Models of Intuitionistic Set Theory
in Subtoposes of Nested Realizability Toposes

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Introduction

Given a *partial combinatory algebra* (pca) $\mathcal{A}$ (see e.g. [16]) together with a subpca $\mathcal{A}_{\#}$ of $\mathcal{A}$ we will construct the *nested realizability topos* $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ as described in [5] (without giving it a proper name there). It is well known (from e.g. [16]) that $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ appears as the exact/regular completion of its subcategory $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ of assemblies. In [5] the authors considered two complementary subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$, namely the *relative realizability topos* $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_{\#})$ and the *modified relative realizability topos* $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_{\#})$, respectively.

Within nested realizability toposes we will identify a class of *small maps* giving rise to a model of intuitionistic set theory IZF (see [6, 13]) as described in [11]. For this purpose we first identify a class of display maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ which using a result of [2] gives rise to the desired class of small maps in the exact/regular completion $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$.

For showing that the subtoposes $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_{\#})$ and $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_{\#})$ also give rise to models of IZF we will prove the following general result. If $\mathcal{E}$ is a topos with a class $\mathcal{S}$ of small maps and $\mathcal{F}$ is a subtopos of $\mathcal{E}$ then there is a class $\mathcal{S}_{\mathcal{F}}$ of small maps in $\mathcal{F}$ which is obtained by closing sheafifications of maps in $\mathcal{S}$ under quotients in $\mathcal{F}$.

As explained in subsections 1.2.2 and 1.2.3 below this covers also the Modified Realizability topos as studied in [15] and the more recent Herbrand topos of van den Berg.

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1 Nested Realizability Toposes and some of their Subtoposes

Given a pca $\mathcal{A}$ in an elementary topos $\mathscr{S}$ we may construct the realizability topos $\mathbf{RT}_{\mathscr{S}}(\mathcal{A})$ relative to $\mathcal{S}$ as described in [16]. If $\mathscr{S}$ is the Sierpiński topos $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$ then a "nested pca", i.e. a pca $\mathcal{A}$ together with a subpca $\mathcal{A}_{\#}$ gives rise to a pca internal to $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$ from which one may construct the "nested realizability topos" $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ as described in [5, 16].[1] Within $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ there is a unique nontrivial subterminal object u giving rise to the *open* subtopos induced by the closure operator $u \to (-)$ and the complementary subtopos induced by the closure operator $u \vee (-)$ as described in [5].

Next we will give more elementary descriptions of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ and the above mentioned subtoposes.

1.1 The Nested Realizability Topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$

Let $\mathcal{A}$ be a pca whose partial application is denoted by juxtaposition and $\mathcal{A}_{\#}$ be a subpca of $\mathcal{A}$, i.e. $\mathcal{A}_{\#}$ is a subset of $\mathcal{A}$ closed under application and there are elements k and s of $\mathcal{A}_{\sharp}$ such that for all $x, y, z \in \mathcal{A}$ it holds that $kxy = x$, $sxyz \simeq xz(yz)$ and sxy is always defined. We write i for skk and $\bar{k}$ for ki which, obviously, satisfy the equations $ix = x$ and $\bar{k}xy = y$, respectively. We write p, p_0 and p_1 for elements of $\mathcal{A}$ such that px_0x_1 is always defined and $p_i(px_0x_1) = x_i$ for $i = 0, 1$. For every natural number n we write $\underline{n}$ for the corresponding numeral as defined in [16]. Notice that k, $\bar{k}$, p, p_0, p_1 and the numerals $\underline{n}$ are all elements of $\mathcal{A}_{\#}$.

Since subsets of $\mathcal{A}$ are the propositions of the realizability topos $\mathbf{RT}(\mathcal{A})$ it is useful to fix some notation for the propositional connectives

$$A \to B = \{a \in \mathcal{A} \mid ax \in B \text{ for all } x \in A\}$$

$$A \wedge B = \{pxy \mid x \in A, y \in B\}$$

$$A \vee B = (\{k\} \wedge A) \cup (\{\bar{k}\} \wedge B)$$

Propositions of the nested realizability topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ will be pairs $A = (A_p, A_a) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}_{\#})$ such that $A_a \subseteq A_p$ where we call A_p and A_a the set of *potential* and *actual* realizers, respectively. We write $\Sigma(\mathcal{A}, \mathcal{A}_{\#})$ for the set of these propositions. The above notation for propositional connectives is adapted to the current class of propositions as follows

$$A \to B = (A_p \to B_p, \mathcal{A}_{\#} \cap (A_p \to B_p) \cap (A_a \to B_a))$$

$$A \wedge B = (A_p \wedge B_p, A_a \wedge B_a)$$

$$A \vee B = (A_p \vee B_p, A_a \vee B_a)$$

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[1] In [5] they do not give a name to this topos and, moreover, write $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ for the relative realizability subtopos of the nested realizability topos.

For the realizability tripos $\mathscr{P}(\mathcal{A})$ induced by the pca $\mathcal{A}$ see [16]. The nested realizability tripos $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ over **Set** induced by the nested pca $\mathcal{A}_\# \subseteq \mathcal{A}$ is defined as follows. For a set I the fibre $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ is given by the set $\Sigma(\mathcal{A}, \mathcal{A}_\#)^I$ preordered by the relation $\vdash_I$ defined as

$$\phi \vdash_I \psi \qquad \text{if and only if} \qquad \bigcap_{i \in I}(\phi(i) \to \psi(i))_a \neq \emptyset$$

for $\phi, \psi \in \mathscr{P}(\mathcal{A}, \mathcal{A}_\#)(I)$. For $u : J \to I$ reindexing along u is given by precomposition with u and denoted as u^*. The fibres are preHeyting algebras where the propositional connectives are given by applying the operations $\to$, $\wedge$ and $\vee$ pointwise. It is easy to check that u^* commutes with the propositional connectives in the fibres. For a map $u : J \to I$, the reindexing u^* has left and right adjoints $\exists_u$ and $\forall_u$, respectively, given by

$$\exists_u(\phi)(i) = \left(\bigcup_{u(j)=i} \phi_p(j), \; \bigcup_{u(j)=i} \phi_a(j) \right)$$

$$\forall_u(\phi)(i) = \left(\bigcap_{j \in J}(Eq(u(j), i) \to \phi(j))_p, \; \bigcap_{j \in J}(Eq(u(j), i) \to \phi(j))_a \right)$$

where $Eq(x, y) = (\{a \in \mathcal{A} \mid x = y\}, \{a \in \mathcal{A}_\# \mid x = y\})$. It is straightforward to check that the so defined quantifiers satisfy the respective Beck-Chevalley conditions. The identity on $\Sigma(\mathcal{A}, \mathcal{A}_\#)$ gives rise to a generic family and, therefore, the fibered preorder $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ is actually a tripos in the sense of [7].

We write $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ for the ensuing topos.

1.2 Some Subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$

In $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ there is a nontrivial subterminal $u = (\mathcal{A}, \emptyset)$ giving rise to two complementary subtoposes induced by the closure operators $o_u(p) = u \to p$ and $c_u(p) = u \vee p$ as in [5]. We denote the open subtopos induced by o_u by $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$ and the complementary subtopos induced by c_u by $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$. In [5] these two subtoposes are referred to as the *relative* and the *modified relative realizability* topos, respectively.

For sake of concreteness and later reference in the following two subsections we give an elementary and explicit construction of triposes inducing $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$ and $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$, respectively.

1.2.1 The Relative Realizability Topos $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$

is induced by the tripos $\mathscr{P}_r(\mathcal{A}, \mathcal{A}_\#)$ over **Set** which we describe next. Let $\Sigma_r(\mathcal{A}, \mathcal{A}_\#) = \mathcal{P}(\mathcal{A})$. The fibre of $\mathscr{P}_r(\mathcal{A}, \mathcal{A}_\#)$ over I is given by the preorder $\left(\mathcal{P}(\mathcal{A})^I, \vdash_I^r\right)$ where

$$\phi \vdash_I^r \psi \qquad \text{if and only if} \qquad \mathcal{A}_\# \cap \bigcap_{i \in I}(\phi(i) \to \psi(i)) \neq \emptyset$$

and as usual reindexing is given by precomposition. At first sight this tripos looks like the tripos $\mathscr{P}(\mathcal{A})$ inducing the realizability topos $\mathbf{RT}(\mathcal{A})$ but notice that entailment in the fibres is defined in a more restrictive way, namely by requiring that the entailment be realized by an element of $\mathcal{A}_\#$ and not just an element of $\mathcal{A}$. Nevertheless, the propositional connectives, quantifiers and the generic family of $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$ can be constructed according to the same recipes as for $\mathscr{P}(\mathcal{A})$ (see [16]).

There is an obvious logical morphism from $\mathscr{P}_r(\mathcal{A}, \mathcal{A}_\#)$ to $\mathscr{P}(\mathcal{A})$ which is the identity on objects. But there is also an injective geometric morphism from $\mathscr{P}_r(\mathcal{A}, \mathcal{A}_\#)$ to $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ sending a family $\phi \in \mathcal{P}(\mathcal{A})^I$ to the family $\lambda i{:}I.(\phi(i), \mathcal{A}_\# \cap \phi(i))$. These morphisms between triposes over $\mathbf{Set}$ extend to morphisms between the associated toposes as described in [16].

1.2.2 The Modified Relative Realizability Topos $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$

is induced by the tripos $\mathscr{P}_m(\mathcal{A}, \mathcal{A}_\#)$ over $\mathbf{Set}$ which is obtained from $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ by restricting the fibre $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ to the set of all $\phi \in \Sigma(\mathcal{A}, \mathcal{A}_\#)^I$ with $\mathcal{A}_\# \cap \bigcap_{i \in I} \phi_p(i) \neq \emptyset$. The logical structure is essentially inherited from $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ though now and then one has to insert the closure operator c_U in order to stay within $\mathscr{P}_m(\mathcal{A}, \mathcal{A}_\#)$. A generic family for $\mathscr{P}_m(\mathcal{A}, \mathcal{A}_\#)$ is given by the identity on $\Sigma_m(\mathcal{A}, \mathcal{A}_\#) = \{u \vee p \mid p \in \Sigma(\mathcal{A}, \mathcal{A}_\#)\}$. The obvious inclusion of $\mathscr{P}_m(\mathcal{A}, \mathcal{A}_\#)$ into $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ gives rise to the inclusion of $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$ into $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$.

Notice that in case $\mathcal{A} = \mathcal{A}_\#$ we obtain the *modified realizability* topos as described in [15, 16] for the case where $\mathcal{A}$ is the first Kleene algebra $\mathcal{K}_1$.

Another prominent example of a modified relative realizability model can be found in a paper by J. R. Moschovakis [14] from 1971 where she constructed a model for a theory INT of Brouwerian intuitionism validating the proposition that all functions on natural numbers are not not recursive, i.e. that there are no non-recursive functions on the natural numbers. Of course, the model of [14] was not constructed in topos-theoretic terms but it is equivalent to the interpretation of the system considered in *loc.cit.* in the topos $\mathbf{RT}_m(\mathcal{K}_2, \mathcal{K}_2^{rec})$ where $\mathcal{K}_2$ is the second Kleene algebra whose underlying set is Baire space $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{K}_2^{rec})$ is the sub-pca of recursive sequences of natural numbers. The ensuing interpretation of INT was called *G-realizability* in *loc.cit.*

1.2.3 The Herbrand Realizability Topos

As shown by J. van Oosten, see Lemma 3.2 of [10], B. van den Berg's *Herbrand realizability topos* over a pca $\mathcal{A}$ arises as a subtopos of $\mathbf{RT}(\mathcal{A}, \mathcal{A})$ induced by some closure operator on $\mathscr{P}(\mathcal{A}, \mathcal{A})$. Moreover, as shown in *loc.cit.* it is disjoint from the open subtopos $\mathbf{RT}_r(\mathcal{A}, \mathcal{A})$ equivalent to $\mathbf{RT}(\mathcal{A})$.

1.3 Assemblies induced by $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$

As described in [16] for every tripos $\mathscr{P}$ (over $\mathbf{Set}$) one may consider the full subcategory $\mathbf{Asm}(\mathscr{P})$ of *assemblies* in $\mathbf{Set}(\mathscr{P})$, i.e. subobjects of objects of

the form $\Delta(S)$ where $S \in \mathbf{Set}$ and $\Delta : \mathbf{Set} \to \mathbf{Set}(\mathscr{P})$ is the *constant objects functor* sending a set S to $(S, \exists_{\delta_S}(\top_S))$.[2]

One can show that the category $\mathbf{Asm}(\mathscr{P}(\mathcal{A}, \mathcal{A}_\#))$ is equivalent to the category $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ whose objects are pairs $X = (|X|, E_X)$ where $|X|$ is a set and $E_X : |X| \to \Sigma(\mathcal{A}, \mathcal{A}_\#)$ with $E_X(x)_p \neq \emptyset$ for all $x \in |X|$. An arrow from X to Y is a function $f : |X| \to |Y|$ such that $E_X \vdash_{|X|} f^* E_Y$.

As follows from [16] Cor. 2.4.5 the topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ appears as the exact/regular completion of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$.

For further reference we note the following

Theorem 1.1 $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ *is a locally cartesian closed Heyting category with stable and disjoint finite sums with a generic monomorphism* $\top : Tr \rightarrowtail Prop$.[3]

Proof: The locally cartesian closed structure is constructed as in the case of $\mathbf{Asm}(\mathcal{A})$, i.e. assemblies within $\mathbf{RT}(\mathcal{A})$ where $\mathcal{A}$ is a pca. Similarly, one shows that $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ is a Heyting category and it has stable and disjoint finite sums.

Finally we exhibit a generic mono $\top : Tr \rightarrowtail Prop$. The object $Prop$ is defined as $\Delta(\Sigma(\mathcal{A}, \mathcal{A}_\#))$. The underlying set of Tr is the subset of $\Sigma(\mathcal{A}, \mathcal{A}_\#)$ consisting of those pairs $A = (A_p, A_a)$ where $A_p \neq \emptyset$ and $E_{Tr}(A) = A$. □

Notice, however, that in general $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ is not well-pointed.

2 Some Facts about Small Maps

A Heyting category is a regular category $\mathcal{C}$ where for all $f : Y \to X$ in $\mathcal{C}$ the pullback functor $f^{-1} : \mathsf{Sub}_\mathcal{C}(X) \to \mathsf{Sub}_\mathcal{C}(Y)$ has a right adjoint $\forall_f$. It is a Heyting pretopos iff, moreover, it has stable disjoint finite sums and every equivalence relation is effective (i.e. appears as kernel pair of its coequalizer).

Definition 2.1 *Let $\mathcal{C}$ be a locally cartesian Heyting category with stable and disjoint finite sums and a natural numbers object N. For a class $\mathcal{S}$ of maps in $\mathcal{C}$ we consider the following properties.*

(A0) *(Pullback Stability) For a pullback square*

$$\begin{array}{ccc} D & \xrightarrow{\ q\ } & B \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\ p\ ]{} & A \end{array}$$
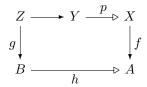
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[2]In [16] the constant objects functor is denoted by ∇ because in case of realizability triposes it is right adjoint to the global elements functor Γ. However, in case of triposes induced by a complete Heyting algebra the constant objects functor is left adjoint to Γ. However, there are also triposes where the constant objects functor is neither left nor right adjoint to Γ. We prefer the notation Δ since $eq_S = \exists_{\delta_S}(\top_S)$ is the (Lawvere) equality predicate on the set S in the sense of the tripos $\mathscr{P}$.

[3]"generic" means that all monos can be obtained as pullbacks of $\top : Tr \rightarrowtail Prop$ but we may have $f^*\top \cong g^*\top$ for different f and g
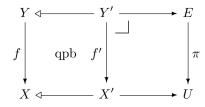
in $\mathcal{C}$ from $f \in \mathcal{S}$ it follows that $g \in \mathcal{S}$.

(A1) *(Descent) If in a pullback square as above p is a cover, i.e. a regular epimorphism, then $f \in \mathcal{S}$ whenever $g \in \mathcal{S}$.*

(A2) *(Sums) If f and g are in $\mathcal{S}$ then $f + g$ is in $\mathcal{S}$.*

(A3) *(Finiteness) The maps $0 \to 1$, $1 \to 1$ and $1 + 1 \to 1$ are in $\mathcal{S}$.*

(A4) *(Composition) Maps in $\mathcal{S}$ are closed under composition.*

(A5) *(Quotient) If $f \circ e$ is in $\mathcal{S}$ and e is a cover then f is in $\mathcal{S}$.*

(A6) *(Collection) Any arrows $p : Y \to X$ and $f : X \to A$ where p is a cover and $f \in \mathcal{S}$ fit into a quasipullback[4]*

$$
\begin{array}{ccccc}
Z & \longrightarrow & Y & \overset{p}{\Longrightarrow} & X \\
{\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\
B & & \longrightarrow & & A \\
& & {\scriptstyle h} & &
\end{array}
$$

where $g \in \mathcal{S}$ and h is a cover.

(A7) *(Representability) There is a universal family $\pi : E \to U$ in $\mathcal{S}$ such that every $f : Y \to X$ in $\mathcal{S}$ fits into a diagram*

$$
\begin{array}{ccccc}
Y & \longleftarrow & Y' & \longrightarrow & E \\
{\scriptstyle f}\downarrow & \text{qpb} \quad {\scriptstyle f'}\downarrow & \lrcorner & & \downarrow{\scriptstyle \pi} \\
X & \longleftarrow & X' & \longrightarrow & U
\end{array}
$$

where the left square is a quasipullback and the right square is a pullback.

(A8) *(Infinity) The terminal projection $N \to 1$ is in $\mathcal{S}$.*

(A9) *(Separation) All monomorphisms are in $\mathcal{S}$.*

*A class $\mathcal{S}$ of maps in $\mathcal{C}$ validating properties **(A0)**–**(A9)** is called a class of small maps.*

The following theorem will be essential later on.

Theorem 2.1 *Let $\mathcal{C}$ be a Heyting category with stable and disjoint finite sums and $\mathcal{S}$ be a class of small maps in $\mathcal{C}$. Let $\bar{\mathcal{C}}$ be the exact/regular completion of $\mathcal{C}$ and $\bar{\mathcal{S}}$ the class of maps f in $\bar{\mathcal{C}}$ which fit into a quasipullback*

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
\cdot & \longrightarrow & \cdot
\end{array}
$$

---

[4]A square is a quasipullback if the mediating arrow to the pullback square is a cover.

with g in the subcategory $\mathcal{C}$ of $\bar{\mathcal{C}}$.

    Then $\bar{\mathcal{S}}$ is a class of small maps within the Heyting pretopos $\bar{\mathcal{C}}$.

Proof: That $\bar{\mathcal{S}}$ validates conditions **(A0)**–**(A8)** follows from Lemma 5.8 and Propositions 6.2 and 6.21 in [2].

    Condition **(A9)** holds for $\bar{\mathcal{S}}$ in $\bar{\mathcal{C}}$ for the following reason. Let $m : B \rightarrowtail A$ be a mono in $\bar{\mathcal{C}}$. Since $\bar{\mathcal{C}}$ is the exact completion of $\mathcal{C}$ there is a cover $p : X \twoheadrightarrow A$ with X in $\mathcal{C}$. Then for the pullback

$$\begin{array}{ccc} Y & \xrightarrow{\ q\ } & B \\ {\scriptstyle n}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ X & \xrightarrow[\ p\ ]{} & A \end{array}$$

in $\bar{\mathcal{C}}$ we know that q is a cover and n is a mono. It follows from Lemma 2.4.4 of [16] that Y is isomorphic to an object in $\mathcal{C}$. $\qquad\square$

3   Small Maps in Nested Realizability Toposes

We will first identify within $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ a class $\mathcal{S}$ of small maps so that we can apply Theorem 2.1 to it in order to obtain a class $\bar{\mathcal{S}}$ of small maps on $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ which is known to arise as the exact/regular completion of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ (see section 2.4 of [16] for more details).

    However, for showing that $\bar{\mathcal{S}}$ is closed under power types we have to appeal to Lemma 27 of [3] guaranteeing that if $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ has *weak* power types under which $\mathcal{S}$ is closed then $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ has power objects under which $\bar{\mathcal{S}}$ is closed.

3.1   Small maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$

For constructing a class of small maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ let us first choose a *strongly inaccessible* cardinal κ exceeding the cardinality of $\mathcal{A}$.

Theorem 3.1 *Let $\mathcal{S}$ be the class of all maps $f : Y \to X$ in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ such that $\mathsf{card}\left(f^{-1}(x)\right) < \kappa$ for all $x \in |X|$. Then $\mathcal{S}$ is a class of small maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ in the sense of Def. 2.1.*

Proof: Conditions **(A0)** and **(A1)** follow from the fact that the forgetful functor from $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ to **Set** preserves finite limits and covers.

    Since the forgetful functor from $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ to **Set** preserves finite sums condition **(A2)** holds.

    Since κ is infinite all maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ with finite fibres are in $\mathcal{S}$. For this reason **(A3)** and **(A9)** trivially hold.

    Condition **(A4)** holds since κ is regular.

For **(A5)** suppose $f \circ e$ is in $\mathcal{S}$ and e is a cover. Then the fibres of f have cardinalities $< \kappa$ since by assumption the fibres of $f \circ e$ have cardinalities $< \kappa$ and the underlying map of e is onto.
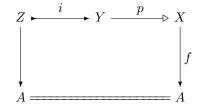
Condition **(A8)** holds since κ exceeds the cardinality of $\mathbb{N}$.

For showing that **(A6)** holds suppose $p : Y \to X$ is a cover and $f : X \to A$ is in $\mathcal{S}$. Since p is a cover the underlying map of p (also denoted by p) is onto and there exists $a \in \mathcal{A}_{\#}$ such that for all $x \in |X|$ it holds that

(1p) if $b \in E_X(x)_p$ then $ab\downarrow$ and $ab \in E_Y(y_{x,b})$ for some $y_{x,b} \in p^{-1}(x)$ and

(1a) if $b \in E_X(x)_a$ then $ab \in E_Y(y_{x,b})_a$.

Let Z be the object of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ whose underlying set $|Z| = \{y_{x,b} \mid x \in |X|, b \in E_X(x)_p\}$ and $E_Z(y) = E_Y(y)$ for $y \in |Z|$. Let $i : Z \hookrightarrow Y$ be the obvious inclusion of Z into Y. Then the rectangle

$$
\begin{array}{ccccc}
Z & \xrightarrow{\;\;i\;\;} & Y & \xrightarrow{\;\;p\;\;} & X \\
\downarrow & & & & \downarrow{\scriptstyle f} \\
A & = \!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & & & A
\end{array}
$$

is a quasipullback since $p \circ i$ is a cover. Since the fibres of $p \circ i$ have cardinality $\leq \mathsf{card}(\mathcal{A}) < \kappa$ the map $p \circ i$ is in $\mathcal{S}$. Thus, by **(A4)** the map $f \circ p \circ i : Z \to A$ is in $\mathcal{S}$, too.

Condition **(A7)** holds in a very strong sense because we can exhibit a generic map $\pi : E \to U$ in $\mathcal{S}$, i.e. $\pi \in \mathcal{S}$ and all maps in $\mathcal{S}$ can be obtained as pullbacks of the generic map π. The codomain U of π is given by
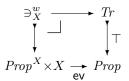
$$\Delta\left(\{X \in \mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#}) \mid |X| \subseteq \kappa, \mathsf{card}(|X|) < \kappa\}\right)$$

and its domain E has underlying set

$$|E| = \{(X, x) \mid X \in |U|, x \in |X|\}$$

and whose existence predicate is given by $E_E(X, x) = E_X(x)$. The map $\pi : E \to U$ is given by projection on the first component, i.e. $\pi(X, x) = X$. Obviously, the map π has fibres of cardinality $< \kappa$ and we leave it as a straighforward exercise for the reader to show that every map in $\mathcal{S}$ can actually be obtained as pullback of π. $\qquad\square$

It is easy to check that the class $\mathcal{S}$ in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ is closed under dependent products, i.e. $\Pi_f g \in \mathcal{S}$ whenever f and g are in $\mathcal{S}$. As a consequence for $a : A \to I$ and $b : B \to I$ in $\mathcal{S}$ their exponential in the fibre over I, i.e. $a \to_I b = \Pi_a a^* b$, is in $\mathcal{S}$, too. Moreover, the generic mono $\top : Tr \rightarrowtail Prop$ constructed in Theorem 1.1 like all monos is also an element of $\mathcal{S}$. Moreover, the terminal projection $Prop \to 1$ is in $\mathcal{S}$, too, since the underlying set of $Prop$ has cardinality $< \kappa$. Accordingly, the object Tr is small, too.

For every object X in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ we may construct a *weak power object* $\exists_X^w \rightarrowtail Prop^X \times X$ as follows

$$
\begin{array}{ccc}
\exists_X^w & \longrightarrow & Tr \\
\downarrow & \lrcorner & \downarrow {\scriptstyle \top} \\
Prop^X \times X & \underset{\mathsf{ev}}{\longrightarrow} & Prop
\end{array}
$$

where $\mathsf{ev} : Prop^X \times X \to Prop$ is the evaluation map. If X is small, i.e. $X \to 1$ is in $\mathcal{S}$, i.e. $\mathsf{card}(X) < \kappa$, then $Prop^X$ is small, too, since $\mathsf{card}\left(Prop^X\right) \le \mathsf{card}(Prop)^{\mathsf{card}(X)} < \kappa$ because κ is inaccessible and $\mathsf{card}(Prop), \mathsf{card}(X) < \kappa$. Notice that this construction of weak power objects also works in all slices.

For future reference we summarize these considerations in the following

Theorem 3.2 *The category $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$ has weak power objects and $\mathcal{S}$ is closed under weak power objects.*

3.2 Small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$

It is well known from [16] (section 2.4) that $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ is the exact/regular completion of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_{\#})$. Let $\bar{\mathcal{S}}$ be the class of maps defined in Theorem 2.1. Now we can show easily that

Theorem 3.3 $\bar{\mathcal{S}}$ *is a class of small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ which is also closed under power objects and thus also under exponentiation.*

Proof: It is an immediate consequence of Theorem 2.1 and Theorem 3.1 that $\bar{\mathcal{S}}$ is a class of small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$. From Lemma 27 of [3] and our Theorem 3.2 it follows that $\bar{\mathcal{S}}$ is also closed under power objects. It is well known that closure under powerobjects and subobjects entails closure under exponentiation. $\square$

As pointed out by J. van Oosten in private communication there is a logical functor $F : \mathbf{RT}(\mathcal{A}, \mathcal{A}_{\sharp}) \to \mathbf{RT}(\mathcal{A})$ which just "forgets the actual realizers". Already in [11] there has been identified for every strongly inaccessible cardinal a class of small maps in $\mathbf{RT}(\mathcal{A})$ from which our class of small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ can be obtained as inverse image under F.

3.3 A Model of IZF in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$

It follows from the previous Theorem 3.3 and Theorem 5.6 of [11] that the class $\bar{\mathcal{S}}$ of small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ gives rise to an "initial ZF-algebra" within $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$. Accordingly, the nested realizability topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#})$ hosts a model of IZF.

It is an open question (raised by J. van Oosten) whether the above mentioned logical functor $F : \mathbf{RT}(\mathcal{A}, \mathcal{A}_{\#}) \to \mathbf{RT}(\mathcal{A})$ preserves the initial ZF-algebras arising from the respective classes of small maps.

4 Small Maps for Subtoposes of $\mathrm{RT}(\mathcal{A}, \mathcal{A}_\#)$

In the previous section we have endowed the nested realizability topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ with a class $\bar{\mathcal{S}}$ of small maps in such a way that it gives rise to a model of IZF in the sense of Algebraic Set Theory as described in [11]. In this section we show how to extend this result to subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$.

4.1 Transferring Classes of Small Maps to Subtoposes

Let $\mathcal{E}$ be an elementary topos and $\mathcal{S}$ a class of small maps in $\mathcal{E}$. Let $\mathsf{a} \dashv \mathsf{i} : \mathcal{F} \hookrightarrow \mathcal{E}$ be a subtopos of $\mathcal{E}$. W.l.o.g. we assume that $\mathcal{F}$ is closed under isomorphisms in $\mathcal{E}$ and that $\mathsf{a}f = f$ for $f \in \mathcal{F}$. We want to endow $\mathcal{F}$ with a class $\mathcal{S}_\mathcal{F}$ of small maps such that $\mathsf{a} : \mathcal{E} \to \mathcal{F}$ sends $\mathcal{S}$ to $\mathcal{S}_\mathcal{F}$. Thus, it is tempting to define $\mathcal{S}_\mathcal{F}$ as $\mathsf{a}\mathcal{F}$ by which we denote the closure under isomorphism in $\mathcal{F}$ of the image of a. But then there are problems with condition **(A5)** because epimorphisms in $\mathcal{F}$ need not be epic in $\mathcal{E}$. In order to overcome this problem we define $\mathcal{S}_\mathcal{F}$ as follows

Definition 4.1 *Let $\mathcal{S}_\mathcal{F}$ be the class of all maps $f : B \to A$ in $\mathcal{F}$ fitting into a quasipullback*

$$
\begin{array}{ccc}
\mathsf{a}Y & \longrightarrow & B \\
{\scriptstyle \mathsf{a}g}\downarrow & \text{qpb} & \downarrow{\scriptstyle f} \\
\mathsf{a}X & \underset{e}{\longrightarrow\!\!\!\!\!\!\rhd} & A
\end{array}
$$

in $\mathcal{F}$ for some $g : Y \to X$ in $\mathcal{S}$, i.e. $e^* f$ is a quotient of some $\mathsf{a}g$ in $\mathcal{F}/\mathsf{a}X$.

The following little observation will be used later on.

Lemma 4.1 *The epis in $\mathcal{F}$ are precisely the sheafifications of epis in $\mathcal{E}$.*

Proof: First recall that epis in toposes are regular. Thus, since a is a left adjoint it preserves regular epis. For the converse direction suppose e is an epi in $\mathcal{F}$. Consider its factorization $e = m \circ p$ in $\mathcal{E}$ where m is monic and p is an epi in $\mathcal{E}$. Then $e = \mathsf{a}(m \circ p) = \mathsf{a}m \circ \mathsf{a}p$ in $\mathcal{F}$. Since a preserves monos and epis and e is epic in $\mathcal{F}$ it follows that $\mathsf{a}m$ is an iso. $\qquad\square$

Now we are ready to prove the main theorem of this subsection.

Theorem 4.1 *Suppose $\mathcal{E}$ is a topos with a natural numbers object N and $\mathcal{S}$ is a class of small maps in $\mathcal{E}$ closed under power objects. If $\mathsf{a} \dashv \mathsf{i} : \mathcal{F} \hookrightarrow \mathcal{E}$ is a subtopos then $\mathcal{S}_\mathcal{F}$ as specified in Def. 4.1 is a class of small maps in $\mathcal{F}$ which is closed under power objects.*

Proof: We will often (implicitly) use the fact that pullbacks in $\mathcal{F}$ preserve epis and maps in $\mathsf{a}\mathcal{S}$.

This ensures for example that quasipullbacks of the form as considered in Def. 4.1 are preserved by pullbacks along morphisms in $\mathcal{F}$. Accordingly, it follows that $\mathcal{S}_\mathcal{F}$ is closed under pullbacks in $\mathcal{F}$, i.e. validates condition **(A0)**.
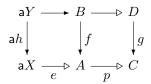
For showing that $\mathcal{S}_\mathcal{F}$ validates **(A1)** suppose that

$$
\begin{array}{ccc}
B & \longrightarrow & D \\
f\downarrow & \lrcorner & \downarrow g \\
A & \xrightarrow{\ p\ } & C
\end{array}
$$

is a pullback in $\mathcal{F}$ where f is in $\mathcal{S}_\mathcal{F}$ and p is a cover in $\mathcal{F}$. Since f is in $\mathcal{S}_\mathcal{F}$ it fits into a quasipullback

$$
\begin{array}{ccc}
\mathsf{a}Y & \longrightarrow & B \\
\mathsf{a}h\downarrow & \mathrm{qpb} & \downarrow f \\
\mathsf{a}X & \xrightarrow{\ e\ } & A
\end{array}
$$

where h is in $\mathcal{S}$ and e is a cover in $\mathcal{F}$. Since quasipullbacks are closed under composition it follows that

$$
\begin{array}{ccccc}
\mathsf{a}Y & \longrightarrow & B & \longrightarrow & D \\
\mathsf{a}h\downarrow & & \downarrow f & & \downarrow g \\
\mathsf{a}X & \xrightarrow{\ e\ } & A & \xrightarrow{\ p\ } & C
\end{array}
$$

is a quasipullback. Thus, since $p \circ e$ is epic, it follows that g is in $\mathcal{S}_\mathcal{F}$ as desired.

That $\mathcal{S}_\mathcal{F}$ validates condition **(A2)** is immediate from the facts that condition **(A2)** holds for $\mathcal{S}$, that a preserves $+$ and that $+$ preserves quasipullbacks.

That $\mathcal{S}_\mathcal{F}$ validates condition **(A3)** is immediate from the fact that that a preserves colimits and finite limits.

That $\mathcal{S}_\mathcal{F}$ validates **(A4)**, i.e. that $\mathcal{S}_\mathcal{F}$ is closed under composition, can be shown by adapting the proof of the analogous Lemma 2.15 of [2].

Obviously, $\mathcal{S}_\mathcal{F}$ validates condition **(A5)** by its very definition since quasipullbacks are closed under horizontal composition.

The proof that $\mathcal{S}_\mathcal{F}$ validates condition **(A6)** is analogous to the proof of case **(A7)** of Proposition 2.14 of [2].

It is easy to check that **(A7)** holds for $\mathcal{S}_\mathcal{F}$. Let π be a universal family for $\mathcal{S}$ then its sheafification $\mathsf{a}\pi$ is universal for $\mathcal{S}_\mathcal{F}$ which can be seen by applying a to the respective diagram in the formulation of **(A7)** and using the fact that quasipullbacks are closed under horizontal composition.

Condition **(A8)** holds for $\mathcal{S}_\mathcal{F}$ since sheafification preserves natural numbers objects.

Condition **(A9)** holds for $\mathcal{S}_\mathcal{F}$ since if m is a mono in $\mathcal{F}$ then it is also a mono in $\mathcal{E}$ and thus by **(A9)** for $\mathcal{S}$ we have $m \cong \mathsf{a}m$ is in $\mathcal{S}_\mathcal{F}$.

For showing that $\mathcal{S}_\mathcal{F}$ is closed under power objects one may adapt the proof of Proposition 6.6 from [2] proving an analogous result. □

4.2 Small Maps in Subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$

As a consequence of Theorem 4.1 we obtain the following result.

Theorem 4.2 *Let $\mathcal{S}$ be the class of small maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ as introduced in Theorem 3.1 and $\bar{\mathcal{S}}$ be the class of small maps in $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ as introduced in Theorem 2.1. Suppose $\mathsf{a} \dashv \mathsf{i} : \mathcal{E} \hookrightarrow \mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ is a subtopos of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ induced by a closure operator j on $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$. Then $\bar{\mathcal{S}}_\mathcal{E}$ as introduced in Theorem 4.1 is a class of small maps in $\mathcal{E}$ closed under power objects and exponentiation.*

Proof: From Theorem 3.3 we know that $\bar{\mathcal{S}}$ is a class of small maps closed under power objects. Thus, we can apply Theorem 4.1 from which it follows that $\bar{\mathcal{S}}_\mathcal{E}$ is a class of small maps in $\mathcal{E}$ which is closed under power objects and, accordingly, also under exponentiation. □

This result applies in particular to the subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ as considered in subsection 1.2 and thus covers most of the examples considered in van Oosten's book [16].

4.3 Models of IZF in Subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$

From the main result of [11] and our Theorem 4.2 it follows that most of the toposes considered in [16] host models of IZF.

Theorem 4.3 *There exist internal models for IZF in subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ induced by local operators on $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$.*

In case $\mathcal{A} = \mathcal{A}_\#$ due to [12] we reobtain the realizability model for IZF as initially introduced by H. Friedman in [6], G. Rosolini in [19] and D. C. McCarty in [13].

In case $\mathcal{A} = \mathcal{A}_\# = \mathcal{K}_1$, the first Kleene algebra (corresponding to number realizability), from Theorem 4.3 it follows that the modified realizability topos $\mathbf{Mod} = \mathbf{Mod}(\mathcal{K}_1) = \mathbf{RT}_m(\mathcal{K}_1, \mathcal{K}_1)$ from [15] hosts a model of IZF. Thus, in IZF one cannot derive Markov's Principle from Church's Thesis.

5 Conclusion

Relying on the main result of [11] we have shown that relative realizability toposes and modified relative realizability toposes host models of IZF. In the unnested case, i.e. $\mathcal{A} = \mathcal{A}_\#$ we reobtain the well known realizability models for IZF and a modified realizability model for IZF which to our knowledge cannot be found in the existing literature. Moreover, as pointed out to us by B. van den

Berg our results also show that his recent Herbrand Realizability topos hosts a model of IZF.

We have obtained these new models for IZF in a quite uniform way using the methods of Algebraic Set Theory. Of course, one could define in each single case these models of IZF in a much more traditional and direct way. Using an appropriate adaptation of the results in [12] one can presumably show that these "hand made" models are equivalent to the ones we have obtained in this paper by more abstract and general means.

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