A Note on "Extensional PERs"

W. P. Stekelenburg

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Abstract

In the paper "Extensional PERs" by P. Freyd, P. Mulry, G. Rosolini and D. Scott, a category $\mathcal{C}$ of "pointed complete extensional PERs" and computable maps is introduced to provide an instance of an *algebraically compact category* relative to a restricted class of functors. Algebraic compactness is a synthetic condition on a category which ensures solutions of recursive equations involving endofunctors of the category. We extend that result to include all internal functors on $\mathcal{C}$ when $\mathcal{C}$ is viewed as a full internal category of the effective topos. This is done using two general results: one about internal functors in general, and one about internal functors in the effective topos.

The paper "Extensional PERs" by P.Freyd, P.Mulry, G.Rosolini and D.Scott [2] identifies a reflective subcategory of the category of PERs, namely the category $\mathcal{C}$ of *pointed CEPERs* – complete extensional partial equivalence relations, implicitly over $\mathbb{N}$ – which is *algebraically compact*. Algebraic compactness ensures the existence of solutions to recursive domain equations. In fact, for any functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ there is an object X with an invertible arrow $F(X; X) \to X$ (see [1]).

One restriction limits the compactness of $\mathcal{C}$: the functor F has to be *realizable*. The category of PERs and its subcategory of pointed CEPERs exist inside the effective topos as internal categories. Any internal functor between these categories comes with a realizer for its functorial properties. Hence the name 'realizable functor'. Unfortunately, the definition in [2] seems more restrictive. It demands that there is a realizer of the functor preserving an index of the identity function. We are not convinced that all internal functors satisfy that property, but the algebraic compactness proof given depends on it.

In the research for my master thesis I found two ways to bypass this problem. Firstly, weakly complete internal categories, like the category of PERs and the category of pointed CEPERs, already satisfy the weaker property of algebraic completeness. We can derive algebraic compactness from a combination of algebraic completeness with other properties of the category of pointed CEPERs. Secondly, any internal functor is isomorphic to some other internal functor with a realizer that *does* preserve indices of identity. Therefore, we can add the requirement without loss of generality.

1 The Category of PERs: Notation

In our discussion of PERs I will employ a more common notation and terminology than in [2].

Definition 1.1. A *PER* is a *partial equivalence relation* on the natural numbers. Spelled out, a PER R is a subset of $\mathbb{N}^2$ such that:

- for all $(n, m) \in R$, $(m, n) \in R$ (*symmetry*)
- for all (n, m) and $(m, p) \in R$, $(n, p) \in R$ (*transitivity*)

Any PER R forms a total equivalence relation on its *domain*:

$$\mathrm{dom}R := \{n|(n,n) \in R\}$$

The quotients $\mathrm{dom}R/R$ are used to define morphisms between PERs. Given $n \in \mathrm{dom}R$, we use $[n]_R$ to denote the equivalence class containing n in $\mathrm{dom}R/R$.

Definition 1.2. A *morphism of PERs* $f : R \to S$, is a function $f : \mathrm{dom}R/R \to \mathrm{dom}S/S$, which is *tracked* by a partial recursive function. This means that there is a partial recursive function ϕ such that for all $n \in \mathrm{dom}R$ ϕn is defined and $f([n]_R) = [\phi n]_S$.

These objects and morphisms form the category of PERs $\mathcal{P}$. This category is cartesian closed: by letting $(n,m) \in [R \to S]$ if n and m are indices of partial recursive functions that track the same function $f : R \to S$, we get a PER, which acts as an internal homset. So any $f : R \to S$ can be identified with the set of those natural numbers that are indices of tracking functions of f. Therefore, I will sometimes use $[n]_{R\to S}$ to refer to the function $R \to S$ that is tracked by the n-th partial recursive function. Finally, I write the application of the n-th partial recursive function to some number m as a simple juxtaposition: nm.

2 Realizable and Monotone Functors

Inspired by the idea that realizable functors are internal functors of the effective topos, we define these as follows.

Definition 2.1. An endofunctor F of the category of PERs is realizable if there is a single partial recursive function ϕ that tracks F. This means: ϕx converges whenever $x \in \mathrm{dom}([R \to S])$ for every pair of PERs R and S, and

$$F([x]_{R\to S}) = [\phi x]_{FR\to FS} \tag{1}$$

The definition is slightly more general than the definition found in [2]. There, ϕ has to preserve an index of the identity map of $\mathbb{N}$, while we do not even require that ϕ maps indices of identity to each other here. Because F as functor has to preserve identities, we know that for any PER R and any index i of the identity function:

$$F([i]_{R\to R}) = [\phi i]_{FR\to FR} = [i]_{FR\to FR}$$

Therefore $i \in \bigcap_R F([i]_{R\to R})$ *does* hold, for any particular realizable functor. This still doesn't guarantee that $\phi i = i'$ for some other index i' of the identity function, however.

Let $\psi i = i$ and $\psi x = \phi x$ if $x \neq i$. ψ is a recursive function, and one might wonder if it can take the place of ϕ, saving the original definition. (1) is satisfied when $x \neq i$ and in the case that $S = R$, the same equation holds when $x = i$. The difficult case is $x = i$ and $S \neq R$. Note that $[i] : R \to S$ iff $R \subseteq S$. Therefore, if $R \subseteq S$ and if $F([i]_{R\to S}) = [\psi i]_{FR\to FS} = [i]_{FR\to FS}$, then $FR \subseteq FS$. This means that all functors which are tracked by an i preserving function are monotone mappings of PERs. On the other hand for any monotone functor tracked by ϕ, the function ψ defined above is another tracking function of the functor that preserves i. So the functors defined in [2] are a special kind of realizable functor.

Definition 2.2. A realizable endofunctor F of the category of PERs is *monotone*, if its object map is monotone with respect to the inclusion ordering on PERs. In other words, if $R \subseteq S$, then $FR \subseteq FS$.

I could not prove (or refute, by the way) that all realizable functors are monotone, or find a proof in the literature. Sadly, in [2] the least fixpoints that monotone functors have, are used in the algebraic compactness proof: for any monotone functor F we have a fixpoint $X := \bigcap\{R|FR \subseteq R\}$, where $FX = X$.

3 Algebraic Completeness

The category of pointed CEPERs is an internal CPO category of the effective topos, and with the theory developed in [3], we can prove that it is algebraically compact *if* it is algebraically complete. The following lemma concerns the algebraic completeness of internal categories.

Lemma 3.1. *For any topos $\mathcal{E}$, and any* weakly *complete internal category $\mathcal{C}$ in $\mathcal{E}$, $\mathcal{C}$ is algebraically complete: for any internal endofunctor F, there is an initial algebra.*

Weak completeness of a category $\mathcal{C}$ means that for arbitrary internal categories $\mathcal{D}$ and each internal functor $\mathcal{D} \to \mathcal{C}$ a limiting cone exists, but that the functor $\mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ that maps objects to constant functors, has no internal right adjoint.

Proof of lemma 3.1. $\mathcal{E}$ allows the construction of the category of algebras of any endofunctor F of $\mathcal{C}$ internally, so both the category of F-algebras F-alg, and the underlying object functor $U : F\text{-alg} \to \mathcal{C}$ are internal to $\mathcal{E}$. This underlying functor creates limits, and since $\mathcal{C}$ is weakly complete (relative to $\mathcal{E}$), F-alg must be weakly complete too. Therefore it has an initial object – the limit of the identity functor on F-alg – and this object is an initial algebra for F. $\square$

The category of CEPERs is weakly complete as an internal category of the effective topos. It inherits that property from the category of PERs (see [10] and [11]), of which it is a reflective subcategory (see [2], the proof doesn't depend on monotonicity of endofunctors). We may conclude that the category of pointed CEPERs is algebraically compact indeed.

3.1 Constructing a Fixmap

For an endofunctor F of a category $\mathcal{C}$, an F-algebra is a pair $(R, a : FR \to R)$, where $a : FR \to R$ is the *structure map* of the algebra. A morphism of F-algebras is a morphism of $\mathcal{C}$ that commutes with structure maps. The structure map of an initial algebra is necessarily an isomorphism. Because of the similarity with fixpoints of monotone endofunctions on a complete poset, I propose that we call the underlying objects of initial algebras *fixobjects*, and the structure maps *fixmaps*. For realizable endofunctors F of the category of pointed CEPERs $\mathcal{C}$, initial algebras always exists, because of combinatory completeness. We can even give a construction for such a fixmap.

An intial algebra is a limit of the identity functor, and this limit is a pair (R_0, a_0) where R_0 is a limit of the underlying PER functor $U : F\text{-alg} \to \mathcal{C}$, and where a_0 is the unique structure map that commutes with the limiting cone. As any limit, R_0 can be constructed as a regular subobject of a product. In $\mathcal{C}$ we can use subsets of PERs to represent regular subobjects. The product we need is an internal product of all PERs over all of the F-algebras. Therefore, given any such product $\prod U$, we can assume: $R_0 \subseteq \prod U$.

If we fix a PER R, then $[FR \to R]$ is a PER of all algebras based on R. Every element $x \in \mathrm{dom}(\prod U)$ restricts to a mapping $x_R : [FR \to R] \to R$:

$$x_R(a) = x_{(R,a)}$$

This is a morphism of PERs, because the category of PERs is a full subcategory of the effective topos. As a consequence x_R itself is an element of the PER $[[FR \to R] \to R]$.

The object of pointed CEPERs $\mathcal{C}_0$ exists in the effective topos, and is uniform. Among other things, this means that any arrow $\mathcal{C}_0 \to N$ is constant. That makes

$\bigcap_{R\in\mathcal{C}_0}[[FR \to R] \to R]$, the intersection of this family of PERs, already its product inside the category of pointed CEPERs (see [4]). Therefore, some limit R_0 of U satisfies:

$$R_0 \subseteq \bigcap_{R\in\mathcal{P}} [[FR \to R] \to R] \tag{2}$$

To find R_0, we only need to select those elements of $\bigcap_{R\in\mathcal{C}_0}[[FR \to R] \to R]$ that commute with all the algebra morphisms. The results in the paper [6] seem to suggest that $R_0 = \bigcap_{R\in\mathcal{P}}[[FR \to R] \to R]$. But in any case, $(f, f') \in R_0$ if and only if for any three algebras (R, a), (S, b) and (T, c) and any pair of morphisms $m : (R, a) \to (T, c)$ and $m' : (S, b) \to (T, c)$, $(m(fa), m('f'b)) \in T$:

$$\begin{tikzcd} R_0 \arrow[r, "f\mapsto fa"] \arrow[d, "f'\mapsto f'b"'] & R \arrow[d, "m"] \\ S \arrow[r, "m'"'] & T \end{tikzcd}$$

Note that I write fa for $f_R(a)$: $f_R(a)$ is constant in R because $\mathcal{C}_0$ is uniform. The projection maps $\pi_{(R,a)}f = fa$ taken together form the limiting cone. Obviously, any structure map a_0 on R_0 has to make the following diagram commute for any algebra (R, a):

$$\begin{tikzcd} FR_0 \arrow[r, "F\pi_a"] \arrow[d, "a_0"] & FR \arrow[d, "a"] \\ R_0 \arrow[r, "\pi_a"] & R \end{tikzcd}$$

That means that for all $(x, y) \in FR_0$, $(a_0xa, a(F\pi_a y)) \in R$. Because R is reflexive, we can let a_0 be any partial recursive function that satisfies $a_0xa = a(\phi\pi_a x)$, where ϕ is some partial recursive function tracking F. The inverse of the initial algebra is a terminal coalgebra of F, since the category of pointed CEPERs is algebraically compact.

This construction shows we can define a functor $\mathcal{C}^{\mathcal{C}} \to \mathcal{C}$ that maps each realizable endofunctors to one of its fixobjects. The existence of such a functor does not follow from weak completeness: it is a peculiar property of the category of pointed CEPERs as an internal category of the effective topos. We need this functor to prove the algebraic completeness of products of $\mathcal{C}$ and $\mathcal{C}^{op}$, and in turn the existence of fixobjects and fixmaps for functors $(\mathcal{C}^{op})^m \times \mathcal{C}^n \to \mathcal{C}$ (for arbitrary m and n). All of this is done in [1].

4 Yoneda

Before we can apply the Yoneda lemma to realizable functors, we need to define what the realizable natural transformations between them are.

Definition 4.1. A natural transformation η between two realizable endofunctors F and G of the category of PERs is realizable if there is a single number e such that $\eta_R = [e]_{FR\to GR}$ for all PERs R. Let $[F \Rightarrow G]$ be the PER of natural transformations F to G: the set of pairs (n, m) where n and m are indices for the same transformation.

Again, realizability makes the transformations internal to the effective topos. The definition given in [2] is correct in this case.

Because natural transformations are represented by natural numbers – or because the category of PERs is weakly complete and internal to the effective topos: it all depends on your perspective – we can construct a PER of natural transformations between any pair of PER valued functors. In fact, categories of PER valued functors are enriched over the category of PERs, as long as the domains are internal categories of the effective topos.

Theorem 4.2. *Every endofunctor of $\mathcal{P}$ is naturally isomorphic to a monotone endofunctor.*

Proof. We know because of Yoneda's lemma that $FX \cong \mathrm{nat}(\mathrm{hom}(X, -), F)$ naturally in both F and X. Given a PER R let R_* be the functor that maps any PER S to $[R \to S]$ and let F_* be the functor that maps S to $[S_* \Rightarrow F]$. F_* happens to be monotone.

If $X \subseteq Y$ and $(n, m) \in [Y \to Z]$, then $(n, m) \in [X \to Z]$ because $(nx, my) \in Z$ whenever $(x, y) \in Y$ and $(x, y) \in Y$ whenever $(x, y) \in X$. Therefore $Y_* \subseteq X_*$ point wise. Furthermore, if i is an index of the identity function, it determines a natural transformation: $(i, i) \in [Y_* \Rightarrow X_*]$.

Let $(i, i) \in [G \Rightarrow G']$ for any two functors G and G', and let $(n, m) \in [G' \Rightarrow F]$. $(n, m) \in [G \Rightarrow F]$, because $n \circ i$ and $m \circ i$ represent the same partial recursive function as n and m. Therefore $[G' \Rightarrow F] \subseteq [G \Rightarrow F]$.

We can see that if $X \subseteq Y$, then $(i, i) \in [Y_* \Rightarrow X_*]$ and therefore $F_*X \subseteq F_*Y$. Consequently, F_* is a monotone functor. □

Although there may be non-monotone internal functors, there is no loss of generality if we assume that realizable functors are. With this information added the original proof suffices to show that the category of pointed complete extensional PERs is indeed algebraically compact.

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