exists h \in c\ h \Vdash \phi(x, y)$$

To apply Strong Collection another time, this needs to be written in a slightly different way:

$$\forall (g, x) \in a \exists Y. \exists c. fg \lhd c \wedge \forall h \in c \exists (pg(fg), y) \in Y\ h \Vdash \phi(x, y) \wedge \forall (g', y) \in Y. g = pg(fg) \wedge \exists h \in c\ h \Vdash \phi(x, y)$$

We get the existence of a set B such that the following hold.

$$\forall (g, x) \in a \exists Y \in B \exists c. fg \lhd c \wedge \forall h \in c \exists (pg(fg), y) \in Y\ h \Vdash \phi(x, y) \wedge \forall (g', y). g = pg(fg) \wedge \exists h \in c\ h \Vdash \phi(x, y)$$

$$\forall Y\in B\exists(g,x)\in a\exists c.fg\lhd c\wedge\forall h\in c\exists(pg(fg),y)\in Y\ h\Vdash\phi(x,y)\wedge\forall(g',y).g=pg(fg)\wedge\exists h\in ch\Vdash\phi(x,y)$$

The set we wish to construct is essentially the union of the sets in B, but more correctly we set

$$b:=S(\{(g,y)\in\bigcap B|y\in V(S)\})$$

For this b, the following hold.

$$\forall(g,x)\in a\exists c.fg\lhd c\wedge\forall h\in c\exists(pg(fg),y)\in b\ h\Vdash\phi(x,y)$$
$$\forall y'\in b\exists y\exists(g,x)\in a.y'=(pg(fg),y)\wedge\exists c.fg\lhd c\wedge\forall h\in c\ h\Vdash\phi(x,y)$$

So indeed $S(b)\in V(S)$ and we can rewrite the above to:

$$\forall(g,x)\in afg\lhd\{h|\exists(pg(fg),y)\in S(b)\ h\Vdash\phi(x,y)\}$$
$$\forall(g',y)\in S(b)\ g'\lhd\{h|\exists x.lh\Vdash x\dot{\in}a\wedge rh\Vdash\phi(x)\}$$

This means that ef is an element of the class

$$\{h|\exists b\in V(S).h\Vdash\forall x\in a\exists y\in S(b)\ \phi(x,y)\wedge\forall y\in S(b)\exists x\in a\phi(x,y)\}$$

In particular, ef is covered by this class, and this was what was to prove. Q.E.D.

Lemma 6. *Let ϕ be a formula. Then*

$$i:=\lambda v.pk(p(\lambda x.p(px(vx))(r(vx)))(\lambda x.p(ly)(r(ry))))$$

realizes

$$\forall a,b\exists c\forall u.\forall x\in a\exists y\in b\phi(x,y,u)\to\exists d\in c.\forall x\in a\exists y\in d\phi(x,y,u)\wedge\forall y\in d\exists x\in a\phi(x,y,u)$$

This is Subset Collection.

Proof. For $a,b\in V(S)$ let

$$\tilde{b}:=\{(g,d)|(l(rg),d)\in b\}\in V(S)$$

For any $d\in\tilde{b}[S]$

$$e\lhd\tilde{b}^{-1}d\Rightarrow l(re)\lhd l(rb^{-1}d)\subseteq b^{-1}d\Rightarrow(e,d)\in\tilde{b}$$

Thus $\tilde{b}\in V(S)$.

Using Subset Collection in the background theory, there exists a $B\subseteq\wp(\tilde{b})$ such that for all $(f,x)\in a$, $u\in V(S)$, $e\in S$ and $S\supseteq v\in r[S]$ the following implication holds: If

$$\forall j\in v\exists(h,y)\in\tilde{b}.j=h\wedge(lh,x)\in a\wedge(l(rh),y)\in b\wedge r(rh)\Vdash\phi[x,y,u]$$

then there is a $\tilde{b}' \in B$ for which $\psi(f, x, u, v, e, \tilde{b}')$ holds, where ψ is the conjunction of the following two formulas:

$$\forall j \in v \exists (h, y) \in \tilde{b}'.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \phi[x, y, u]$$

and

$$\forall (h, y) \in \tilde{b}' \exists j \in v.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \phi[x, y, u]$$

Subset Collection now needs to be applied again and for each $e \in S$. It yields a set $C_e \subseteq \wp(B)$, such that for all $u \in V(S)$ the following implication holds:

$$\forall (f, x) \in a \exists \tilde{b}' \in B \exists v \in r[S].pf(ef) \lhd v \wedge \phi(f, x, u, v, e, \tilde{b}')$$

implies the existence of a $B' \in C$ with

$$\forall (f, x) \in a \exists \tilde{b}' \in B' \exists v \in r[S].pf(ef) \lhd v \wedge \phi(f, x, u, v, e, \tilde{b}')$$

and

$$\forall \tilde{b}' \in B' \exists (f, x) \in a \exists v \in r[S].pf(ef) \lhd v \wedge \phi(f, x, u, v, e, \tilde{b}')$$

Note that for each $e \in S$ there exists such a set C_e, but the choice is not canonical, which is to mean that there needn't be a function which assigns such a set to each $e \in S$, so the indexing may be seen as too suggestive. Nevertheless the important thing works, namely that by Strong Collection there is a set C^* such that for each $e \in S$ there is some set C_e in C^* for which the implication above holds for all $u \in V(S)$ and all elements of C^* are such an C_e for some element $e \in S$.

We would like to define

$$c = S(\{(k, S(\{(l, y) | \exists \tilde{b}' \in B'.(l, y) \in \tilde{b}'\})) | B' \in \bigcap_{e \in S} C_e\})$$

but as the relationship between the $e \in S$ and their C_es isn't functional, we can only write

$$c = S(\{(k, S(\{(l, y) | \exists \tilde{b}' \in B'.(l, y) \in \tilde{b}'\})) | B' \in \bigcap C^*\})$$

Still, this set c is a witness for the instance of Subset Collection under consideration.

To see this, take $u \in V(S)$ and $e \Vdash \forall x \in a \exists y \in b \phi(x, y, u)$. This means that for all $(f, x) \in a$

$$\exists v \in r[S].ef \lhd v \wedge \forall j \in v \exists (lh, y) \in b.j = h \wedge (lh, y) \in b \wedge rh \Vdash \phi[x, y, u]$$

This in turn implies

$$\exists v \in r[S].pf(ef) \lhd v \wedge \forall j \in v \exists (h, y) \in \tilde{b}.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \phi[x, y, u]$$

Remembering our first application of the Subset Collection Scheme,

$$\exists v \in r[S].pf(ef) \lhd v \wedge \exists \tilde{b}' \in B.\psi(f,x,u,v,e,\tilde{b}')$$

So using the introduction rule for universal quantification, we arrive at

$$\forall (f,x) \in a \exists \tilde{b}' \in B \exists v \in r[S].pf(ef) \lhd v \wedge \phi(f,x,u,v,e,\tilde{b}')$$

This is the hypothesis of one of the implications above, implying the existence of a $B' \in C$ for which the following two formulae hold:

$$\forall (f,x) \in a \exists \tilde{b}' \in B' \exists v \in r[S].pf(ef) \lhd v \wedge \psi(f,x,u,v,e,\tilde{b}')$$
$$\forall \tilde{b}' \in B' \exists (f,x) \in a \exists v \in r[S].pf(ef) \lhd v \wedge \psi(f,x,u,v,e,\tilde{b}')$$

Taking b' to be $S(\{(l,y) | \exists \tilde{b}' \in B'.(l,y) \in \tilde{b}'\})$ and recalling that $l(ef) \trianglelefteq k$ we have $(l(ef), b') \in c$. It remains to show that

$$r(ie) \Vdash \forall x \in a \exists y \in b' \phi(x,y,u) \wedge \forall y \in b' \exists x \in a \phi[x,y,u]$$

Assume $(f,x) \in a$. The first of the two required properties of B' implies the existence of some $v \in r[S]$ with $pf(ef) \lhd v$ and $\psi(f,x,u,v,e,\tilde{b}')$ for suitable $\tilde{b}' \in B'$. In particular this means that for each $j \in v$ there exists $(h,y) \in \tilde{b}'$ such that

$$j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \phi[x,y,u]$$

So $pf(ef) \lhd \{j | \exists (j,y) \in b' \wedge r(r(j)) \Vdash \phi[x,y,u]\}$. And thus the first part of the required statement follows:

$$r(r(ie))f \trianglelefteq p(pf(ef))(r(ef)) \Vdash \exists y \in b' \phi[x,y,u]$$

On the other hand, assume $(g', y) \in b'$. This implies the existence of some $\tilde{b}'$ with $(g,y) \in \tilde{b}'$ and $g' \lhd \tilde{b}'^{-1} y$. We want to show that all g with $(g,y) \in \tilde{b}'$ fulfill

$$l(r(ie))g \Vdash \exists x \in a \phi[x,y,u]$$

As truth values are closed with respect to $\lhd$, this is then certainly also true for g', i.e.

$$l(r(ie))g' \Vdash \exists x \in a \phi[x,y,u]$$

So if $(g,y) \in \tilde{b}'$ then the second of the two required properties of B' implies the existence of some $(f,x) \in a$ such that

$$\exists v \in r[S].pf(ef) \lhd v \wedge \forall (h,y') \in \tilde{b}' \exists j \in v.j = h \wedge (lh,x) \in a \wedge (l(rh'),y) \in b \wedge r(rh) \Vdash \phi[x,y,u]$$

One should note that only the second conjunct of ψ was needed here[8]. So if $(h,y') := (g,y)$, this entails

$$(lg, x) \in a \wedge r(rg) \Vdash \phi[x,y,u]$$

[8]I should like this opportunity to give my thanks to Nicola Gambino, who took the time to discuss with me his consistency proof regarding Subset Collection for Heyting models, where a similar trick was used.

As desired, we conclude

$$l(r(ie))g \trianglelefteq p(lg)(r(rg)) \Vdash \exists x \in a\phi[x, y, u]$$

Q.E.D.

These lemmata prove the main theorem.

4.1 Conclusion and Outlook

It is always satisfying to be able to see two previously different entities as merely different aspects of the same concept. But nevertheless the main hope in generalisations is for obtaining not only the vague feeling of new insights, but also new hard results. Fortunately, these are available.

In [13], some minor new independance results were presented using applicative topologies arising neither from pca's nor from Heyting algebras and it may be hoped that further ones could be found.

But besides finding new models, the more general framework also enables to discover new facts about the already existing models. For example, [13] and [14] describe how generalising Rathjens method to prove the absoluteness of the the important regular extension axiom REA for standard realizability models makes it possible to prove this absoluteness for the new models. This then automatically shows REA to be absolute for Heyting models, which was not previously known, although conjectured and taken to strengthen the position of REA as constructive axiom [1] on the very conference where these results where presented.

The results stated above have been concerned with the theory of CZF and shown to work for this theory. Of course the question about the scope of this method for obtaining models of constructive set theories arises: Will it also work for other theories?

The results from [13] show that they do work in many other set theories, although axioms dealing with functions (like Replacement) give difficulties. Choice principles can of course not be absolute in the general case, in fact one of the aims for develloping forcing constructions (of which the herein presented machinery is a generalisation) was just to find models where choice principles do not always hold, even if they held in the background theory. But they are absolute for a large class of applicative topologies.

Much attention has been dedicated to making these methods work in a predicative setting. However, they still work in an impredicative setting. For example, when using IZF in the background (one of the most prominent non-predicative set theories with intuitionistic logic), $V(S)$ proves to be a model of IZF again [13].

While these enlargements of the scope might lead to interesting future work, I would intend to first delve into general absoluteness results and concrete models. This requires finding new applicative topologies which don't arise from formal topologies nor from and which would lead to new, and hopefully interesting, independance results.

References

[1] Aczel, P., On avoiding dependent choices in formal topology. Talk at the Third Workshop on Formal Topology in Padua

[2] Aczel, Peter, The type theoretic interpretation of constructive set theory: inductive definitions. In: R.B. Marcus *et al.*, eds., *Logic, Methodology, and Philosophy of Science VII.* North–Holland, Amsterdam, 1986, pp. 17–49

[3] Aczel, Peter, The type theoretic interpretation of constructive set theory. In: A. Macintyre, L. Pacholski, J. Paris, eds., *Logic Colloquium '77.* North–Holland, Amsterdam, 1978, pp. 55–66

[4] Aczel, P., A note on interpreting intuitionistic higher order logic. Unpublished, undated

[5] Aczel, P, Rathjen, M., *Notes on Constructive Set Theory.* Institut Mittag–Leffler Preprint 40, 2000/01

[6] Gambino, N., Heyting-valued interpretations for Constructive Set Theory. *Ann. Pure Appl. Logic* 137 (2006), 164–188

[7] Hofstra, P., van Oosten, J., Ordered Partial Combinatory Algebras. Available at Jaap van Oosten's website.

[8] Kuhnen, K., Set Theory. North–Holland, Amsterdam, 1980

[9] McCarty, D.C., D. C. McCarty. Realizability and Recursive Mathematics. PhD thesis, Oxford University, 1984

[10] Rathjen, M., Realizability for Constructive Zermelo-Fraenkel Set Theory. In: Stoltenberg-Hansen, V. and Väänänen, J. eds., *Logic Colloquium '03.* Lecture Notes in Logic 24

[11] Rathjen, M., Generalized inductive definitions in constructive set theory. Available at the author's website.

[12] Troelstra, A.S., van Dalen, D., Constructivism in Mathematics. North-Holland, 1988

[13] Ziegler, A., *Eine gemeinsame Verallgemeinerung von Realisierbarkeitsmodellen und Heyting-Algebra bewerteten Modellen.* Diplomarbeit, available from the author

[14] Ziegler, A., The absoluteness of *REA. Typoscript, available from the author, 2007