Completions in Realizability

Completeringen in de Realizeerbaarheid

(met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

Introduction

The subject matter of this thesis is a systematic study of a certain class of toposes, called *realizability toposes*, and the various ways of constructing them.

In this introduction, I will first give some background concerning topos theory (section 1) and categorical logic (section 2). Then, in section 3, realizability is introduced. The category-theoretic approach to realizability will be explained in section 4; in particular, I will introduce the most widely known example of a realizability topos, namely the Effective Topos. Section 5 is concerned with presentations of realizability toposes; such presentations are often formulated in terms of free constructions, aiming at a description of a realizability topos as the free completion of some simpler category. Finally, in section 6, I will explain what my contribution to the subject is, and I will provide a motivation for the research that this thesis is a report of.

1.1 Topos theory

The origin of topos theory lies in algebraic geometry, and in sheaf theory in particular. It was Grothendieck who realized that one could generalize the notion of a sheaf on a topological space by replacing the lattice of open sets of the space by an arbitrary category, and the notion of an open cover by that of a Grothendieck topology. Given a small category and a Grothendieck topology on it (this is called a *site*), one can still define the notion of a sheaf, and the category of sheaves is then called a *Grothendieck topos.* Because of their close resemblance to categories of sheaves over a space, one often encounters the phrase that Grothendieck toposes are "generalized spaces". Indeed, many properties of spaces, such as (local) connectedness, compactness, or discreteness, can be directly generalized to arbitrary Grothendieck toposes.

In the beginning of the 1970's, Lawvere discovered that all Grothendieck toposes share some important properties with the category of sets, and this led him to the definition of an *elementary topos*: this is a category with finite limits, exponentials and a subobject classifier. I will not explain the first two conditions (see any textbook on category theory, such as [51]), but let me give a definition of the last. A *subobject classifier* in a category $\mathcal{C}$ is an object Ω together with a monomorphism $1 \xrightarrow{t} \Omega$, such that for any other monomorphism $A \xrightarrow{m} X$, there is a unique ("characteristic") map $\chi : X \to \Omega$ for which the square

$$\begin{tikzcd}
A \arrow[r, tail, "m"] \arrow[d] & X \arrow[d, "\chi"] \\
1 \arrow[r, tail, "t"] & \Omega
\end{tikzcd}$$

is a pullback. We may think of Ω as the object of truth-values, and of $1 \xrightarrow{t} \Omega$ as the value "true". In the category of sets, the object Ω is the two-element set $\{t, f\}$ and given a subset $A \longrightarrow X$, the map χ is the characteristic function

$$\chi(x) = \begin{cases} t & \text{if } x \in A; \\ f & \text{otherwise.} \end{cases}$$

Next, let me give some important examples of toposes.

- The category **Set** of sets and functions is a topos.
- For any category $\mathcal{C}$, the category of *presheaves* on $\mathcal{C}$ (contravariant functors from $\mathcal{C}$ to **Set**) is a topos.
- $G-$**Set**, the category of sets equipped with an action from a group G and equivariant maps between such, is a topos.
- $M -$ **Set** (sets equipped with an action from a monoid M) is a topos.
- If G is a topological group then the subcategory of $G -$ **Set** on the *continuous* G-sets is topos.
- Every topological space X gives a topos of sheaves, $\mathbf{Sh}(X)$.
- A *locale*, or *complete Heyting algebra*, is a partially ordered set $(A, \leq)$ with finite meets and arbitrary suprema, such that the infinite distributive law $x \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a_i \wedge x)$ is satisfied; the motivating example is the lattice of open subsets of a topological space. Any locale H gives a topos of sheaves on the locale, $\mathbf{Sh}(H)$.

All the examples given so far are Grothendieck toposes; the most interesting examples of toposes which are not Grothendieck are realizability toposes, which we will discuss in detail later on.

The notion of an elementary topos comes together with a notion of a morphism between toposes; such a morphism is called a *geometric morphism*, and is modeled on continuous maps of topological spaces. If $\mathcal{E}$ and $\mathcal{F}$ are toposes, then a geometric morphism f from $\mathcal{E}$ to $\mathcal{F}$ consists of a pair $(f^* : \mathcal{F} \to \mathcal{E}, f_* : \mathcal{E} \to \mathcal{F})$ of functors, such that f^* is left adjoint to f_* and f^* preserves finite limits. For example, every continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $f : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ between the toposes of sheaves.

By now, topos theory is a large research field, which has intimate connections to various other fields of mathematics, such as (algebraic) topology, geometry, and mathematical logic. Standard textbooks are: [42, 52, 44].

1.2 Categorical logic

The key idea behind categorical logic is that, given a category $\mathcal{C}$ with certain structure, one can reason about the objects and arrows of $\mathcal{C}$ as if they were sets and functions, provided one uses the right kind of reasoning. What this right kind of reasoning is, depends on the category in question, but in general, this reasoning should be *constructive* and should not use the *axiom of choice*. (Recall that the axiom of choice is the statement that given a family of non-empty sets $(X_i)_{i \in I}$ we can choose a family of elements $(x_i)_{i \in I}$ with each $x_i \in X_i$. There are many equivalent formulations, such as: every set can be well-ordered, or: every surjective function has a section.)

Let me give a very rough sketch of some of the basic ingredients of categorical logic. First of all, given an object X of a category $\mathcal{C}$, one has the set $Sub_{\mathcal{C}}(X)$ of subobjects (isomorphism classes of monos) of X. This set is in fact partially ordered by inclusion, and it has a maximal element, X itself. Now the idea is, to think of a subobject $A \rightarrowtail X$ as a *predicate* on X, i.e. a formula with a free variable of type X. To say that one such predicate A entails another predicate B, comes down to saying that $A \leq B$ in the partially ordered set $Sub_{\mathcal{C}}(X)$. Informally, we think of this entailment as expressing that for all $x \in X$: if $A(x)$ holds, then so does $B(x)$. Furthermore, we say that a predicate A is true, if it is the maximal element of $Sub_{\mathcal{C}}(X)$.

To interpret logical connectives, such as "and", "or", "implies", we need to make assumptions on the structure of the category $\mathcal{C}$ that we are working in. In fact, there is a fairly tight correspondence between certain classes of categories and various fragments of logic, but I leave that aside for now, and just give some idea of the mechanism. Suppose that our category $\mathcal{C}$ has finite limits. This implies that the poset $Sub_{\mathcal{C}}(X)$ has binary meets, since for each pair of subobjects $A, B \in Sub_{\mathcal{C}}(X)$, we can compute their meet $A \wedge B$ as the pullback

$$\begin{tikzcd}
A \wedge B \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\
A \arrow[r, tail] & X.
\end{tikzcd}$$

Now we think of $A \wedge B$ as the conjunction of the predicates A and B, and indeed, it satisfies the usual rules of logic, for example if C entails A and C entails B, then also C entails $A \wedge B$.

Similarly, if our category has some right exactness properties we can interpret disjunction: each $Sub_{\mathcal{C}}(X)$ then has binary suprema. And if $\mathcal{C}$ is cartesian closed, then $Sub_{\mathcal{C}}(X)$ has exponents, which serve to interpret implication. Finally, quantification is dealt with by using adjoints along pullback functors; if $A \in Sub_{\mathcal{C}}(X \times Y)$ is a predicate $A(x, y)$ then $\exists(y).A(x, y)$ is interpreted by applying the left adjoint to the projection $X \times Y \to X$ to A. Similarly, one uses the right adjoint to form $\forall y.A(x, y)$. (Hence we get the familiar slogan from categorical logic: "quantifiers as adjoints".)

Provided the structure of the category $\mathcal{C}$ allows the interpretation of all connectives, the poset $Sub_{\mathcal{C}}(X)$ will be a *Heyting algebra*; this is a poset with binary meets and binary suprema, a top and a bottom element, and implication. Negation can then be defined by putting $\neg A = A \Rightarrow \perp$. In general, we will only find $A \leq \neg\neg A$, but not $\neg\neg A \leq A$. In words: the principle of reductio ad absurdum does not always hold! In some categories (such as the category of sets), however, $\neg\neg A = A$ is true, in which case $Sub_{\mathcal{C}}(X)$ is called a *Boolean algebra*; the logic of the category is then classical, and the category is called Boolean as well.

We have already introduced toposes, and because of their rich structure, they play a special role in categorical logic. Because of the presence of exponentials and a subobject classifier, a topos allows for the interpretation of *higher-order* logic, i.e. logic where quantification over subsets is possible. This gives substance to the idea that a topos may be seen as a "generalized universe of sets", a view advocated by Lawvere. Indeed, in a topos we can perform many constructions that we know from naive set theory, such as the formation of powersets, function spaces and what more; to prove facts about these constructions, one can just prove them for sets, as long as one uses constructive reasoning. For clarity, let me list some aspects in which the logic of toposes in general is different from that of sets. In a general topos:

- the principle of *excluded middle*, $A \vee \neg A$ is not always valid;
- the axiom of choice (formulated as: every surjection has a right-inverse) is not always valid;
- there may be many subterminal objects, i.e. subobjects of 1;
- an object X is not always determined by its *global elements* $1 \to X$.

Because a topos may be viewed as a universe of sets, one can develop quite a lot of constructive mathematics in it, such as arithmetic or analysis. It will not come as a surprise that the models that one obtains may be quite different from the classical ones. For example, there are toposes in which there are only countably many functions from $\mathbb{N}$ to $\mathbb{N}$, or where every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous!

1.3 Realizability

In the beginning of the 1940's, Stephen Cole Kleene originated a whole new field of research, called realizability. His motivation for developing his definition of numerical realizability (which we shall present in a moment) was first and foremost to make precise in what sense there was a relation between intuitionism on the one hand, and effective computability on the other. (For more on the philosophy of mathematics in general, and intuitionism in particular, see e.g. [9, 72, 71]. For more on effective computability, see any textbook on recursion theory, e.g. [65].) One of the main ideas of intuitionism, although this was a far from clear-cut philosophy at that time, was that, in order to prove a mathematical theorem, one has to present an effective procedure to verify this theorem; for example, if an intuitionist is to prove a statement of the form "There exists a natural number x such that the property $P(x)$ holds," then it is not enough (as it would be for the classical mathematician) to show that the assumption that such a number does not exist leads to a contradiction. Rather, one has to establish the truth of the statement by presenting a *construction* of such a number. This idea of proofs consisting of effective constructions certainly suggests that one could use the theory of computability in order to make the notion of intuitionistic provability precise.

For concreteness, let us look at Kleene's original 1945 definition ([47]): realizability takes the form of a relation between natural numbers and sentences of arithmetic. By induction on the complexity of a sentence ϕ, we define what it means that a natural number n realizes ϕ. We use the following notation: $n \bullet m$ denotes the result of applying the n-th partial recursive function to m, and the symbol $\downarrow$ stands for "is defined". Moreover, we have assumed that there is a "pairing function", i.e. a recursive bijection $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ with unpairings $(-)_0, (-)_1$.

Definition 1.3.1 (Kleene's Numerical Realizability)

- n realizes $t = s$ iff $t = s$ is a true equation;
- n realizes $\phi \wedge \psi$ iff n_0 realizes ϕ and n_1 realizes ψ;
- n realizes $\phi \vee \psi$ iff either $n_0 = 0$ and n_1 realizes ϕ, or $n_0 = 1$ and n_1 realizes ψ;
- n realizes $\phi \rightarrow \psi$ iff for all m, if m realizes ϕ, then $n \bullet m\downarrow$ and $n \bullet m$ realizes ψ;
- n realizes $\exists x.\phi(x)$ iff n_0 realizes $\phi[n_1/x]$;
- n realizes $\forall x.\phi(x)$ iff for all m, $n \bullet m\downarrow$ and $n \bullet m$ realizes $\phi[m/x]$.

One may think of "n realizes ϕ" as: "n is a witness for the constructive truth of ϕ". Indeed, if we look at the clause for existential quantification, for instance, then we see that a witness for an existential statement $\exists x.\phi(x)$ consists of a pair, the first component of which gives a number m for which $\phi[m/x]$ holds, and the second of which is a witness for this fact. Also, the clause for universal quantification shows, that a witness for $\forall x.\phi(x)$ should be a (total) recursive function that gives, for each number m, a witness for $\phi[m/x]$.

Now if a sentence ϕ is derivable from the axioms of arithmetic using constructive logic, then there exists a realizer for ϕ. This states precisely, that realizability is a *sound* interpretation of constructive arithmetic. But this is not all: there are many arithmetical statements which are not provable, or even classically false, but which are true under the realizability interpretation (meaning that there is a realizer for such a statement). The most important example is Church's Thesis , which states that every function from the natural numbers to the natural numbers is computable:

$$CT_0 : \forall x\, \exists y.A(x,y) \to \exists m\, \forall x(m \bullet x{\downarrow}\ \wedge A(x, m \bullet x))$$

The fact that this principle is realizable implies that it is consistent with constructive arithmetic (even though it is inconsistent with classical arithmetic). This is one of the fascinating aspects of constructive systems: many strange and counterintuitive principles may be added without running into contradictions.

Kleene realizability may have been the first realizability interpretation, but it was certainly not the last. Many variations, extensions and modifications have been devised over the past sixty years, not only for arithmetic, but also for extensions of arithmetic, analysis and set theory. For a survey, the reader may consult [69, 70, 77]. I will only briefly describe a few variants that will occur in this thesis.

First of all, there is *modified realizability* . The essence of this definition is that there are both actual and potential realizers for a sentence; every actual realizer is a potential realizer, but not vice versa. This version of realizability can be used to show the consistency of the *Independence of Premisses* schema:

$$(\neg A \to \exists x.B) \to \exists x(\neg A \to B)$$

(where x is not free in B).

Second, there is *Lifschitz realizability.* This complicated interpretation was used to show that Church's Thesis is strictly stronger then

$$CT_0! : \forall x\, \exists! y.A(x,y) \to \exists m\, \forall x(m \bullet x{\downarrow}\ \wedge A(x, m \bullet x)).$$

A third modification is *extensional realizability* where one does not only inductively define what a realizer for a sentence is, but at the same time gives an equivalence relation on these realizers (intuitively, two realizers are equivalent if they show the same computational behaviour). Under this realizability, CT_0 is not valid.

Finally, let us indicate an important generalization of the original definition. It is clear that the domain of realizers must have some combinatorial structure, because realizers may be applied to other realizers, and this application should somehow result in a model for combinatory logic. *Partial Combinatory Algebras* (see chapter 2, section 2.3) certainly satisfy this requirement, so it is natural to "do realizability" over any such partial combinatory algebra.

1.4 Realizability toposes

In the 1970's sheaf semantics (i.e. models in Grothendieck toposes) was thoroughly studied and used for showing independence results, for example the independence of the axiom of choice, or of the continuum hypothesis. (See [23, 55].) An important type of Grothendieck toposes that was most frequently employed for these purposes was that of the form: sheaves over a locale (see section 1.1). Higgs (see [32]) had shown that such a topos is equivalent to the category of H-valued sets, Heyting-valued sets for the complete Heyting algebra H. Heyting-valued semantics is a straightforward generalization of Boolean-valued semantics ([31]), which is familiar to logicians.

Around 1979, Martin Hyland developed the idea that the H-valued sets construction could be mimicked in order to produce a topos for realizability, replacing the complete Heyting algebra by the object $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers. The topos resulting from this was dubbed "The Effective Topos", and described in detail in the paper [36]. The connection between Kleene's realizability and this topos can be stated very precisely: a sentence in the language of arithmetic is realizable in Kleene's sense precisely when it is true in the logic of the topos. To put this briefly: the logic of the natural numbers of the Effective Topos *is* Kleene realizability.

Let me give a brief description of the Effective Topos. First, we define two operations $\wedge, \Rightarrow$: $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ on the powerset of the natural numbers. Let U, V be subsets of $\mathbb{N}$. Define

$$\begin{array}{rll} U \wedge V & =_{def} & \{j(u,v) | u \in U, v \in V\} \quad \text{where } j : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is pairing} \\ U \Rightarrow V & =_{def} & \{n \in \mathbb{N} | \forall u \in U. n \bullet u{\downarrow} \;\; \& \, n \bullet u \in V\} \end{array}$$

Now we can introduce the following notation: for a set X, and functions $\alpha, \beta : X \to \mathcal{P}\mathbb{N}$, write

$$\alpha(x) \vdash_x \beta(x) \text{ if and only if } \exists n \in \bigcap_{x \in X} (\alpha(x) \Rightarrow \beta(x)) \quad (*)$$

This notation will be extended to several variables, for example, if we have maps $\gamma : X \times Y \to \mathcal{P}\mathbb{N}$ and $\delta : Y \times Z \to \mathcal{P}\mathbb{N}$, then the expression $\alpha(x,y) \vdash_{x,y,z} \beta(y,z)$ means $\exists n \in \bigcap_{x \in X, y \in Y, z \in Z} (\alpha(x,y) \Rightarrow \beta(y,z))$.

Definition 1.4.1 (Effective Topos) The Effective Topos is the category with **Objects**: pairs $(X, =_X)$, where X is a set and $=_X$ a function $X \times X \to \mathcal{P}\mathbb{N}$ satisfying

- $x =_X x' \vdash_{x,x'} x' =_X x$
- $x =_X x' \wedge x' =_X x'' \vdash_{x,x',x''} x =_X x''$.

Morphisms: given two objects $(X, =_X), (Y, =_Y)$, a *functional relation* from $(X, =_X)$ to $(Y, =_Y)$ is a function $F : X \times Y \to \mathcal{P}\mathbb{N}$ satisfying

- $F(x,y) \vdash_{x,y} x =_X x \wedge y =_Y y$
- $F(x,y) \wedge x =_X x' \wedge y =_Y y' \vdash_{x,x',y,y'} F(x',y')$
- $F(x,y) \wedge F(x,y') \vdash_{x,y,y'} y =_Y y'$
- $x =_X x \vdash_x \bigcup_{y \in Y} F(x,y)$.

A morphism from $(X, =_X)$ to $(Y, =_Y)$ is an equivalence class of functional relations from $(X, =_X)$ to $(Y, =_Y)$, where two $F, G : X \times Y \to \mathcal{P}\mathbb{N}$ are equivalent if

$$F(x,y) \vdash_{x,y} G(x,y) \text{ and } G(x,y) \vdash_{x,y} F(x,y).$$

Both from a logical and from a topos-theoretical point of view the Effective Topos turned out to have many fascinating properties. First of all, because toposes are a setting for studying intuitionistic higher-order logic, the logic of the Effective Topos is a natural generalization of Kleene realizability to higher-order logic. For example, one can look at the logic of the finite type structure over the natural numbers, or second-order arithmetic.

The Effective Topos is very interesting *qua* topos as well: for one thing, it is not Grothendieck (there aren't that many interesting non-Grothendieck toposes around), and it has the category of sets as sheaf subtopos for the double negation topology. Furthermore, it has a small internal full subcategory, called the category of modest sets , which is *complete* but not a poset ([37, 40]); classically, this is impossible. The non-classical behaviour of modest sets, or PERs, as they are also called, can be exploited by constructing models for second-order lambda-calculus or for programming languages. Two good books in which such "PER-models" are described, are: [2, 58]. For related issues, see for example [67, 61, 25, 39, 26, 62, 64, 4].

Triposes. Around 1980, Hyland, Johnstone and Pitts developed a general framework for the construction of "Effective Topos-like" categories ([38, 59]). They defined the notion of a tripos (which is an acronym for Topos Representing Indexed PreOrdered Set); this is a special kind of indexed category, with a structure that is rich enough to interpret higher order logic without equality. Out of a tripos one can construct a topos, very roughly speaking by adding non-standard equality predicates. This procedure generalizes the construction of the topos of H-valued sets for a complete Heyting algebra H, and simultaneously that of the Effective Topos.

For many variations on realizability, such as the ones we mentioned earlier: modified realizability, Lifschitz realizability and extensional realizability, triposes and toposes have been constructed ([29, 30, 76, 75, 74]) and this has provided insight into the relations between these different realizability notions. Moreover, the framework of tripos theory (which has been further developed in detail in Pitts' thesis [59]) has made it possible to study realizability over an arbitrary base category, and not just the category of sets. In particular, the iteration results in this work give rise to new kinds of previously unstudied toposes, by combining different triposes. Altogether, tripos theory must be seen as a major step towards a more systematic approach to realizability.

1.5 Analysis of realizability toposes

Presentations in terms of Completions. From the end of the 1980's on, various researchers have tried to obtain good presentations for realizability toposes. Their main motivation was the fact that it would be beneficial to view these complicated structures as a result of a universal construction. This would present realizability toposes in terms of categories that are easier to understand, and would give better insight to the categorical and topos-theoretical properties of the toposes.

The universal constructions at issue are known as *completions*. A completion is a way of freely adding structure to a category (for a detailed description, examples and references, see chapter 2).

The first result in this direction was obtained by Carboni, Freyd and Scedrov in their 1988 paper "A categorical approach to realizability and polymorphic type theory", [18]. They showed that the Effective Topos is the universal solution to the problem of making its full subcategory of $\neg\neg$-separated objects, or Assemblies, exact (this universal construction is called the *ex/reg*-completion of a regular category).

Then, in 1990, Robinson and Rosolini ([63]) gave a presentation of the Effective Topos which was analogous to a construction of presheaf toposes; if $\mathcal{C}$ is a small category, then one may construct the presheaf topos $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ by first freely adding all small coproducts to $\mathcal{C}$, and then adding quotients of equivalence relations. Robinson and Rosolini build the Effective Topos by first adding all *recursively indexed* coproducts to the category of sets, and then adding all quotients of equivalence relations. The result of the first step is **PAss**, the category of Partitioned Assemblies, so the second part of the construction tells us that the Effective Topos is the free exact completion of the category **PAss**.

In his comprehensive survey "Some free constructions in realizability and proof theory" from 1995 ([17]), Carboni presents an overview of the results about completions in connection to realizability. One of the questions that he raised in this paper is for which toposes the exact completion is again a topos.

In order to exploit the knowledge that some categories arise as free completions of others, research in the second half of the 1990's was often concentrated on characterizing properties of the completion of a category in terms of the category itself. There is a manifest pattern in all the results obtained here: almost always one determines the *weak structure* that a category is required to have in order for the exact completion to have certain (strong) structure. For example, the exact completion of a category $\mathcal{C}$ is locally cartesian closed if and only if $\mathcal{C}$ has *weak dependent products* (weak right adjoints to pullback functors). This was proved by Carboni and Rosolini in their 1995 paper "Locally cartesian closed exact completions" ([20]).

The PhD-thesis ([57]) by Matías Menni (2000) contains several results in this area, the most important one being a characterization of those left exact categories of which the exact completion is a topos. Using the result on locally cartesian closure in exact completions, he establishes that the key ingredient is the notion of a *generic proof.* Roughly speaking, a proof is a weakening of the notion of a subobject, and proofs in $\mathcal{C}$ correspond to subobjects in the exact completion of $\mathcal{C}$. Then a generic proof (a "weak proof classifier") in $\mathcal{C}$ is precisely the structure corresponding to a subobject classifier in $\mathcal{C}_{ex}$, the exact completion of $\mathcal{C}$ (using local cartesian closure).

Menni also exhibits a correspondence between universal closure operators on $\mathcal{C}_{ex}$ and *topologies* on $\mathcal{C}$ (this is not the same as a Grothendieck topology, although there are definite similarities), again a perfect example of translating structure of the exact completion to the original category.

As a partial result to Carboni's question on the exact completion of toposes, Menni shows that a presheaf topos has a generic proof precisely when the underlying site is a groupoid.

Relations between Realizability Toposes. Whereas completions are used to present realizability toposes in terms of simpler categories, there has also been research aiming for a better understanding of the relation between different realizability toposes.

An interesting piece of theory was developed by John Longley in his 1995 thesis ([54]). He introduces a category for partial combinatory algebras, where the notion of a morphism is quite different from that in standard combinatory logic ([5, 12]). There, one usually takes a homomorphism of PCAs to be a function preserving the application and the combinators. From the point of view of combinatory logic this makes sense, but for realizability, this is needlessly restrictive; Longley shows that, for realizability purposes, it is much more useful to consider *relations* which only preserve the application *up to a realizer.* He then shows that the construction of a realizability topos out of a PCA is functorial, and that a morphism of PCAs gives rise to an exact functor of realizability toposes.

Next, I want to mention relative realizability. Although the key ingredient was already in [48], and was generalized to triposes in [59], it was fully developed by Lars Birkedal (see [13, 3]). The idea is as follows: if $\mathbb{A}$ is a PCA, then we say that $\mathbb{A}_\sharp$ is a sub-PCA if it contains the combinators k, s and is closed under the application. Then we can define a realizability tripos on $\mathbb{A}$ where the difference with the ordinary tripos lies in the fact that the realizers have to come from $\mathbb{A}_\sharp$. Formally, the definition of logical entailment for the usual tripos for $\mathbb{A}$ is

$$\alpha \vdash_x \beta \text{ iff } \exists a \in \mathbb{A}\, \forall x \in X, a \in (\alpha(x) \Rightarrow \beta(x)).$$

where $\alpha, \beta : X \to \mathcal{P}\mathbb{A}$. Now we relativise this to the sub-PCA $\mathbb{A}_\sharp$ by:

$$\alpha \vdash_x \beta \text{ iff } \exists a \in \mathbb{A}_\sharp\, \forall x \in X, a \in (\alpha(x) \Rightarrow \beta(x)).$$

Intuitively, the elements of $\mathbb{A}$ are thought of as continuous realizers, and those of $\mathbb{A}_\sharp$ as computable realizers. If we write $\mathbf{RT}[\mathbb{A}, \mathbb{A}_\sharp]$ for the associated topos, then the relation between the ordinary realizability toposes $\mathbf{RT}[\mathbb{A}]$, $\mathbf{RT}[\mathbb{A}_\sharp]$ and $\mathbf{RT}[\mathbb{A}, \mathbb{A}_\sharp]$ can be formulated by saying that there is a logical functor $\mathbf{RT}[\mathbb{A}, \mathbb{A}_\sharp] \to \mathbf{RT}[\mathbb{A}]$, and a local geometric morphism $\mathbf{RT}[\mathbb{A}_\sharp] \to \mathbf{RT}[\mathbb{A}, \mathbb{A}_\sharp]$.

1.6 This thesis

After this brief, and very incomplete, overview of the field of research that this thesis aims to be a contribution to, I will give a motivation for the problems that will be addressed and the results that will be obtained.

First of all, although Longley's work on PCAs is certainly a step towards a more systematic understanding of realizability toposes and their relations, there are still unanswered questions. Most importantly, Longley gave a correspondence between relations between PCAs on the one hand, and certain exact functors between the realizability toposes on the other hand; but in topos theory, we are usually more interested in *geometric morphisms* between toposes. How can we characterize those? In chapter 3, we present a solution to this problem. Our proposal is to consider a generalization of PCAs which we call *Ordered PCAs.* Although the notion is more general, we can still build a tripos and thus a topos. (This was already worked out in van Oosten's paper on Extensional Realizability [75].) For these Ordered PCAs, we have a natural notion of morphism based on functions rather than relations. The key observation is that there is a monad on the category of Ordered PCAs, such that the Kleisli category for that monad is dual to the category of realizability toposes and exact functors. Moreover, we give an explicit condition on morphisms (called *computational density*) which characterizes those morphisms that induce a geometric morphism of realizability toposes. We also study various applications, such as relative realizability and local maps, the effective monad, and hierarchies of realizability toposes.

Secondly, I have already stressed that much energy has been spent on presentations of realizability toposes, and that this has given us a much better conceptual understanding of these toposes. One serious drawback of these techniques, however, is the fact that they rely on an essential appeal to the axiom of choice in **Set**[1]. This raises a conceptual question: is it a mere coincidence that the Effective Topos is the exact completion of the category of Partitioned Assemblies, a coincidence that has drawn our attention to completions but that breaks down as soon as we work over an arbitrary base topos? Does the fact that this approach does not work in the absence of choice teach us that these completions are, from a systematic point of view, not the right kind of tools for analysing realizability toposes? After all, we are interested in realizability because it gives a semantics for constructive systems. It would at least be inconsistent to insist on working over a classical (i.e. non-constructive) base topos, then. Of course, this line of argumentation is quite tentative, since it is a perfectly acceptable standpoint to study constructive systems from a classical viewpoint, but still, from a topos-theoretic point of view it is odd not to develop theory over an arbitrary base topos. Surely, one should aim at understanding the Effective Topos-construction in generality and not just one special instance!

In chapter 4 on Relative Completions we address this question. Even though

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[1] To be more precise, the presentations in terms of regular and exact completions use choice, but the presentation in terms of the ex/reg-completion does not.

we know that the Effective Topos, constructed over an arbitrary base, is seldom an exact completion of Partitioned Assemblies, we can still investigate the relationship between the two categories. It turns out, that we can define a refined notion of exact completion, which is called the *relative* exact completion since it is relative to the base topos, so that the Effective Topos is the exact completion of **PAss** relative to the base topos. In the special case when the base category satisfies the axiom of choice the construction reduces to the "absolute" exact completion. This shows that it is not a mere coincidence that the "classical" presentations work; the only coincidence there is that the axiom of choice allows for a simplified presentation which leaves out information that is essential in the general context.

It will turn out, however, that the relative version of the completion is much harder to handle; in order to show the above result, we develop quite some machinery, which is related to Menni's work on topologies ([57]).

Next, I come to another drawback of the use of completions for presenting realizability toposes: although they provide us with a succinct and conceptually compelling way of introducing such toposes, there is not much concrete information to be extracted. In particular, if one wants to make any detailed calculation about some logical aspect of the topos, the completions will not be of any help, and one has to revert to tripos theory. So, is there perhaps an eclectic approach to the matter, combining the conceptual advantages of the completions with the logical transparency of the theory of triposes? Surprisingly, the idea to present the Effective *tripos* as a completion of a simpler type of indexed preorder, has not come up so far. We work this idea out in the chapter 5 on Indexed Preorders and completions, where we make precise in which sense realizability triposes are *free triposes*, parallel to the fact that the toposes are free exact completions. In fact, it will be shown that the fact that realizability toposes are free exact completions (relative to the base topos) actually is a consequence of the fact that the triposes from which they are built are free completions.

Interestingly enough, in the main result of this chapter, which characterizes when applying a free construction to an indexed preorder gives a tripos, gives a close connection with the ordered PCAs that we introduced in chapter 3.

The framework which we study encompasses many interesting triposes, and we will also exhibit several hierarchies of triposes (and hence of toposes) which were not discovered before. Moreover, this framework, and the operations on it, brings some structure in the wild variety of realizability triposes.

Finally, in chapter 6, I will consider the problem posed by Aurelio Carboni, about the characterization of the class of toposes for which the exact completion is again a topos. Although I was not able to give a full answer, I will obtain several partial results, which I think are interesting in themselves. Among these are: a characterization of those Grothendieck toposes which are an exact completion, a condition on the geometric morphism to sets which is equivalent to the fact that the exact completion of a topos is a Grothendieck topos, and a characterization of the class of toposes which arise as a coproduct completion of a small category.

Chapter 2

Preliminaries

This chapter is meant to introduce the reader to the basic concepts and theory that will be used in this thesis. Some basic knowledge about category theory and toposes is required. a good reference for general category theory is Mac Lane's book "Categories for the Working Mathematician" [51]. As an introduction to topos theory there is "Sheaves in Geometry and Logic", by Mac Lane and Moerdijk ([52]), whereas Johnstone's "Sketches of an Elephant" ([44]) is by far the most comprehensive survey (but, because of its conciseness, a bit less suitable as a first introduction). Some aspects of enriched category theory and the theory of 2-categories and bicategories will also be used. For introduction to these matters, we also refer to [51, 44] or [16].

First, we introduce two important classes of categories: regular and exact categories (section 1); we discuss some of their most important properties and look at some examples. Then, in section 2, we treat the basic theory of completions. We recall the constructions, theorems and characterizations, without giving all the proofs. Section 3 is devoted to partial combinatory algebras, which form the key ingredient for realizability toposes. From a partial combinatory algebra, various categories can be constructed, such as Partitioned Assemblies, Assemblies and Modest Sets. This is reviewed in section 4. Finally, section 5 deals with indexed category theory; the main concept that we introduce there is that of a tripos, which is a special kind of indexed category, suitable for interpreting higher-order constructive logic without equality. A tripos gives rise to a topos, and this construction is also explained.

At the end of each section we give references to the most important publications on the subject.

2.1 Regularity and Exactness

In this section, the essentials of *regular* and *exact* categories are explained.

Regular Categories and Relations. Let $\mathcal{C}$ be a category with finite limits (sometimes we will briefly call this a *lex* or left exact category). An epimorphism

$e : X \to Y$ is called a *regular epimorphism* if it is a coequalizer. A regular epimorphism e is called *stable* if pullbacks of e along arbitrary maps are again regular.

We say that $\mathcal{C}$ is *regular* if it satisfies the following two conditions: (i) every map f can be factored as $f = me$, where m is a mono and e is a regular epimorphism, and (ii) regular epimorphisms are stable under pullback.

A functor between regular categories is said to be a *regular* functor if it preserves finite limits and regular epimorphisms. Equivalently, one can say that a left exact functor is regular if it preserves *exact sequences*, i.e. diagrams of the form

$$\begin{tikzcd} R \arrow[r, shift left, "q_0"] \arrow[r, shift right, "q_1"'] & X \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

where q coequalizes the pair q_0, q_1 and where R is the kernel pair of q.

Regular categories form a suitable setting for the study of relations; given two relations $\langle m_0, m_1 \rangle : R \to X \times Y$, and $\langle n_0, n_1 \rangle : S \to Y \times Z$ on can define their composition $R * S$ as follows: first form the pullback

$$\begin{tikzcd} R \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_R"'] & R \arrow[d, "m_1"'] \\ S \arrow[r, "n_0"'] & Y. \end{tikzcd}$$

There is a map $\langle m_0 \pi_R, n_1 \pi_S \rangle : R \times_Y S \to X \times Z$, and the image of this map is $R * S \subseteq X \times Z$.

This composition is associative and has a unit, namely the diagonal Δ. Thus for any regular category $\mathcal{C}$, we can consider the category $\mathbf{Rel}(\mathcal{C})$, which has the same objects as $\mathcal{C}$ but where the morphisms $X \to Y$ are now relations $R \subseteq X \times Y$. $\mathbf{Rel}(\mathcal{C})$ is locally ordered, and for two relations $R, S \subseteq X \times Y$, we write $R \leq S$ if R is smaller than S in $Sub_{\mathcal{C}}(X \times Y)$. Note also, that there is a self-duality $(-)^{\mathrm{op}}$ on $\mathbf{Rel}(\mathcal{C})$ which is the identity on objects, and which sends a relation R, given as $\langle m_0, m_1 \rangle : R \to X \times Z$ to R^{op}, given by $\langle m_1, m_0 \rangle : R \to Z \times X$.

There is an inclusion $\mathcal{C} \hookrightarrow \mathbf{Rel}(\mathcal{C})$, which is the identity on objects, and which sends a morphism f to its graph. A morphism $R \subseteq X \times Y$ in $\mathbf{Rel}(\mathcal{C})$ is the graph of a map in $\mathcal{C}$ if and only if it has a right adjoint. This is the same as saying that the inequalities $1_X \leq R * R^{\mathrm{op}}$ and $R^{\mathrm{op}} * R \leq 1_Y$ hold. The functor $\mathcal{C} \hookrightarrow \mathbf{Rel}(\mathcal{C})$ has a right adjoint precisely when $\mathcal{C}$ is a topos. In this case, $\mathbf{Rel}(\mathcal{C})$ is the Kleisli category for the covariant powerset monad on $\mathcal{C}$.

An *equivalence relation* in a left exact category $\mathcal{C}$ is a relation $R \subseteq X \times X$ which is reflexive, symmetric and transitive. These requirements can be formulated as follows:

- $\Delta \leq R$,
- $R \leq R^{\mathrm{op}}$ (equivalently, $R = R^{\mathrm{op}}$),

- $R * R \leq R$ (equivalently, $R * R = R$).

As usual, there are straightforward diagrammatic counterparts to these demands. Note that the last condition says that R, considered as an endomorphism $R : X \to X$ in $\mathbf{Rel}(\mathcal{C})$, is idempotent.

Given two equivalence relations $R \subseteq X \times X$, $S \subseteq Y \times Y$, a *functional relation* F from R to S is a relation $F \subseteq X \times Y$ such that the following hold:

- $R * F = F = F * S$,
- $R \leq F * F^{\mathrm{op}}$,
- $F^{\mathrm{op}} * F \leq S$.

Intuitively, F represents a map from R-equivalence classes of X to S-equivalence classes of Y.

Note that since $R * R = R$ says that R is an idempotent, the first clause tells us that F is a map between idempotents in the splitting. The other two clauses say that F is the graph of a map up to equivalence.

Exact Categories. In a regular category $\mathcal{C}$, every kernel pair has a coequalizer. But in general, not every equivalence relation has a quotient. We say that an equivalence relation is *effective* if it is (isomorphic to) the kernel pair of some arrow. We say that a regular category $\mathcal{C}$ is *exact* if every equivalence relation is effective. The notion of an *exact functor* is the same as for regular categories: one requires preservation of finite limits and regular epimorphisms.

Let us give some important examples of exact categories: first of all, any topos is exact. Second, any category of algebras over **Set**, such as monoids, groups, rings, etc. is exact. This also includes **CSL**, the category of complete sup-lattices, which is the category of algebras for the covariant powerset-monad. This actually holds for any elementary topos; the category $\mathbf{CSL}(\mathcal{E})$ of complete sup-lattices in a topos $\mathcal{E}$ is always exact. Another class of examples is provided by *Abelian* categories (see [24]).

Finally, a warning is in order: in some texts, the terminology “exact functor” is used for functors that preserve finite colimits, so as a dual to “left exact”. In our terminology, exact functors need not preserve coproducts. In other texts, the term “exact category” is abandoned in favor of “effective regular category”. Although we sympathize with this terminological shift, we stick to exactness, since it is much more common (and shorter).

Literature on Regular and Exact Categories. As a standard reference, we give the book “Exact Categories and Categories of Sheaves”, by Barr, Grillet and van Osdol ([6]). Other expositions of regular and exact categories can be found in [44, 27] or [16]. For more on relations in regular categories, we refer to [27] or [22].

2.2 Completions

Everywhere in mathematics one finds free constructions: one has the free group on a set, the free abelian group on a group, the free ring on a monoid, the free operad on a collection, etcetera (for a large list of examples, see [51]). All these constructions admit a very compact description: one has a left adjoint to a forgetful functor. Free constructions in category theory differ from these algebraic examples in at least one important respect: whereas we are interested in groups, rings, etc. up to isomorphism (and in their elements up to identity), we are interested in categories only up to *equivalence* (and in their objects up to isomorphism). This implies, that, in dealing with completions of categories, we have to take into account that these are organised in a 2-categorical fashion.

Typically, the picture is as follows: one has a "large" 2-category $\mathbf{S}$ where the objects are categories which have a certain structure, and where the morphisms are functors preserving that structure. The 2-cells are simply natural transformations. And one has a forgetful 2-functor K to another large 2-category $\mathbf{C}$, where the objects are not required to have that specific structure.

$$\begin{tikzcd}
\mathbf{S} \arrow[r, shift right, "K"'] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift right, "F"']
\end{tikzcd}$$

Then one constructs a left bi-adjoint to this forgetful functor; this means that for every $\mathcal{C}$ in $\mathbf{C}$, one has a category $F(\mathcal{C})$ in $\mathbf{S}$ and a functor $y : \mathcal{C} \to F(\mathcal{C})$ with the following universal property: given any functor $g : \mathcal{C} \to K(\mathcal{D})$ there is an extension $\hat{g} : F(\mathcal{C}) \to \mathcal{D}$ in $\mathbf{S}$ for which $K(\hat{g}) \circ y \cong g$. Moreover, the functor $\hat{g}$ is required to be unique up to natural isomorphism. In a picture:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "y"] \arrow[dr, "g"'] & \mathcal{F}(\mathcal{C}) \arrow[d, dotted, "\hat{g}"] \\
& \mathcal{D}.
\end{tikzcd}$$

The fact that the extension $\hat{g}$ is unique up to isomorphism implies that the free category $F(\mathcal{C})$ is unique up to equivalence of categories, as desired.

Category-theoretic completions often take the form of a so-called Kock-Zöberlein-monad[1] (KZ-monad, for short); this is, up to some 2-categorical subtleties which will not bother us here, a 2-monad (T, η, μ) on a 2-category $\mathcal{C}$ with the property that the multiplication μ is naturally left adjoint to the unit η. Algebras for such a monad are then adjoint to units; hence algebra structures, if they exist, are unique up to isomorphism. Another property is, that free algebras are characterized by the fact that there is a further left adjoint to the algebra map. Both the regular and the exact completion that will be described below are instances of KZ-monads.

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[1] these also go under the heading of a KZ-doctrine, but I stick to monad, since this is what it is (well, a 2-categorical version of a monad).

I make one philosophical remark of minor significance about this subject: in the algebraic case we add extra structure to an object (as in the free group on a set case) or we add extra properties (e.g. abelianize a group). In the category-theoretic situation it is not always clear what it is that we add. Take finite limits, for instance: the fact that a certain category has finite limits is certainly a property of that category, but one is inclined to say that it is more than that. Because finite limits are determined only up to isomorphism, it is not fully accurate to regard them as extra *structure* on a category, however. Anyhow, after having noticed it, I will from now on ignore this subtlety since it is of no practical importance to the rest of the work, and we will loosely think and speak of finite limits as additional structure on a category.

When one wishes to add structure (certain types of limits or colimits, say) to a category $\mathcal{C}$, there is a general recipe for doing so: as objects of $F(\mathcal{C})$ one takes diagrams of the required type, and as morphisms natural transformations of these diagrams. When one wants to add certain properties, such as effectiveness, the constructions are usually more involved.

Next, I describe a number of important completions. For every completion that we will consider, I will do the following: first, I give the construction and formulate the universal property. Then, I will give a categorical characterization of the free categories.

Free Colimit Completion. This completion, which freely adds all small colimits to a category, is probably the most well-known completion there is. Starting with a small category $\mathcal{C}$, one forms the presheaf category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. There is the Yoneda embedding $y : \mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ which has the following universal property: for any cocomplete category $\mathcal{D}$ and any functor $g : \mathcal{C} \to \mathcal{D}$ there is a colimit-preserving functor $\hat{g} : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ such that $\hat{g} \circ y \cong g$; this $\hat{g}$ is unique up to natural isomorphism with these properties. So the assignment $\mathcal{C} \mapsto \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the object part of a left biadjoint to the forgetful functor from **CoComp**, the large category of cocomplete categories, to **Cat**, the category of categories.

Later, we will see that this colimit completion may be broken up into two steps, namely one in which one adds small sums, and one in which quotients are added.

The objects of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ in the image of the Yoneda functor have a categorical characterization: they are precisely the *indecomposable projectives* in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Recall that an object X of a category $\mathcal{E}$ is called indecomposable, or connected, if $X \cong Y + Z$ implies that either $X \cong Y$ or $X \cong Z$; in other words, X has no non-trivial coproduct decompositions. Recall also that X is called (regular) projective if for every regular epi $e : Y \to Z$ and any map $f : X \to Z$ there is a lifting $\hat{f} : X \to Y$. (We will usually drop the adjective "regular" in "regular projective".)

Now in a presheaf topos every object X can be covered by a sum of representables; so presheaf categories have the property that every object can be covered by a sum of indecomposable projectives, and in fact, this property is also sufficient: thus we find

Theorem 2.2.1 (Characterization of presheaf toposes) *Let $\mathcal{D}$ be a category with all small colimits, and denote by $\mathcal{C}$ the full subcategory of indecomposable projective objects of $\mathcal{D}$. If $\mathcal{C}$ is small, then the following are equivalent:*

1. $\mathcal{D} \simeq \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$*;*
2. *every object of $\mathcal{D}$ can be covered by a sum of $\mathcal{C}$-objects.*

Free Coproduct Completion. Let $\mathcal{C}$ be any category, and consider the category $\mathcal{C}_+$ with objects families $(C_i)_{i \in I}$, I a set, C_i objects of $\mathcal{C}$; a map from $(C_i)_{i \in I}$ to $(D_j)_{j \in J}$ is a pair $(\phi, (f_i)_{i \in I})$ with $\phi : I \to J$, and $f_i : C_i \to D_{\phi(i)}$. The category $\mathcal{C}_+$ has all **Set**-indexed coproducts, there is an obvious embedding $y : \mathcal{C} \to \mathcal{C}_+$, and for any other category $\mathcal{D}$ with small coproducts and any functor $g : \mathcal{C} \to \mathcal{D}$ we have a coproduct- preserving extension $\hat{g} : \mathcal{C}_+ \to \mathcal{D}$ defined by $\hat{g}((C_i)_{i \in I}) = \coprod_{i \in I} g(C_i)$.

Note that existing coproducts in $\mathcal{C}$ are destroyed by this construction. The objects of the form $y(C)$ are the *indecomposable* objects of $\mathcal{C}_+$, and one has

Theorem 2.2.2 (Characterization of free coproduct completions) *Let $\mathcal{D}$ be a category with all small, disjoint and stable coproducts, and $\mathcal{C}$ denote the full subcategory of indecomposable objects of $\mathcal{D}$. Then the following are equivalent:*

1. $\mathcal{D} \simeq \mathcal{C}_+$*;*
2. *every object of $\mathcal{D}$ can be written as a sum of $\mathcal{C}$-objects.*

Free Regular Completion. Denote the 2-category of left exact categories and left exact functors between them (and arbitrary natural transformations as 2-cells) by **LEX**, and let **REG** denote the 2-category of regular categories and regular functors. The *free regular completion* $\mathcal{C}_{reg}$ of a left exact category $\mathcal{C}$ is constructed as follows: an object is a map $(X \xrightarrow{p} Y)$ in $\mathcal{C}$. For two such objects $(X \xrightarrow{p} Y)$, $(U \xrightarrow{q} V)$, an arrow $[f] : p \to q$ is an equivalence class of maps $f : X \to U$ with the property $qfp_0 = qfp_1$, where p_0, p_1 is the kernel pair of p, and where two such f, f' are considered equivalent if their composites with q are equal.

We think of an object $(X \xrightarrow{p} Y)$ as the *image* of p, i.e. of the coequalizer of the kernel pair of p.

It is routine to check that the category $\mathcal{C}_{reg}$ inherits finite limits from $\mathcal{C}$. Furthermore, given a map $[f] : (X \xrightarrow{p} Y) \to (U \xrightarrow{q} V)$, we form the image factorization of $[f]$ as:

$$\begin{tikzcd} \begin{pmatrix} X \\ p\downarrow \\ Y \end{pmatrix} \arrow[r, two heads, "{[1]}"] & \begin{pmatrix} X \\ p\downarrow \\ Y \\ f\downarrow \\ U \end{pmatrix} \arrow[r, tail, "{[f]}"] & \begin{pmatrix} U \\ q\downarrow \\ V \end{pmatrix}. \end{tikzcd}$$

It is easy to verify that such factorizations are pullback-stable, so that $\mathcal{C}_{reg}$ is indeed a regular category. There is an embedding $y : \mathcal{C} \to \mathcal{C}_{reg}$, putting $yC = (C \xrightarrow{1} C)$. Given a left exact functor $g : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is regular, define $\hat{g} : \mathcal{C}_{reg} \to \mathcal{D}$ by sending $(X \xrightarrow{p} Y)$ to the image of the map $g(p) : g(X) \to g(Y)$.

Now objects in the image of y are precisely the *projective* objects of $\mathcal{C}_{reg}$. Moreover, any object $(X \xrightarrow{p} Y)$ of $\mathcal{C}_{reg}$ can be fitted into a diagram

$$\begin{tikzcd} y(X) \arrow[r, two heads, "{[1]}"] & \begin{pmatrix} X \\ p\downarrow \\ Y \end{pmatrix} \arrow[r, tail, "{[p]}"] & y(Y). \end{tikzcd}$$

Hence, every object in $\mathcal{C}_{reg}$ can be covered by a projective and can be embedded into a projective. Finally, the projective objects are closed under finite limits in $\mathcal{C}_{reg}$. Thus, we have the following characterization:

Theorem 2.2.3 (Characterization of free regular categories) *Let $\mathcal{D}$ be a regular category, and denote by $\mathcal{C}$ the full subcategory of projective objects of $\mathcal{D}$. Then the following are equivalent:*

1. $\mathcal{D} \simeq \mathcal{C}_{reg}$;
2. *$\mathcal{C}$ is closed under finite limits in $\mathcal{D}$, and every object of $\mathcal{D}$ can be covered by and embedded into an object of $\mathcal{C}$.*

Free Exact Completion. As before, let **LEX** be the 2-category of left exact categories and functors, and let **EX** be the 2-category of exact categories and exact functors. The *exact completion* defines a left biadjoint to the forgetful 2-functor **EX** → **LEX**. Given $\mathcal{C}$ left exact, build $\mathcal{C}_{ex}$ as follows: an object of $\mathcal{C}_{ex}$ is a pseudo-equivalence relation in $\mathcal{C}$ (i.e. a pair of maps $r_0, r_1 : R \to X$ satisfying the axioms for an equivalence relation, but not necessarily being jointly monic). A map $[f] : (R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X) \to (S \underset{s_1}{\overset{s_0}{\rightrightarrows}} Y)$ is an equivalence class of maps $f : X \to Y$ for which there is a lifting $\bar{f} : R \to S$ such that $s_0\bar{f} = fr_0$, and $s_1\bar{f} = fr_1$. Two such f, f' are equivalent if there is a $g : X \to S$ for which $s_0 g = f$ and $s_1 g = f'$.

The Yoneda embedding sends an object C of $\mathcal{C}$ to the trivial equivalence relation $y(C) = (C \underset{1}{\overset{1}{\rightrightarrows}} C)$ on C. This is full and faithful and left exact.

If $\mathcal{D}$ is exact, then every pseudo-equivalence relation in $\mathcal{D}$ has a coequalizer. Therefore a left exact functor $g : \mathcal{C} \to \mathcal{D}$ can be extended along the Yoneda functor by putting

$$g\ (R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X) = Coeq\ (g(R) \underset{g(r_1)}{\overset{g(r_0)}{\rightrightarrows}} g(X))\ .$$

The characterization of the free exact categories in **EX** is analogous to the case of the regular completion; again the objects in the image of the embedding $y : \mathcal{C} \to \mathcal{C}_{ex}$ are precisely the projectives. So we get:

Theorem 2.2.4 (Characterization of free exact categories) *Let $\mathcal{D}$ be an exact category, and denote by $\mathcal{C}$ the full subcategory of projective objects of $\mathcal{D}$. Then the following are equivalent:*

1. $\mathcal{D} \simeq \mathcal{C}_{ex}$;
2. *$\mathcal{C}$ is closed under finite limits in $\mathcal{D}$, and every object of $\mathcal{D}$ can be covered by an object of $\mathcal{C}$.*

One remark concerning notation: the notation $\mathcal{C}_{reg}$, $\mathcal{C}_{ex}$ leaves out the information that we form the free regular or exact completion of a *lex* category. A better notation therefore is $\mathcal{C}_{ex/lex}$, but if confusion is unlikely, we prefer the shorter notation (see next paragraph, though).
Next we list some conditions under which a category and its exact completion coincide: this is certainly well-known, but it is convenient to have these at hand.

Lemma 2.2.5 *Let $\mathcal{C}$ be an exact category. Then the following are equivalent:*

1. $\mathcal{C} \simeq \mathcal{C}_{ex}$
2. $y : \mathcal{C} \to \mathcal{C}_{ex}$ *preserves regular epimorphisms*
3. *every epi splits in $\mathcal{C}$.*

Proof. $1 \Rightarrow 2$ is trivial; for $2 \Rightarrow 3$, consider an epi $e : A \longrightarrow B$. This gives an epi in $\mathcal{C}_{ex}$, $ye : yA \longrightarrow yB$. Since any object in the image of y is projective, this epi splits. But $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{ex}$, hence the splitting also works in $\mathcal{C}$.

Finally, assume 3. We show that any object in $\mathcal{C}_{ex}$ is isomorphic to an object in the image of y. So take an object $R \rightrightarrows X$ of $\mathcal{C}_{ex}$. The functor a sends this to the coequalizer in $\mathcal{C}$, call this Q. Now there is section for this coequalizer, say $m : Q \to X$, which induces a map $[m] : yQ \to (R \rightrightarrows X)$. This is easily seen to be an isomorphism.

$\square$

Free Exact completion of a Regular Category. When one starts with a regular category $\mathcal{C}$, on can freely add effectiveness in the following manner: $\mathcal{C}_{ex/reg}$ has equivalence relations in $\mathcal{C}$ as objects, and functional relations as morphisms. Since this gives a left biadjoint to the forgetful functor $\mathbf{EX} \to \mathbf{REG}$, it follows easily that the exact completion $\mathcal{C}_{ex}$ of a left exact category $\mathcal{C}$ can also be obtained in two steps, namely $\mathcal{C}_{ex} \simeq (\mathcal{C}_{reg})_{ex/reg}$, because biadjoints compose.

There are some respects in which the behaviour of $(-)_{ex/reg}$ is quite different from that of the other completions. For one thing, the construction is *idempotent*, in the sense that if $\mathcal{C}$ is already exact, then $\mathcal{C} \simeq \mathcal{C}_{ex/reg}$. This means that we should not try to characterize thoses exact categories which are of the form $\mathcal{C}_{ex/reg}$, because every exact category is trivially of this form.

Furthermore, whereas the embeddings $\mathcal{C} \hookrightarrow \mathcal{C}_{reg}$ and $\mathcal{C} \hookrightarrow \mathcal{C}_{ex}$ destroy all existing regular structure in $\mathcal{C}$, in the sense that these embeddings will not preserve regularity of epimorphisms unless they split, the embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{ex/reg}$ does preserve the regular structure.

There is an alternative description of the free exact completion on a regular one; it is based on the fact that if a regular category is exact, then in its category of relations, the reflexive, transitive idempotents split. Thus, given a regular category $\mathcal{C}$, one may consider $\mathbf{Rel}(\mathcal{C})$, split this class of idempotents and then retrieve $\mathcal{C}_{ex/reg}$ as the subcategory on the maps. For more details, we refer to [21].

Presheaves again. As said before, the free colimit completion may be broken up into two separate steps: for any category $\mathcal{C}$ with finite limits we have $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \simeq (\mathcal{C}_+)_{ex}$. The proof is easy, because both categories have the same universal property.

In case $\mathcal{C}$ does not have finite limits, there is an obstacle, because then $\mathcal{C}_+$ doesn't have finite limits, either, so we cannot form the exact completion of $\mathcal{C}_+$. The next paragraph explains however, that this obstruction is more apparent than real, and that a suitable modification of the above fact holds for any category $\mathcal{C}$.

Weak Versions. Although we will not encounter these frequently, we do mention that the construction of $\mathcal{C}_{ex}$ does not make full use of the fact that $\mathcal{C}$ has finite limits. In fact, one can weaken the requirement on $\mathcal{C}$ by asking that $\mathcal{C}$ has *weak finite limits*; this just means that there for any diagram there is a cone with the property that for any other cone there exists a (not necessarily unique map) to this cone. If $\mathcal{C}$ has weak finite limits, then the exact completion is denoted $\mathcal{C}_{ex/wlex}$, and one gets:

Theorem 2.2.6 (Characterization of weak free exact categories) *Let $\mathcal{D}$ be an exact category, and denote by $\mathcal{C}$ the full subcategory of projective objects of $\mathcal{D}$. Then the following are equivalent:*

1. $\mathcal{D} \simeq \mathcal{C}_{ex/wlex}$;
2. *every object of $\mathcal{D}$ can be covered by an object of $\mathcal{C}$.*

Note, that if there are enough projectives in a category $\mathcal{D}$, then it automatically follows that these are closed under weak finite limits: for one can form their genuine limit in $\mathcal{D}$ and then take a projective cover of this limit.

The weak version of the exact completion has some subtleties; in formulating the universal property, finite limit-preserving functors need to be replaced by so-called left-covering functors. The construction is not a KZ-monad, unlike the ordinary exact completion.

Now take any small category $\mathcal{C}$, left exact or not. Then it holds that $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \simeq (\mathcal{C}_+)_{ex/wlex}$. Note, however, that the objects of $\mathcal{C}_+$ need not be *all* projectives in the presheaf topos; this is only the case if all idempotents split in $\mathcal{C}$.

Another case in which we really need the weak version is that of complete sup-lattices: as mentioned earlier, this is an exact category, since it is a category of algebras (for the covariant powerset-monad). Because the free algebras (complete sup-lattices of the form $\mathcal{P}(X)$) are projective and because every algebra is a quotient of a free algebra, we find that the category of complete sup-lattices is the exact completion of its subcategory of free algebras[2], i.e. of the category of relations. But this category never has finite limits in the stong sense, only in the weak sense (products exist, but only weak equalizers).

More generally, if we have a monad on a power of $\mathbf{Set}$, then the category of algebras is exact, and the free algebras are projective; since every algebra is a quotient of a free algebra, it follows that the category of algebras is the weak exact completion of the Kleisli category for the monad.

Literature on Completions. All of the above completions are well-known, and there are many papers where they are studied in great detail. We mention that the free regular and exact completions were introduced by Celia Magno in 1981 in his Ph.D. thesis; the relevant publication is the paper with Carboni: "The Free Exact Category on a Left Exact One" [19]. The ex/reg-completion seems to have been known at least since 1973, but the first paper in which the construction appeared was Carboni, Freyd and Scedrov's "A Categorical Approach to Realizability and Polymorphic Types" [18]. An overview of the various completions and their relations and applications is given in [17].

For a thorough treatment of KZ-monads, one can consult Kock's original paper [49], or a summary of the main results in [44].

There are a lot of publications relating properties of $\mathcal{C}$ to properties of $\mathcal{C}_{ex}$. Carboni and Rosolini [20] characterize those left exact categories which have a locally cartesian closed exact completion. Rosický [66] does the same for ordinary cartesian closedness. Menni's paper [56] contains the important result that the exact completion of a left exact category $\mathcal{C}$ is a topos if and only if the category $\mathcal{C}$ has weak dependent products and a generic proof (for an explanation, see section2.4).

Completions over weak finite limits appeared on the stage in 1995, in a paper [21] by Carboni and Vitale entitled "Regular and Exact Completions". Here, there are applications to presheaf toposes, localizations of those (i.e. Grothendieck toposes) and categories of algebras.

Although we will not encounter this in our work, for completeness' sake we point out that there is a construction called "epi-monic completion", which freely adds epi-monic factorizations to a category. This construction is described by Grandis in [28], where the relation with homotopy theory is explored.

Finally we mention (also solely for completeness) that there are infinitary generalizations of the regular and exact completions; for those who are interested

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[2] A bizarre fact is, that it is at the same time the *regular* completion of the category of free algebras. It rarely occurs that the regular and the exact completion coincide.

in these variants, we refer to [35, 50].

2.3 Partial Combinatory Algebras

We introduce one of the main building blocks in the study of realizability, namely the notion of a Partial Combinatory Algebra (PCA). First, we define what an *applicative structure* is, and then we look at combinatory completeness.

Definition 2.3.1 A *partial applicative structure* (PAS for short) $\mathbb{A} = (A, \bullet)$ consists of a set A together with a partial, binary function $\bullet : A \times A \rightharpoonup A$.

As is common when dealing with applicative stuctures we write $a \bullet b{\downarrow}$ or $ab{\downarrow}$ if the pair (a, b) is in the domain of $\bullet$, in which case $a \bullet b$ or ab denotes the value. Furthermore, in writing abc we use associativity to the left, i.e. abc is shorthand for $(ab)c$. Also, we write $a \simeq b$ to express that a is defined iff b is, and $a = b$ if both defined.

A partial applicative structure is called *total* if for all a, b, $ab{\downarrow}$, i.e. if $\bullet$ is a total function $A \times A \to A$.

Given a PAS $\mathbb{A}$, we define the *set of terms over* $\mathbb{A}$ as follows: let $x_0, x_1, \ldots$ be a countable sequence of fresh variables. Then $\mathcal{T}(\mathbb{A})$ is the smallest set with the properties

1. $\mathbb{A} \subseteq \mathcal{T}(\mathbb{A})$
2. $x_i \in \mathcal{T}(\mathbb{A})$, for all $i \in \mathbb{N}$
3. if $t, s \in \mathcal{T}(\mathbb{A})$, then so is (ts).

It is convenient to think of the elements of $\mathcal{T}(\mathbb{A})$ as polynomials, or algebraic functions over $\mathbb{A}$.

If t is a term over $\mathbb{A}$, then the set of (free) variables, $FV(t)$, is by definition the set of those x_i that occur in t. A *valuation* is a function assigning elements of $\mathbb{A}$ to the variables $x_0, x_1, \ldots$. Hence, every valuation gives a partial function $\mathcal{T}(\mathbb{A}) \to \mathbb{A}$. For a valuation ρ and an element $t \in \mathcal{T}(\mathbb{A})$, we write $t^\rho{\downarrow}$ if this function if defined, and then the value is denoted t^ρ.

We say that $\mathbb{A} \models t \simeq s$, for terms t, s, if for every valuation ρ we have that $t^\rho \simeq s^\rho$. Now we can give the key definition of this section:

Definition 2.3.2 (Partial Combinatory Algebra, version 1) We say that a partial applicative structure $\mathbb{A} = (A, \bullet)$ is a *Partial Combinatory Algebra*, if for every term $t \in \mathcal{T}(\mathbb{A})$ in free variables $x_0, \ldots, x_n$ there is $a_t \in \mathbb{A}$ such that for every valuation ρ:

$$(a_t x_0 \ldots x_{n-1})^\rho{\downarrow} \quad \text{and } \mathbb{A} \models (a_t x_0 \ldots x_n)^\rho \simeq t.$$

This defining property is called *combinatory completeness*, and states that every partial algebraic function over $\mathbb{A}$ can be represented by an element of $\mathbb{A}$.

It is a well-known result by Curry that combinatory completeness is already ensured by the existence of two special combinators k, s. This means that we may redefine the notion of a PCA as follows:

Definition 2.3.3 (Partial Combinatory Algebra, version 2) We say that a partial applicative structure $\mathbb{A}$ is a *Partial Combinatory Algebra* if there are elements $k, s \in A$, for which

1. $kab = a$, for all $a, b \in A$,
2. $sab\downarrow$, $sabc \simeq ac(bc)$ for all $a, b, c \in A$.

The second version of the definition is in fact the one that is encountered most frequently in the modern literature, undoubtedly because it is more compact. The first version, however, is slightly more conceptual; moreover, the set of terms $\mathcal{T}(\mathbb{A})$ is of some independent interest.

The prime example of a PCA is Kleene's first PCA, where the underlying set is $\mathbb{N}$ and where $n \bullet m$ is the application of the n-th partial recursive function to the number m. Another important example is $\mathcal{P}\omega$, Scott's *graph model*. This PCA is in fact *total*. There is one trivial PCA, namely the one-point structure[3].

Some remarks about the definition are in order: first, notice that (in the second definition) we require the existence of k, s and that we do not need or want these elements to be part of the structure. In general, there may be (infinitely) many elements of a PCA that can serve as k or s. Secondly, although the word algebra occurs in the name, there is not much algebraic about PCAs; in fact, it is an unrelenting source of frustration that almost all techniques, concepts and constructions that one is naturally inclined to use in algebra fail for the bluntest of reasons in the case of PCAs.

Any non-trivial PCA contains a copy of the natural numbers; these can be constructed from the combinators k and s. Any partial recursive function is then representable. Different choices for k and s may give different copies; that is why we use the terminology *choice of numerals* for a particular copy of $\mathbb{N}$ inside the PCA.

From a categorical point of view, it may seem strange that we have not yet formulated an appropriate notion of a homomorphism of PCAs, so that we have a good category of PCAs. This is not as straightforward as it may seem at first sight, however, and the answer to the problem of finding the right definition of morphism of PCAs is highly dependent on the applications one has in mind. In the chapter on Ordered PCAs, we will propose a definition, and compare this with other definitions; therefore we will leave this matter for the moment.

The notion of a PCA makes sense in any elementary topos; note that we consider the domain of the application map as a subobject of $\mathbb{A} \times \mathbb{A}$, i.e. as structure, while the existence of combinators k and s is taken to be a property of $\mathbb{A}$. Although the axioms for PCAs cannot be formulated in purely geometric

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[3] In some treatments of this subject, one takes nontriviality to be part of the definition of a PCA. In our context, we don't see a particular reason for doing so.

logic, inverse image functors preserve PCAs (see [15]), generalizing the special case of natural number objects.

Literature on PCAs. A classic text, in which PCAs are introduced and in which their role in combinatory logic and models of the lambda calculus is exposed, is Barendregt's [5]. Another standard reference for combinatory logic is the book by Hindley and Seldin: [33]. In her thesis [12], Bethke discusses various special types of PCAs (such as topological PCAs), constructions on PCAs and embedding properties. It should be noted, however, that the notion of homomorphism of PCAs which is in both of these references is a strict one which is of little use for our purposes.

Different is the Ph.D. thesis by John Longley, "Realizability Toposes and Language Semantics" [54]. He proposes a much more relaxed notion of homomorphism of PCAs, suited for studying functors between categories arising in realizability.

2.4 Partitioned Assemblies, Assemblies, PERs

This section deals with representational categories for PCAs; for us, the categories of Partitioned Assemblies and Assemblies will be most important, but we also devote a little space to PERs.

Partitioned Assemblies. Let $\mathbb{A}$ be a PCA. Define the category of *Partitioned Assemblies over* $\mathbb{A}$, denoted $\mathbf{PAss}(\mathbb{A})$, by taking objects (X, α), where $\alpha : X \to \mathbb{A}$, and morphisms $f : (X, \alpha) \to (Y, \beta)$, where $f : X \to Y$ is a map for which $\exists a \in \mathbb{A}\, \forall x \in X. a \bullet \alpha(x) = \beta(f(x))$.

The category $\mathbf{PAss}(\mathbb{A})$ has finite limits, and there is an adjointness

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend right, "\nabla"'] \arrow[r, phantom, "\perp"] & \mathbf{PAss}(\mathbb{A}) \arrow[l, bend right, "\Gamma"'] \end{tikzcd}$$

The functor ∇ is thought of as a constant, or discrete, objects functor, sending a set X to the partitioned assembly $(X, \top_X)$ with $\top_X$ a constant function. The functor Γ is a faithful forgetful functor, sending (Y, β) to Y, but may also be thought of as global sections. Because limits in $\mathbf{PAss}(\mathbb{A})$ are computed as in $\mathbf{Set}$, Γ preserves all existing limits. Finally, the counit of the adjunction is an isomorphism, and the unit is a monomorphism.

Without proof, we mention some of the properties that categories of the form $\mathbf{PAss}(\mathbb{A})$ enjoy: they have finite, disjoint stable coproducts, they have stable epi - regular mono factorizations (these are inherited from the underlying category of sets), and they have so-called *weak dependent products*. This means that pullback functors between slices have weak right adjoints, and may therefore be viewed as a weak version of local cartesian closedness. Weak dependent products are precisely what is needed to show that the exact completion is locally cartesian closed.

Moreover, the object $(\mathbb{A}, Id)$ is a *generic object* in $\mathbf{PAss}(\mathbb{A})$. This means that for any other object (X, α), there is a (non-unique) cartesian map[4] from (X, α) to $(\mathbb{A}, Id)$ (we can take $\alpha : X \to \mathbb{A}$).

If we are prepared to use the axiom of choice in **Sets**, then there is also a *generic proof*; recall that for any category $\mathcal{C}$ there is a functor Prf, called the proof-theoretic powerset functor, from $\mathcal{C}^{\mathrm{op}}$ to the category of posets, sending each object X to the poset reflection of the slice $\mathcal{C}/X$, and which acts on arrows by pullback. An element $[f]$ of $Prf(X)$ is called a *proof*; sometimes one also encounters the term *weak subobject*. Now a generic proof is a map $\phi : \Theta \to \Lambda$ with the property that for any object X and any map $f : Y \to X$ there is a map χ for which the square in the diagram

$$\begin{tikzcd}
Y \arrow[r, shift left, "\alpha"] \arrow[dr, "f"'] & P \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, "g"] & \Theta \arrow[d, "\phi"] \\
 & X \arrow[r, "\chi"] & \Lambda
\end{tikzcd}$$

is a pullback, and where f and g represent the same proof, i.e. there are α, β as in the diagram, for which $f = g\alpha, g = f\beta$. The arrow χ is not required to be unique[5].

To find such a generic proof in Partitioned Assemblies, let Θ be the object $(\{(U, a) | a \in U \subseteq \mathbb{A}\}, \pi_2)$, and $\Lambda = \nabla(\mathcal{P}\mathbb{A})$. There is an evident projection $\pi_1 : \Theta \to \Lambda$, and, using choice, one easily shows that this is a generic proof.

We wish to emphasize at this point, that generic proofs are by no means unique, and that it is an unsolved question whether one can establish the existence of a generic proof in $\mathbf{PAss}(\mathbb{A})$ without an appeal to the axiom of choice. It is not too difficult to show, however, that if the above map is a generic proof, then choice must hold.

Assemblies. Again, let $\mathbb{A}$ be a PCA, and consider the category of *Assemblies over* $\mathbb{A}$, denoted $\mathbf{Ass}(\mathbb{A})$, with objects (X, α), where α is now a map from X to $\mathcal{P}_i(\mathbb{A})$, the object of *inhabited* subsets of $\mathbb{A}$. A morphism $f : (X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ for which $\exists a \in \mathbb{A}\, \forall x \in X\, \forall m \in \alpha(x). a \bullet m \in \beta(f(x))$.

The singleton map $a \mapsto \{a\}$ induces a full embedding $\mathbf{PAss}(\mathbb{A}) \hookrightarrow \mathbf{Ass}(\mathbb{A})$; we do not give this inclusion a name, and we will simply treat partitioned assemblies as assemblies if this is convenient.

Again, there is an adjointness

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift right, "\nabla"'] \arrow[r, phantom, "\perp"] & \mathbf{Ass}(\mathbb{A}) \arrow[l, shift right, "\Gamma"']
\end{tikzcd}$$

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[4] The term "cartesian" will be explained in section 2.5 on fibrations; for now, one can just take it for granted that the cartesian maps in **PAss** are maps in the image of the functor ∇, and pullbacks of such. Menni introduces the terminology "pre-embedding" for such maps, but since we will be interested in the fibrational aspects, we stick to the term "cartesian".

[5] If a category has a generic proof for which classifying maps are unique, then the axiom of choice holds, as Menni has shown.

where we have used the same names Γ, ∇, since they are again global sections and constant objects, respectively. In fact, we have a commutative diagram of adjunctions

$$\begin{tikzcd}
\mathbf{PAss}(\mathbb{A}) \arrow[rr] \arrow[dr, shift left, "\Gamma"] & & \mathbf{Ass}(\mathbb{A}) \arrow[dl, shift right, "\Gamma"'] \\
& \mathbf{Set} \arrow[ul, shift left, "\nabla"] \arrow[ur, shift right, "\nabla"'] &
\end{tikzcd}$$

The category $\mathbf{Ass}(\mathbb{A})$ is a regular category; if $f : (X, \alpha) \to (Y, \beta)$ is a morphism, then we may first factor f in $\mathbf{Set}$ as $f = me$, with m mono, e epi, and then endow the image of f, $Im(f) \subseteq Y$, with the function $\exists_e \alpha : Im(f) \to \mathcal{P}_i\mathbb{A}$, $\exists_e\alpha(y) = \bigcup_{e(x)=y} \alpha(x)$. Then in $\mathbf{Ass}(\mathbb{A})$, we have a factorization

$$\begin{tikzcd}
(X, \alpha) \arrow[r, two heads, "e"] & (Im(f), \exists_e\alpha) \arrow[r, tail, "m"] & (Y, \beta).
\end{tikzcd}$$

As a special instance of this, consider an epi e in $\mathbf{Set}$, and apply the constant objects functor ∇. Then $\nabla(e)$ is a regular epi in $\mathbf{Ass}(\mathbb{A})$, and so is every pullback of $\nabla(e)$. In fact, these epis are already regular epi in $\mathbf{PAss}(\mathbb{A})$, although this category is not regular in general. We thus see that the constant objects functor is a regular functor $\mathbf{Set} \to \mathbf{Ass}(\mathbb{A})$.

The category $\mathbf{Ass}(\mathbb{A})$ is not only regular, but using the characterization given above (theorem 2.2.3) we see that it is in fact the free regular completion of the category $\mathbf{PAss}(\mathbb{A})$. We emphasize, however, that this makes an essential appeal to the axiom of choice in $\mathbf{Set}$: for take any surjection $e : X \to Y$ in $\mathbf{Set}$, and consider $\nabla(e) : \nabla(X) \to \nabla(Y)$ in $\mathbf{Ass}(\mathbb{A})$. If partitioned assemblies are projective in $\mathbf{Ass}(\mathbb{A})$, then in particular so are the objects of the form $\nabla(Z)$. But then the epi $\nabla(e) : \nabla(X) \to \nabla(Y)$ splits, and the splitting induces a splitting of the epi e in $\mathbf{Set}$.

Regular Epis in Partitioned Assemblies? This is a tricky question: can we characterize which epis are regular in $\mathbf{PAss}$? Are there any regular epis which are not cartesian (see section 2.5)? In fact, the latter can be answered affirmatively: note first that, assuming the axiom of choice in $\mathbf{Set}$, every cartesian epi in $\mathbf{PAss}$ is split. We claim that the following construction gives a regular epi which does not split, and therefore is not cartesian: Denote the halting set by H, and its characteristic function by $\phi : \mathbb{N} \to \mathbf{2}$. So there is a morphism in $\mathbf{PAss}$ induced by this function $\phi : (\mathbb{N}, Id) \to \nabla\mathbf{2}$. This is in fact a coequalizer: for if we have any object (X, α) and a morphism $f : (\mathbb{N}, Id) \to (X, \alpha)$ for which $n, n' \in H \Rightarrow f(n) = f(n')$ and $n, n' \notin H \Rightarrow f(n) = f(n')$, then we can derive that the tracking for f must be a constant function. For, if we write $f(n) = x_0$ for all $n \in H$, and $f(n) = x_1$ for all $n \notin H$, then this tracking a of f has the property that $a \bullet n = \alpha(x_0)$ for all $n \in H$, and $a \bullet n = \alpha(x_1)$ for all $n \notin H$. So a is constant, otherwise we could recursively decide H. Thus, for all $m, n \in \mathbb{N}$, $\alpha(m) = \alpha(n)$, and now it is easy to factor f through ϕ. Finally, ϕ cannot be a split epi, because any map from $\nabla\mathbf{2}$ to $(\mathbb{N}, Id)$ must be constant.

Additional Structure. Apart from being regular, the category of Assemblies has quite a lot of structure: it is locally cartesian closed, and it has finite, disjoint stable coproducts. In fact, it is a quasi-topos (see [44]), so the only thing that prevents it from being a topos is that it has no subobject classifier, or, equivalently, that it is not balanced. The object $\nabla\mathbf{2}$ is a classifier for regular subobjects, however.

Furthermore, Assemblies has, like Partitioned Assemblies, a generic object, and, again with the help of choice, a generic proof. We will give a conceptual explanation of the fact that **Ass** has many of the same properies as **PAss**, in the chapter on Ordered PCAs.

Working over an Arbitrary Base. As a final remark on the categories $\mathbf{PAss}(\mathbb{A})$ and $\mathbf{Ass}(\mathbb{A})$, I would like to emphasize that most of what was said above still goes through when we replace the base topos **Set** by an arbitrary elementary topos $\mathcal{E}$. Only the constructions where choice was used (for example, to show the equivalence between $\mathbf{PAss}(\mathbb{A})_{reg}$ and $\mathbf{Ass}(\mathbb{A})$) do not work in general. For this general situation we adopt the notation $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$. We mention the following straightforward change of base result; if $p : \mathcal{E} \to \mathcal{F}$ is a geometric morphism and $\mathbb{A}$ a PCA in $\mathcal{F}$, then p^* induces functors $\mathbf{PAss}_{\mathcal{F}}(\mathbb{A}) \to \mathbf{PAss}_{\mathcal{E}}(p^*\mathbb{A})$ and $\mathbf{Ass}_{\mathcal{F}}(\mathbb{A}) \to \mathbf{Ass}_{\mathcal{E}}(p^*\mathbb{A})$; the first of these preserves finite limits, and the second is regular. Moreover, the diagrams

$$\begin{tikzcd}
\mathbf{PAss}_{\mathcal{E}}(p^*\mathbb{A}) \arrow[d, "\Gamma"'] & \mathbf{PAss}_{\mathcal{F}}(\mathbb{A}) \arrow[l, "p^*"] \arrow[d, "\Gamma"] \\
\mathcal{E} & \mathcal{F} \arrow[l, "p^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Ass}_{\mathcal{E}}(p^*\mathbb{A}) \arrow[d, "\Gamma"'] & \mathbf{Ass}_{\mathcal{F}}(\mathbb{A}) \arrow[l, "p^*"] \arrow[d, "\Gamma"] \\
\mathcal{E} & \mathcal{F} \arrow[l, "p^*"]
\end{tikzcd}$$

commute, and so do the diagrams where we have replaced Γ by its right adjoint ∇.

PERs, Modest Sets. Besides the representational categories $\mathbf{PAss}(\mathbb{A})$ and $\mathbf{Ass}(\mathbb{A})$, there is at least one other interesting category that we can associate to a PCA, and that is the category of *Modest Sets*. The category $\mathbf{Mod}(\mathbb{A})$ is the full subcategory of $\mathbf{Ass}(\mathbb{A})$ on the objects (X, α) with the following property: for all $x, x' \in X : \alpha(x) \cap \alpha(x') \neq \emptyset \Rightarrow x = x'$. Thus, a partitioned assembly (X, α) is modest precisely if the map α is an injection, and so we can identify the modest partitioned assemblies with subsets of $\mathbb{A}$. The category $\mathbf{Mod}(\mathbb{A})$ admits an alternative description via PERs, partial equivalence relations on $\mathbb{A}$. It is easy to see that each modest set (X, α) determines and is determined by the partial equivalence relation on $\mathbb{A}$ given by $a \sim b \Leftrightarrow \exists x \in X.a \in \alpha(x) \& b \in \alpha(x)$. The description in terms of PERs makes it easy to see that $\mathbf{Mod}(\mathbb{A})$ is a *small* category, i.e. only has a set of objects.

Literature. Menni's thesis [57] contains lots of facts about Partitioned Assemblies, Assemblies and the special properties that the inclusion of Sets has. Among the papers in which completions are applied to these categories are [17, 18, 63].

For more on modest sets we refer to [67].

2.5 Fibrations and Indexed Categories

Fibrations. Let $p : \mathcal{C} \to \mathcal{E}$ be a functor. A map $f : X \to Y$ in $\mathcal{C}$ is said to be *cartesian* if for any $g : Z \to Y$ in $\mathcal{C}$, and any $v : p(Z) \to p(X)$ for which $p(f) \circ v = p(g)$, there is a unique $w : Z \to X$ in $\mathcal{C}$ for which $f \circ w = g$, and $p(w) = v$. The functor p is said to be a *fibration* if every map $m : I \to p(Y)$ in $\mathcal{E}$ has a cartesian lifting, i.e. a map $f : X \to Y$ in $\mathcal{C}$ with $p(f) = m$. Cartesian liftings, if they exist, are unique up to isomorphism, by which we mean that if $f : X \to Y$ and $f' : X' \to Y$ are two cartesian liftings of some arrow, then f and f' are isomorphic as objects of the slice category $\mathcal{C}/Y$.

As an example of a fibration that will be of use to us, consider the forgetful functor $\Gamma : \mathbf{PAss}(\mathbb{A}) \to \mathbf{Set}$. A cartesian lifting of $m : X \to \Gamma(Y, \beta) = Y$ given by $m : (X, \alpha) \to (Y, \beta)$, with $\alpha(x) = \beta(m(x))$. Similarly, the functor $\Gamma : \mathbf{Ass}(\mathbb{A}) \to \mathbf{Set}$ is a fibration.

Indexed Categories.

Definition 2.5.1 Let $\mathcal{E}$ be a base category. An *$\mathcal{E}$-indexed category* is a pseudofunctor $\mathbb{P} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$. Explicitly, we have, for each $X \in \mathcal{E}$, a (small) category $\mathbb{P}(X)$, and for each map $f : X \to Y$ a reindexing functor $\mathbb{P}(f) : \mathbb{P}(Y) \to \mathbb{P}(X)$, such that there are coherent natural isomorphisms $\mathbb{P}(f) \circ \mathbb{P}(Id) \cong \mathbb{P}(f) \cong \mathbb{P}(Id) \circ \mathbb{P}(f)$ and $\mathbb{P}(g) \circ \mathbb{P}(f) \cong \mathbb{P}(fg)$.

Every fibration gives rise to an indexed category, and vice versa; if $p : \mathcal{C} \to \mathcal{E}$ is a fibration, then the *fibre of p over $I \in \mathcal{E}$* is the subcategory of $\mathcal{C}$ with as objects those X with $p(X) = I$, and as morphisms those $f : X \to Y$ for which $p(f) = Id$ (such morphisms are called *vertical*). If $u : I \to J$ is any map in $\mathcal{E}$ then we obtain a reindexing functor $p^*(u)$ from the fibre over J to the fibre over I by sending an object X over J to the domain of the cartesian lifting of u.

Conversely, if $\mathbb{P}$ is an $\mathcal{E}$-indexed category, we define the *total category,* or *category of elements* $\mathcal{G}(\mathbb{P})$ of $\mathbb{P}$ to have objects (X, α), with $X \in \mathcal{E}$, $\alpha \in \mathbb{P}(X)$; maps $(X, \alpha) \to (Y, \beta)$ are pairs (f, m) with $f : X \to Y$ in $\mathcal{E}$, and $m : \alpha \to \mathbb{P}(f)(\beta)$ in $\mathbb{P}(X)$. The forgetful functor sending (X, α) to X is then a fibration.

Under the equivalence of indexed categories and fibrations, the fibrations for which the functor $p : \mathcal{C} \to \mathcal{E}$ is faithful correspond to indexed preorders, i.e. indexed categories for which each $\mathbb{P}(X)$ is a preorder. In this case, we denote the preorder in $\mathbb{P}(X)$ by $\vdash_X$. The presentation of the total category $\mathcal{G}(\mathbb{P})$ can then be simplified: morphisms $(X, \alpha) \to (Y, \beta)$ are now simply maps $f : X \to Y$ for which $\alpha \vdash_X \mathbb{P}(f)(\beta)$.

Some indexed categories have the special property that they have a *canonical presentation*. This means, that there is an object Σ of the base category $\mathcal{E}$, such that $\mathbb{P}(X) = \mathcal{E}(X, \Sigma)$, and where reindexing is given as composition. The object Σ is then called a generic object for the indexed category.

Logic of Indexed Preorders. If we have a canonically presented $\mathcal{E}$-indexed preorder $\mathcal{E}(-, \Sigma)$, then we think of the objects of $\mathcal{E}$ as types, and of the elements of $\mathcal{E}(X, \Sigma)$ as predicates with a free variable $x : X$. The preorder on $\mathcal{E}(X, \Sigma)$

is now thought of as *entailment*. We will often use the following notation, which facilitates reasoning in/about the logic of the indexed preorder: for $\alpha, \beta : X \to \Sigma$, we write $\alpha(x) \vdash_x \beta(x)$ for $\alpha \vdash_X \beta$. This notation has the advantage that it easily deals with predicates with more free variables, e.g. instead of $\alpha \circ \langle \pi_2, \pi_1 \rangle \vdash_{X \times Y \times Z} \beta \circ \pi_3$ we simply say $\alpha(y, x) \vdash_{x,y,z} \beta(z)$.

Completeness and Cocompleteness. If $\mathbb{P}$ is an $\mathcal{E}$-indexed category, we say that $\mathbb{P}$ is *finitely $\mathcal{E}$-complete*, or has $\mathcal{E}$-indexed finite limits, if each $\mathbb{P}(X)$ has finite limits, and each reindexing functor preserves those finite limits. Similarly, finite $\mathcal{E}$-cocompleteness is defined.

Next, define $\mathbb{P}$ to have $\mathcal{E}$-indexed (co)products if every reindexing functor $\mathbb{P}(f)$ has a right (left) adjoint $\forall_f$ ($\exists_f$), such that the Beck-Chevalley condition is satisfied (shown here only for $\exists_f$): if the left square below is a pullback in $\mathcal{E}$, then the canonical natural transformation $\exists_f \mathbb{P}(g) \Rightarrow \mathbb{P}(h)\exists_k$ is an isomorphism:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & Q \arrow[d, "h"] \\
X \arrow[r, "k"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{P}(P) \arrow[r, "\exists_f"] & \mathbb{P}(Q) \\
\mathbb{P}(X) \arrow[r, "\exists_k"] \arrow[u, "\mathbb{P}(g)"] & \mathbb{P}(Y). \arrow[u, "\mathbb{P}(h)"']
\end{tikzcd}$$

Finally, we say that $\mathbb{P}$ is *$\mathcal{E}$-(co)complete* if $\mathbb{P}$ has both $\mathcal{E}$-indexed finite (co)limits and $\mathcal{E}$-indexed (co)products.

Triposes. A *tripos* $\mathbb{P}$ on a cartesian category $\mathcal{E}$ is a special kind of indexed preorder, namely one that satisfies:

- $\mathbb{P}$ is $\mathcal{E}$-complete and $\mathcal{E}$-cocomplete;
- each preorder $\mathbb{P}(X)$ is a *Heyting Pre-Algebra*, i.e. has $\wedge, \vee, \top, \bot, \Rightarrow$, all preserved by reindexing, and
- $\mathbb{P}$ has a *generic predicate*, i.e. for each X in $\mathcal{E}$ there are objects PX in $\mathcal{E}$ and ϵ_X in $\mathbb{P}(X \times PX)$, with the property that for α in $\mathbb{P}(X \times Y)$, there is a map $\{\alpha\} : Y \to P(X)$ such that $\mathbb{P}(Id_X \times \{\alpha\})(\epsilon_X) \dashv\vdash_{X \times Y} \phi$.

Again, we think of the objects of the base category $\mathcal{E}$ as types, and of the elements of $\mathbb{P}(X)$ as predicates with a free variable of type X.

Note that $\alpha \vdash_X \beta$ iff $\top_X \vdash_X \alpha \Rightarrow \beta$ iff $\top_1 \vdash_1 \forall_x(\alpha \Rightarrow \beta)$. Elements in the fibre over 1 for which $\top_1 \vdash_1 p$ are called *designated truth-values.* Thus we see that the tripos-structure is determined up to isomorphism by the set of designated truth-values, and by $\Rightarrow$ and $\forall$.

In case the base category $\mathcal{E}$ is a topos, we may assume that the tripos $\mathbb{P}$ is canonically presented, i.e. that there is an object Σ, such that each $\mathbb{P}(X)$ is isomorphic to $\mathcal{E}(X, \Sigma)$. We may think of Σ as the type of truth-values, and if $\alpha : X \to \Sigma$ is a predicate over X, then $\alpha(x)$ denotes the extent to which $x \in \alpha$ is true. If Σ is a generic predicate, then we may transport the propositional structure to Σ and obtain maps $\wedge : \Sigma \times \Sigma \to \Sigma$, $\top : 1 \to \Sigma$, etcetera, such that the propositional structure of each fibre $\mathbb{P}(X)$ is given by composing with these maps. Reindexing then preserves the propositional structure on the nose, rather than up to isomorphism. We cannot always do the same with the quantifiers, in the sense that we cannot hope that, for example, existential quantification is determined by a map $\bigvee : \Sigma^{\Sigma} \to \Sigma$. When this is the case, the tripos is said to have *fibrewise quantification.*

Examples. I will only give some brief examples here, since we will encounter many more in the rest of the thesis. First of all, starting with a topos $\mathcal{E}$, the assignment $X \mapsto Sub_{\mathcal{E}}(X)$ defines a tripos, and we may take the subobject classifier as a generic object for this tripos.

Second, if H is a locale in $\mathcal{E}$, then the assignment $X \mapsto \mathcal{E}(X, H)$ is a canonically presented tripos, where all structure on $\mathcal{E}(X, H)$ is taken pointwise.

Finally, if $\mathbb{A}$ is a PCA in $\mathcal{E}$ then we can define the realizability tripos associated with $\mathbb{A}$, to be $X \mapsto \mathcal{E}(X, \mathcal{P}(\mathbb{A}))$ preordered by

$$\alpha \vdash_X \beta \Leftrightarrow \exists m \in \mathbb{A}\, \forall x \in X\, \forall a \in \alpha(x) : m \bullet a \in \beta(x).$$

Tripos-to-topos Construction. One of the points of the definition of a tripos is, that we can build a topos out of it. For an $\mathcal{E}$-tripos $\mathbb{P}$, this topos is denoted $\mathcal{E}[\mathbb{P}]$, and is constructed as follows:

Objects: pairs $(X, =_X)$, where X is an object of $\mathcal{E}$ and $=_X \in \mathbb{P}(X \times X)$ satisfying

- $x =_X x' \vdash_{x,x'} x' =_X x$
- $x =_X x' \wedge x' =_X x'' \vdash_{x,x',x''} x =_X x''$

Morphisms: a morphism from $(X, =_X)$ to $(Y, =_Y)$ is an equivalence class of functional relations $F \in \mathbb{P}(X \times Y)$, satisfying

- $F(x, y) \vdash_{x,y} x =_X x \wedge y =_Y y$
- $F(x, y) \wedge x =_X x' \wedge y =_Y y' \vdash_{x,x',y,y'} F(x', y')$
- $F(x, y) \wedge F(x, y') \vdash_{x,y,y'} y =_Y y'$
- $x =_X x \vdash_x \exists_y F(x, y)$,

and where two such F, G are equivalent iff $F(x, y) \dashv\vdash_{x,y} G(x, y)$.

There is a constant objects functor $\nabla : \mathcal{E} \to \mathcal{E}[\mathbb{P}]$. Every object in the topos $\mathcal{E}[\mathbb{P}]$ is a subquotient of an object of the form $\nabla(X)$. Constant objects have the property that $Sub_{\mathcal{E}[\mathbb{P}]}(\nabla X) \cong \mathbb{P}(X)$.

If we apply the above construction to the canonical tripos $\mathcal{E}(-, H)$ for a locale H, then the resulting topos is equivalent to $\mathbf{Sh}(H)$, the topos of sheaves over H. In this case, the constant objects functor is the inverse image of the unique geometric morphism $\mathbf{Sh}(H) \to \mathbf{Set}$.

For a PCA $\mathbb{A}$, application of the tripos-to-topos construction to the canonical tripos $\mathcal{E}(-, \mathcal{P}(\mathbb{A}))$ yields the *realizability topos* $\mathbf{RT}(\mathbb{A})$. Now (except for the case when $\mathbb{A}$ is a trivial one-point PCA) this topos is not Grothendieck, and the constant objects functor is *right* adjoint to the global sections functor, presenting the topos $\mathcal{E}$ as a sheaf subtopos of $\mathbf{RT}(\mathbb{A})$.

The Effective Topos. The main example of a topos of the form described in the above paragraph, is of course the Effective Topos, **Eff**. We state some properties that will be of interest to us. First of all, both the categories **PAss** and **Ass** are full subcategories of **Eff**; the category of Partitioned Assemblies is equivalent to the full subcategory of projectives in **Eff**, whereas **Ass** is equivalent to the full subcategory on the separated objects for the double negation topology, and also to the full subcategory on the subobjects of the objects in the image of the functor **Set** → **Eff**. From this, we can derive two important presentations of the Effective Topos:

Theorem 2.5.2

1. $\mathbf{Eff} \simeq \mathbf{Ass}_{ex/reg}$
2. $\mathbf{Eff} \simeq \mathbf{PAss}_{ex}$.

Literature. The most comprehensive survey on fibrations and indexed categories is “Categorical Logic and Type Theory”, by Bart Jacobs [41]. Another useful, introductory text is Thomas Streicher’s lecture notes [68], based on work by Bénabou.

Concerning triposes, there is the paper “Tripos Theory”, by Hyland, Johnstone and Pitts [38], and Pitts’ Ph.D. thesis [59]. The thesis contains a large amount of examples of triposes, among which various interesting combinations of localic- and realizability triposes. We also mention his “Tripos Theory in Retrospect”, [60].

For the Effective Topos, [36] is the classical reference. The first part of theorem 2.5.2 is shown in [18], the second in [63]. Carboni’s [17] also sheds some light on these matters.

Chapter 3

Ordered Partial Combinatory Algebras

This chapter is a slightly extended and improved version of a joint paper with Jaap van Oosten, which has been published in the Mathematical Proceedings of the Cambridge Philosophical Society ([34]). As soon as the paper was published, I discovered a gap in one of the results; luckily, the essence of the result remained true, only the formulation had to be made a bit more subtle. The flaw concerned the characterization of geometric morphisms between realizability toposes in terms of maps between ordered PCAs. This characterization was formulated in terms of computational density, and the definition of the latter was a bit too rigid, with the consequence that not all geometric morphisms corresponded to a computationally dense map. The relaxed definition (definition 3.3.5) is presented in this version, and now lemma 3.3.7 is true.

Two pieces of text are added to the paper; the first is a short comment on the effective monad (section 3.5.2). The second is a bit more elaborate, and consists of a rudimentary calculation of a colimit (in a weak sense) for the hierarchy of realizability toposes that arise from the theory of ordered PCAs.

Otherwise, we have only changed some of the notation in order to make the text cohere with the preliminaries and the other chapters in the thesis.

3.1 Introduction

Partial Combinatory Algebras, models for a form of Combinatory Logic with partial application, have been studied for the last thirty years because of their close connection to Intuitionistic Logic (see, for example, [71]).

From the "algebraic" side, Partial Combinatory Algebras gave rise to the construction of *elementary toposes*: for every partial combinatory algebra $\mathbb{A}$ we have the *realizability topos* $\mathbf{RT}[\mathbb{A}]$. See chapter 2 for details.

This work is motivated by the question: what would be a good category for partial combinatory algebras (PCAs), such that the construction of a realizability topos **RT**[$\mathbb{A}$] out of $\mathbb{A}$ becomes a functor with nice properties? Of course, this depends on one's point of view as to which category these realizability toposes live in. Some functoriality is obtained in John Longley's thesis [54]; he defines a 2-category of PCAs, such that morphisms in this category correspond to certain exact functors between realizability toposes.

In this chapter, we are mainly interested in *geometric morphisms* between realizability toposes. Our approach is both a refinement and an analysis of Longley's. First, we propose the notion of an *ordered partial combinatory algebra* (OPCA), a generalization of PCAs. The standard construction of realizability toposes goes through for these ordered PCAs. This is reviewed in the first section.

However, the context of OPCAs allows some constructions which are not avaliable for PCAs. This becomes apparent when we introduce a 2-category for ordered PCAs, **OPCA+**. On this category, there is a 2-monad, the non-empty downset monad, T. Whereas Longley's morphisms are certain total relations, we are able to work with functions and recover his category as follows: Longley's 2-category of PCAs is a full subcategory of the Kleisli category **Kl**(T) for our monad T, on objects with are in fact genuine PCAs. There is a 2-functor from **Kl**(T) to the 2-category of realizability triposes and exact functors between them; this functor is locally an equivalence, so that, up to 2-isomorphism, maps in **Kl**(T) between two fixed OPCAs are the same as exact functors between the associated triposes.

The next step is to impose a restriction on OPCA-maps, obtaining a subcategory **OPCA**, to which the monad T restricts. The idea is that the maps in **OPCA** are precisely the maps which induce geometric morphisms between triposes. Then we obtain a 2-functor from the Kleisli category for the monad on **OPCA** to the 2-category of triposes and geometric morphisms, and this 2-functor is again a local equivalence.

In the third section we focus on (pseudo-) algebras for our monad, and we consider the category **Pass**($\mathbb{A}$) of Partitioned Assemblies associated to an ordered PCA $\mathbb{A}$. We obtain the result that **Pass**($\mathbb{A}$) is regular if and only if $\mathbb{A}$ has a pseudo-algebra structure. Moreover, this category is a regular completion (of a category that is again of the form **Pass**($\mathbb{B}$)) if and only if $\mathbb{A}$ is equivalent to a free algebra $T\mathbb{B}$.

Then we discuss some applications of our framework. The first one concerns relative realizability (see [3]); the main result is a characterization of those sub-OPCAs $\mathbb{A}$ of some $\mathbb{B}$ for which there is a local map from **RT**[$\mathbb{B}$] to **RT**[$\mathbb{A}$]. In other words, we give a necessary and sufficient condition so that the relative realizability topos **RT**[$\mathbb{B}, \mathbb{A}$] coincides with **RT**[$\mathbb{B}$].

Then, we have a look at the Effective Monad from the perspective of our category of ordered PCAs, and it is shown that the Effective Monad is induced by a comonad on the category of ordered PCAs.

As a third application we give a slight generalization of a theorem by Johnstone and Robinson, stating that the Effective Topos is not equivalent to any topos obtained from a total combinatory algebra.

In section 6 we study iteration of the downset-construction, in order to give a presentation of a hierarchy of realizability toposes, induced by the sequence of OPCAs $\mathbb{A}, T\mathbb{A}, T^2\mathbb{A}, \ldots$. The fact that certain hierarchies can be presented in this tripos-theoretic way was already conjectured by Menni [57].

Then we ask ourselves whether there is a colimit for such hierarchies. There don't seem to be reasons why the colimit should exist, but we give a presentation of an interesing cocone, which still has a useful universal property.

3.2 Definitions and Basic Properties

This section sets out the definitions and reviews basic properties. We define ordered PCAs, the standard realizability tripos $I(\mathbb{A})$ for an ordered PCA $\mathbb{A}$ and the associated categories of assemblies and partitioned assemblies. Most of the well-known properties of these structures for ordinary PCAs carry over easily to the ordered case; proofs are omitted.

3.2.1 Ordered PCAs

Definition 3.2.1 An *ordered PCA* is a triple $\mathbb{A} = (A, \leq, \bullet)$, where $\leq$ partially orders the set A, and where $\bullet$ is a partial function from $A \times A$ to A. We write $a \bullet b\!\downarrow$ or $ab\!\downarrow$ if (a, b) is in the domain of $\bullet$, in which case $a \bullet b$ or ab denote the value. We require that the following conditions are satisfied:

1. For all $a, b \in A$: if $ab\!\downarrow$, $a' \leq a$ and $b' \leq b$, then $a'b'\!\downarrow$ and $a'b' \leq ab$.
2. There are elements k and s of A that satisfy
   - for all $a, b \in A$: $ka\!\downarrow$ and $kab\!\downarrow$ and $kab \leq a$,
   - for all $a, b, c \in A$: $sa\!\downarrow$ and $sab\!\downarrow$ and if $(ac)(bc)\!\downarrow$, then $sabc\!\downarrow$ and $sabc \leq (ac)(bc)$.

Of course, every ordinary PCA can be seen as an ordered PCA, by taking the discrete ordering.

The motivating example for the definition of ordered PCAs in [75] (where they are called $\leq$-PCAs; however, this terminology is hard to pronounce) is the following: given a PCA A, the set of nonempty subsets of A (or the set of nonempty finite subsets of A) forms an ordered PCA (but not a PCA!) by putting

$$\alpha \bullet \beta \;=\; \{xy \,|\, x \in \alpha, y \in \beta\}$$

(This is defined if for all $x \in \alpha$ and $y \in \beta$, $xy\!\downarrow$)

A fundamental property of PCAs is their so-called *combinatorial completeness.* Up to $\leq$, this remains true for ordered PCAs:

Proposition 3.2.2 (Combinatorial completeness) *Let $\mathbb{A}$ be an ordered PCA. For any term t composed of elements of A, application and variables x, $x_1, \ldots, x_n$,* *there is a term* $[\Lambda x.t]$*, containing at most the variables* $x_1, \ldots, x_n$*, such that for all elements* $a, a_1, \ldots, a_n \in A$*: if* $t[a/x, a_1/x_1, \ldots, a_n/x_n]\downarrow$ *then*

$$([\Lambda x.t][a_1/x_1, \ldots, a_n/x_n])a\downarrow$$

and

$$([\Lambda x.t][a_1/x_1, \ldots, a_n/x_n])a \leq t[a/x, a_1/x_1, \ldots, a_n/x_n]$$

As was already remarked in [75], the proof is an easy adaptation of the standard case.

From Proposition 3.2.2 it follows that there are pairing operations, written j, j_0, j_1 that satisfy

$$j_0(j(a,b)) \leq a, \qquad j_1(j(a,b)) \leq b.$$

It is well-known that every PCA is either infinite or consists of only one element (One way of understanding this is to observe first that, using k and s one can construct all the *numerals* $\bar{0}, \bar{1}, \ldots$, and then to remark that these all have to be distinct, if $k \neq s$). For ordered PCAs there are other possibilities, as becomes apparent after the following definition:

Definition 3.2.3 An ordered PCA is called *trivial* if it has a least element, and it is called *pseudo-trivial* if there is an element that serves both as k and as s.

An example of a pseudo-trivial ordered PCA that is not trivial is provided by a meet-semilattice (without a least element, of course; application is given by meet). We have the following characterization:

Lemma 3.2.4 *For any ordered PCA* $\mathbb{A}$ *the following statements are equivalent:*

1. $\mathbb{A}$ *is pseudo-trivial,*
2. *there is an element* u *such that* $u \leq k = true$ *and* $u \leq sk = false$*,*
3. *any two elements have a lower bound (not necessarily a meet),*
4. *there are natural numbers* n, m *such that* $n \neq m$*, but* $\overline{n}$ *and* $\overline{m}$ *have a lower bound (*$\overline{n}$ *denotes the element that corresponds to* n *for some coding of the natural numbers).*

Proof. (1) $\Rightarrow$ (3): consider the element $u = skkk = kskk$. We have $skkk \leq kk(kk) \leq k$, but also $kskk \leq sk$. Now $kxy \leq x$, so $(skkk)xy \leq x$. And $skxy \leq y$, so $(kskk)xy \leq y$, and we have found that $(skkk)xy = (kskk)xy = uxy$ is a lower bound of any x and y.

(2) $\Rightarrow$ (1): take u with $u \leq k$ and $u \leq sk$. Then uks is a lower bound for k and s, and this lower bound serves both as k and as s.

(3) $\Rightarrow$ (1), (2), (4) are trivial.

(4) $\Rightarrow$ (2): suppose $m > n$ and $x \leq \overline{m}$ and $x \leq \overline{n}$. We have, by combinatorial completeness, terms *zero* and *pred*, that test for zero and take the predecessor.

To be more precise: $zero \bullet \overline{p} \leq k$ if $p = 0$, and $zero \bullet \overline{p} \leq sk$ if $p \neq 0$, $pred {\bullet} \overline{p} \leq \overline{p-1}$: Now we find that $zero(pred^n {\bullet} \overline{m}) \leq sk$ and $zero(pred^n {\bullet} \overline{m}) \leq k$. So for x this implies $zero(pred^n \bullet x) \leq sk$ and $zero(pred^n \bullet x) \leq k$.

□

3.2.2 Triposes for Ordered PCAs

Recall from chapter 2, section 4 the construction of a tripos, and hence of a realizability topos out of a partial combinatory algebra. We give the straightforward generalization to ordered PCAs.

Given a ordered PCA $\mathbb{A} = (A, \leq, \bullet)$, define $I(\mathbb{A})$ as the set of all *downsets* in A, that is,

$$I(\mathbb{A}) = \{\alpha \subseteq A \mid \forall a \in \alpha, \forall a' \in A(a' \leq a \rightarrow a' \in \alpha)\}.$$

The *standard realizability tripos* on $\mathbb{A}$, also denoted $I(\mathbb{A})$ assigns to any set X the set of functions $I(\mathbb{A})^X$; reindexing is given by composition. The tripos structure is a straightforward generalisation of the PCA case: for $\phi, \psi \in I(\mathbb{A})^X$, we put

$$\phi \vdash \psi \quad \text{iff} \quad \exists a \in A \; \forall x \in X \; \forall b \in \phi(x) : ab{\downarrow} \;\; \& \; ab \in \psi(x)$$

We leave the rest of the structure to the reader.

The topos represented by the tripos $I(\mathbb{A})$ is denoted by $\mathbf{RT}[\mathbb{A}]$.

Remark. It is easily seen that $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{Set}$ if $\mathbb{A}$ is trivial. Moreover, if $\mathbb{A}$ is a meet-semilattice, then $\mathbf{RT}[\mathbb{A}]$ is a filter quotient of the presheaf topos $\mathbf{Set}^{\mathbb{A}^{op}}$ (see [75]).

Remark. A possible confusion might arise if one considers PCAs like Scott's $P(\omega)$ or some examples from domain theory, which have a partial order such that requirement 1. of Definition 3.2.1 is satisfied. Considered as OPCA, $P(\omega)$ is trivial, so $\mathbf{RT}[P(\omega)] \simeq \mathbf{Set}$, in contrast to the realizability topos over $P(\omega)$ as PCA!

3.2.3 Toposes, Assemblies and Partitioned Assemblies

This section shows that it is straightforward to associate categories of Partitioned Assemblies and of Assemblies to an ordered PCA, just as was done in chapter 2, section 4 for an ordinary PCA.

The category of *assemblies* over $\mathbb{A}$, $\mathbf{Ass}(\mathbb{A})$, has as objects pairs of form (X, ϵ_X) where X is a set and $\epsilon_X : X \rightarrow I(\mathbb{A})$ a function such that $\epsilon_X(x) \neq \emptyset$ for each $x \in X$; a morphism $(X, \epsilon_X) \rightarrow (Y, \epsilon_Y)$ is a function $f : X \rightarrow Y$ such that there is an $a \in \mathbb{A}$ such that for all $x \in X$ and all $b \in \epsilon_X(x)$, $ab{\downarrow}$ and $ab \in \epsilon_Y(f(x))$ (one says that f is *tracked* by a).

The category of *partitioned assemblies* over $\mathbb{A}$, $\mathbf{Pass}(\mathbb{A})$, is the full subcategory of $\mathbf{Ass}(\mathbb{A})$ on objects (X, ϵ_X) where for each $x \in X$, $\epsilon_X(x)$ is a principal downset ${\downarrow}(a) = \{b \in \mathbb{A} \mid b \leq a\}$. When working in $\mathbf{Pass}(\mathbb{A})$ we will simply take ϵ_X to be a function $X \rightarrow \mathbb{A}$.

Ass($\mathbb{A}$) and **Pass**($\mathbb{A}$) are full subcategories of **RT**[$\mathbb{A}$] and closed under finite limits. We have the usual results, that **Ass**($\mathbb{A}$) is equivalent to the category of $\neg\neg$-separated objects in **RT**[$\mathbb{A}$], and **Pass**($\mathbb{A}$) is equivalent to the category of projective objects of **RT**[$\mathbb{A}$]. **RT**[$\mathbb{A}$] has enough projectives, and is therefore the exact completion of **Pass**($\mathbb{A}$); **Ass**($\mathbb{A}$) is the regular completion of **Pass**($\mathbb{A}$).

3.3 A 2-Category for Ordered PCAs

In Longley's thesis [54], we find a description of a 2-category of PCAs. The definition of a morphism between two PCAs is chosen in such a way, that there is a correspondence between such morphisms and certain exact functors between the associated realizability toposes.

In Longley's framework, a morphism $\phi : \mathbb{A} \to \mathbb{B}$ of PCAs is defined to be a total relation from $\mathbb{A}$ to $\mathbb{B}$, for which there is an element $r \in \mathbb{B}$ such that if $\phi(a, b)$, $\phi(a', b')$ and $aa'{\downarrow}$ hold, then $rbb'{\downarrow}$ and $\phi(aa', rbb')$ hold.

In the context of ordered PCAs, we can redefine this with functions (instead of relations), and recover Longley's definition with the help of the monad structure on ordered PCAs, discussed in 3.3.2.

Now the succes of Longley's definition is easily seen to depend crucially on the following theorem by Pitts (see [59], Remark 4.10 (ii)):

Theorem 3.3.1 *Let $\mathbb{A}$ and $\mathbb{B}$ be PCAs. There is a one-to-one correspondence between*

1. **Set**-*indexed functors from $I(\mathbb{A})$ to $I(\mathbb{B})$ that preserve $T, \wedge$ and $\exists$, and*
2. *functions $f : \mathbb{A} \to P(\mathbb{B})$ such that $f(a) \neq \emptyset$ for all a, and moreover $\bigcap_{a,a' \in Dom(\bullet)} f(a) \to (f(a') \to f(aa')) \neq \emptyset$.*

We will also base our definition on this theorem ourselves, but we are more interested in geometric morphisms than in exact functors, so an important part of our approach will be a characterization of those functions between ordered PCAs that induce geometric morphisms between the realizability toposes.

3.3.1 The category OPCA+

As a first approximation, we present a category for ordered PCAs, that is suitable for studying exact functors between realizability triposes, and generalizes Longley's 2-category for PCAs. The objects are, of course, ordered PCAs. For morphisms, we introduce the following definition:

Definition 3.3.2 Let $\mathbb{A}$ and $\mathbb{B}$ be ordered PCAs, and let $f : \mathbb{A} \to \mathbb{B}$ be a function. We say that f is a *morphism of ordered PCAs* (or *OPCA-map*) if:

- there exists an element $r \in \mathbb{B}$ such that $aa'{\downarrow} \Rightarrow (r \bullet f(a)) \bullet f(a'){\downarrow}$ and $(r \bullet f(a)) \bullet f(a') \leq f(aa')$.

- there exists an element $u \in \mathbb{B}$ such that $a \leq a' \Rightarrow u \bullet f(a){\downarrow}\ \ \& \, u \bullet f(a) \leq f(a')$

It is easily verified that composition is well-defined. We will write **OPCA+** for this category.

Next, we observe that the Hom-sets of this category are pre-ordered sets if we define, for $f, g : \mathbb{A} \to \mathbb{B} : f \leq g$ iff $\exists b \in \mathbb{B} : b \bullet f(a){\downarrow}\ \ \& \ b \bullet f(a) \leq g(a)$ for all $a \in \mathbb{A}$. Since composition of morphisms preserves this ordering, in the sense that $f \leq g \Rightarrow fh \leq gh$ and $kf \leq kg$, we see that **OPCA+** is a pre-order enriched category. We write $f \sim g$ for $f \leq g \ \& \ g \leq f$, and we say that f and g are *isomorphic* as morphisms.

It is good to observe that a map $f : \mathbb{A} \to \mathbb{B}$ provides us with a description of $\mathbb{A}$ as an *internal* ordered PCA in the topos $\mathbf{RT}[\mathbb{B}]$. The underlying set of this (canonically projective) object is the underlying set of $\mathbb{A}$, and the existence predicate is given by $E_f(a) = {\downarrow}(f(a))$. Moreover, if we have $f, g : \mathbb{A} \to \mathbb{B}$, then $f \leq g$ iff, internally in $\mathbf{RT}[\mathbb{B}]$, the identity on $\mathbb{A}$ is a map $(\mathbb{A}, E_f) \to (\mathbb{A}, E_g)$.

Remarks. The structure of the category **OPCA+** is not particularly impressive. We mention the following:

1. (This generalizes an observation by Longley.) The terminal object in **OPCA+** is the one-point ordered PCA. For any other trivial $\mathbb{A}$, there is, for any $\mathbb{B}$, always a morphism $f : \mathbb{B} \to \mathbb{A}$. This f is unique up to isomorphism. Trivial ordered PCAs are also pseudo-initial, in the sense that for any other ordered PCA $\mathbb{B}$, there is always a map into $\mathbb{B}$, and any two such maps are isomorphic.

   Apart from this, we can observe that any constant function between ordered PCAs is a morphism, and that any two constant maps are isomorphic.

2. The category **OPCA+** has products: given $\mathbb{A}$ and $\mathbb{B}$, we define $\mathbb{A} \times \mathbb{B}$ as $\mathbb{A} \times \mathbb{B} = (A \times B, \bullet, \leq)$ with $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$, $(a, b) \bullet (a', b'){\downarrow}$ iff $aa'{\downarrow}$ and $bb'{\downarrow}$, in which case $(a, b) \bullet (a', b') = (aa', bb')$. The pairs $(k_A, k_B), (s_A, s_B)$ serve as k and s in the product.

3. Monos and epis are just injective and surjective maps, respectively. For, consider a map $f : \mathbb{A} \to \mathbb{B}$ that is not injective, say $f(a) = f(a')$. Then we take two (different) maps $1 \to \mathbb{A}$ sending the unique element to a and a', respectively. Their composites with f are obviously equal.

   If $f : \mathbb{A} \to \mathbb{B}$ is not surjective, then there is some element $b_0 \in \mathbb{B}$ that is outside the image of f. Consider the trivial structure $\mathbb{P}$ consisting of two elements p, q with $p \leq q$. Now define maps $g, h : \mathbb{B} \to \mathbb{P}$ by

   $$g(b) = \begin{cases} q & \text{if } \ b_0 < b \\ p & \text{otherwise,} \end{cases} \qquad h(b) = \begin{cases} q & \text{if } \ b_0 \leq b \\ p & \text{otherwise.} \end{cases}$$

   It is not hard to verify that these are indeed morphisms in our category, and that $gf = hf$, but not $g = h$.

4. Equalizers do not exist in **OPCA+**. The reason is simple: if we have two structures $\mathbb{A}$, $\mathbb{B}$, then we can take two different constant maps. Their equalizer would have to have the empty set as underlying set, but no such ordered PCA exists.

3.3.2 The Downset-monad

Now we describe a monad $(T, \delta, \cup)$ on **OPCA+**. On objects, we define

$$T\mathbb{A} = (\{\alpha \mid \alpha \in I\mathbb{A}, \alpha \neq \emptyset\}, \subseteq, \bullet).$$

So the underlying set of $T\mathbb{A}$ consists of all nonempty downsets in $\mathbb{A}$. It is ordered by inclusion, and partial application is defined by $\alpha \bullet \beta\downarrow$ iff $\forall a \in \alpha\ \forall b \in \beta\ ab\downarrow$, and if $\alpha \bullet \beta\downarrow$ then $\alpha \bullet \beta = \downarrow\{ab \mid a \in \alpha, b \in \beta\}$. It is not hard to verify that this gives again a ordered PCA, with $\downarrow(k)$ and $\downarrow(s)$ serving as combinators. Also, there is a map $\delta : \mathbb{A} \to T\mathbb{A}$, given by $\delta(a) = \downarrow(a)$.

For a morphism $f : \mathbb{A} \to \mathbb{B}$, we put $Tf(\alpha) = \bigcup_{a\in\alpha} \downarrow(f(a))$. It is easily verified that this is well-defined. Finally, it is clear that composition and identities are preserved, so T is indeed an endofunctor. Actually, it is an endo-2-functor, since it preserves the ordering on morphisms (in fact it also reflects the order).

Now let $\cup : T^2\mathbb{A} \to T\mathbb{A}$ be the map given by union: $\cup\xi = \{a \in \mathbb{A} \mid \exists\alpha \in \xi : a \in \alpha\}$. The verifications that both δ and $\cup$ are natural transformations, and that the monad identities are satisfied are left to the reader.

Lemma 3.3.3 *If $f : \mathbb{A} \to T\mathbb{B}$ is a morphism in* **OPCA+**, *then f is equivalent to a morphism that preserves the ordering on the nose.*

Proof. Let u be a realizer such that $a \leq a' \Rightarrow u \bullet f(a) \leq f(a')$. Put $g(a) = \cup_{a'\leq a} f(a)$. This clearly preserves the ordering. Since $f(a) \subseteq g(a)$, $f \leq g$. And if $b \in g(a)$, that is, $b \in f(a')$ for some $a' \leq a$, then $u \bullet b \in f(a)$; hence $g \leq f$.

□

The theorem by Pitts (3.3.1) that we stated at the beginning of this section can now be strengthened as follows: let **Kl(T)** denote the Kleisli category for the monad $(T, \delta, \cup)$ (this is a 2-category, since the pre-ordering of the arrows is inherited from **OPCA+**). Let **RTripExact** denote the 2-category of realizability triposes of the form $I(\mathbb{A})^{(-)}$, with exact functors as arrows, and natural transformations pre-ordering those exact functors. Then we obtain:

Theorem 3.3.4 *Every map $f : \mathbb{A} \to T\mathbb{B}$ induces a Set-indexed functor from $I(\mathbb{A})^{(-)}$ to $I(\mathbb{B})^{(-)}$, that commutes with $\wedge, \top$ and $\exists$. Moreover, every such Set-indexed functor is, up to isomorphism, induced by a map $f : \mathbb{A} \to T\mathbb{B}$. Hence we have a 2-functor from the Kleisli category* **Kl(T)** *to* **RTripExact**. *This 2-functor is bijective on objects and a local equivalence: it induces equivalences on the Hom categories.*

Proof. Given $f : \mathbb{A} \to T\mathbb{B}$, define the tripos map $\bar{f} : I(\mathbb{A}) \to I(\mathbb{B})$ as $\bar{f}(\alpha) = \bigcup_{a \in \alpha} f(a)$.

Conversely, take $\phi : I(\mathbb{A}) \to I(\mathbb{B})$ with the mentioned properties. By 3.3.1 it follows that there is a map $\lambda : \mathbb{A} \to I^*\mathbb{B}$ such that ϕ is naturally isomorphic to $\bar{\lambda}$, and $\bigcap_{a,a' \in Dom(\bullet)} \lambda(a) \to (\lambda(a') \to \lambda(aa')) \neq \emptyset$. This map λ preserves the ordering up to a realizer: consider the object $X = \{(a', a) | a' \leq a\}$, and the two projections $\pi_1, \pi_2 \in I(\mathbb{A})^X$. Clearly $\pi_1 \vdash \pi_2$. Hence also $\lambda \circ \pi_1 \vdash \lambda \circ \pi_2$, so there is a realizer $c \in \bigcap_{a' \leq a} (\lambda(a') \to \lambda(a))$. Therefore, λ is a map of ordered PCAs.
□

This theorem shows, in effect, that our approach is an extension of Longley's, because Longley's 2-category of PCAs is a full sub-2-category of **KL**(T).

A final observation for this section: just as a map $f : \mathbb{A} \to \mathbb{B}$ presents $\mathbb{A}$ as a projective internal ordered PCA in **RT**$[\mathbb{B}]$, a map $g : \mathbb{A} \to T\mathbb{B}$ presents $\mathbb{A}$ as a *separated* internal ordered PCA in **RT**$[\mathbb{B}]$.

3.3.3 The 2-category OPCA

For reasons that are about to become transparent, we introduce the following definition:

Definition 3.3.5 A morphism $f : \mathbb{B} \to \mathbb{A}$ is said to be *computationally dense* (cd) iff there exists an element $m \in \mathbb{A}$ such that the following condition holds:

$$\forall a \in \mathbb{A} \exists b \in \mathbb{B} \forall b' \in \mathbb{B} : a \bullet f(b')\downarrow \Rightarrow bb'\downarrow \ \ \& \ m \bullet f(bb') \leq a \bullet f(b') \qquad \text{(cd)}$$

The terminology is explained by the fact that the condition actually tells us that any representable function from $\mathbb{B}$ to $\mathbb{A}$ (representable by some element in $\mathbb{A}$, that is), is, modulo the realizer m, bounded below by a function which is representable by some element in $\mathbb{B}$.

Let us first check that the composition of two computationally dense maps is again such a map. So, suppose that $g : \mathbb{C} \to \mathbb{B}$ and $f : \mathbb{B} \to \mathbb{A}$ are cd, witnessed by realizers $n \in \mathbb{B}$, and $m \in \mathbb{A}$, respectively, so that we have

$$\forall a \in \mathbb{A} \exists b \in \mathbb{B} \forall b' \in \mathbb{B} : a \bullet f(b')\downarrow \Rightarrow bb'\downarrow \ \ \& \ m \bullet f(bb') \leq a \bullet f(b')$$

and

$$\forall b \in \mathbb{B} \exists c \in \mathbb{C} \forall c' \in \mathbb{C} : b \bullet g(c')\downarrow \Rightarrow cc'\downarrow \ \ \& \ n \bullet g(cc') \leq b \bullet g(c').$$

Let u be a realizer such that $u \bullet f(b') \leq b$ for all $b' \leq b$, and r a realizer such that $r \bullet f(b) \bullet f(b') \leq f(bb')$ for all b, b' for which $bb'\downarrow$. Then we claim that $\lambda x.m(u(r \bullet f(n) \bullet x))$ is a witness for the computational density of fg. For, take any $a \in \mathbb{A}$, we get a $b \in \mathbb{B}$ from the computational density of f; suppose $a \bullet fg(c')\downarrow$. Taking $b' = g(c')$, we find

$$b \bullet g(c')\downarrow \ \ \& \ m \bullet f(b \bullet g(c')) \leq a \bullet fg(c').$$

From the computational density of g, we find that there is a $c \in \mathbb{C}$ such that

$$cc'{\downarrow} \quad \& \; n \bullet g(cc') \leq b \bullet g(c')$$

Now $\lambda x.m(u(r \bullet f(n) \bullet x)) \bullet fg(cc') \leq m(u(r \bullet f(n) \bullet fg(cc'))) \leq m(u(f(n \bullet g(cc')))) \leq m(f(b \bullet g(c'))) \leq a \bullet fg(c')$.

Next, the identity map is computationally dense, too, so we can form the subcategory **OPCA** of **OPCA+** which has the same objects, but only the computationally dense maps.

Moreover, the structure maps of the monad δ and $\cup$ are both cd, and if f is cd, then so is Tf. Therefore, the monad $(T, \delta, \cup)$ restricts to a monad on **OPCA**. We shall not distinguish notationally between the two uses of T; relying on context to make clear in which category we work.

Let us now explain what the relevance of computational density is.

Lemma 3.3.6 *Consider a morphism* $f : \mathbb{B} \to T\mathbb{A}$ *in* **OPCA**. *Then* f *induces a geometric morphism of triposes:*

$$\begin{tikzcd} I(\mathbb{A}) \arrow[r, shift right, "\hat{f}"'] \arrow[r, phantom, "\perp"] & I(\mathbb{B}). \arrow[l, shift right, "\bar{f}"'] \end{tikzcd}$$

Proof. We define the arrows $\bar{f}$ and $\hat{f}$ as

$$\bar{f}(\beta) = \bigcup_{b \in \beta} f(b), \qquad \hat{f}(\alpha) = \{b \in \mathbb{B} \mid m \bullet f(b) \subseteq \alpha\},$$

and where m is the witness for the computational density of f (intuitively, $\hat{f}$ is f^{-1}, but up to the realizer m). Note that $\hat{f}(\alpha)$ is downwards closed because $b' \leq b$ implies $f(b') \subseteq f(b)$ and thus $m \bullet f(b') \subseteq m \bullet f(b) \subseteq \alpha$.

First, the existence of the functor $\bar{f}$ that preserves finite limits follows from theorem 3.3.4.

Second, let us see why $\hat{f}$ is order-preserving. Suppose $d \in \phi \to \psi$. Put $a = \lambda x.d(m \bullet x)$, and use (cd) to find a $b \in \mathbb{B}$ with $\forall b' \in \mathbb{B} : a \bullet f(b'){\downarrow} \Rightarrow bb'{\downarrow}$ & $m \bullet f(bb') \subseteq a \bullet f(b')$, i.e.

$$\forall b' \in \mathbb{B} : d \bullet (m \bullet f(b')){\downarrow} \Rightarrow bb'{\downarrow} \quad \& \; m \bullet f(bb') \subseteq d \bullet (m \bullet f(b')) \qquad \dagger$$

We claim that this b is a realizer for $\hat{f}(\phi) \vdash \hat{f}(\psi)$: take $x \in \hat{f}(\phi)$, i.e. $m \bullet f(x) \subseteq \phi$. Because d is a realizer for $\phi \vdash \psi$, we get $d \bullet (m \bullet f(x)) \subseteq \psi$, and from $\dagger$ it follows that $bx{\downarrow}$ and $m \bullet f(bx) \subseteq d \bullet (m \bullet f(x))$. Hence $m \bullet f(bx) \subseteq \psi$, so $bx \in \hat{f}(\psi)$.

Third, we show that there is a natural transformation $Id \vdash \hat{f} \circ \bar{f}$, that is, we show the existence of a realizer b for $\beta \vdash \hat{f}\bar{f}(\beta)$, uniformly in $\beta \in I(\mathbb{B})$. For this b, apply (cd) to $\lambda x.x$, the identity on $\mathbb{A}$, as to obtain

$$\forall b' \in \mathbb{B} : bb'{\downarrow} \quad \& \; m \bullet f(bb') \subseteq f(b').$$

From this description it is straightforward to see that $b' \in \beta$ implies $bb' \in \hat{f}\bar{f}(\beta)$.

Finally, we have a natural transformation $\bar{f}\hat{f} \vdash Id$. The verification of this fact is also straightforward.

$\square$

Note in particular that for any map $g : \mathbb{B} \to \mathbb{A}$ in **OPCA**, composition with the structure map $\delta : \mathbb{A} \to T\mathbb{A}$ of the monad induces a geometric morphism.

The next step is to show, that, up to isomorphism, any geometric morphism of realizability triposes is induced by a morphism in **OPCA**.

Lemma 3.3.7 *Suppose we have a geometric morphism*

$$\begin{tikzcd} I(\mathbb{A}) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & I(\mathbb{B}). \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

Then there is a map $f : \mathbb{B} \to T\mathbb{A}$ such that $\bar{f} \dashv\vdash f^*$, and f is computationally dense.

Proof. As has already been shown by Pitts, putting $f(b) = f^*(\downarrow(b))$ is the only choice we have, since this gives $f^*(\beta) \dashv\vdash \bigcup_{b\in\beta} f(b) = \bar{f}(\beta)$, because f^*, as a left adjoint, preserves unions. Again from theorem 3.3.4, it follows that this is a morphism in **OPCA+**.

Now consider the counit of the adjunction, $\bar{f}f^* \vdash Id$. This means that there is a realizer m such that for all $\alpha \in T(\mathbb{A})$, and for all $x \in \bar{f}f^*(\alpha), mx \in \alpha$. Now take any $a \in \mathbb{A}$, and put $D = \{b' \in B | a \bullet f(b')\downarrow\}$. Then

$$\bar{f}(\downarrow(b')) \vdash_{b'\in D} \downarrow(a \bullet f(b')),$$

and transposing along the adjunction we get

$$\downarrow(b') \vdash_{b'\in D} f^*(\downarrow(a \bullet f(b'))).$$

This says that there is a $b \in \mathbb{B}$ such that for all $b' \in D : bb'\downarrow$ and $bb' \in f^*(\downarrow (a \bullet f(b')))$. It follows that for all $b' \in D$: $f(bb') \subseteq \bar{f}f^*(\downarrow(a \bullet f(b')))$. Apply m to get $m \bullet f(bb') \subseteq a \bullet f(b')$.

$\square$

This establishes, that geometric morphisms $I(\mathbb{B})^{(-)} \to I(\mathbb{A})^{(-)}$, are, up to isomorphism, the same as ordered PCA morphisms $\mathbb{A} \to T\mathbb{B}$ that are computationally dense. But the latter are precisely the morphisms from $\mathbb{A}$ to $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(T)$ for the monad T on **OPCA**.

Let **RTrip** denote the 2-category with as objects triposes of the form $I(\mathbb{A})^{(-)}$ for some ordered PCA $\mathbb{A}$, and as arrows geometric morphisms of triposes. For two geometric morphisms $(f^*, f_*), (g^*, g_*)$ from $I(\mathbb{B})^{(-)}$ to $I(\mathbb{A})^{(-)}$, we say that $(f^*, f_*) \leq (g^*, g_*)$ iff for every set X and any $\phi : X \to I\mathbb{A}$, $f^*\phi \vdash g^*\phi$. This makes **RTrip** into a preorder-enriched category. Moreover, let **RTop** be the 2-category of toposes of the form $\mathbf{RT}[\mathbb{A}]$ for some ordered PCA $\mathbb{A}$, with geometric morphisms commuting with the inclusion of **Set**, and natural transformations between them. It is known that these categories are equivalent when we forget about the 2-categorical structure. The following lemma shows that there is also a correspondence between natural transformations on the tripos-level and on the topos-level.

Lemma 3.3.8 *Let* $\mathbb{A}, \mathbb{B}$ *be ordered PCAs, an let* $f, g : \mathbb{A} \to T\mathbb{B}$ *be two maps in* **OPCA**. *Then* $\bar{f} \leq \bar{g}$ *in* **RTrip** *iff there is a (necessarily unique) natural transformation* $\eta : \bar{f} \to \bar{g}$ *in* **RTop**.

Proof. The idea of the proof is, first to establish this for separated objects, and then to use the fact that every object can be covered by a separated object. Details are left to the reader.

□

Now we relate the preorder on Hom-sets in **OPCA** to the one on the Hom-Sets in **RTrip**.

Lemma 3.3.9 *Let* $f, g : \mathbb{A} \to T\mathbb{B}$ *be two maps in* **OPCA**, *inducing two geometric morphisms of triposes,* $(\bar{f}, f^{-1})$ *and* $(\bar{g}, g^{-1})$. *Then* $f \leq g$ *iff* $(\bar{f}, f^{-1}) \leq (\bar{g}, g^{-1})$.

Proof. If $f \leq g$ then there is an element $b \in \mathbb{B}$ with the property that $b \in \bigcap_{a \in \mathbb{A}} f(a) \to g(a)$. This implies that $b \in \bigcap_{\alpha \in I\mathbb{A}} \bar{f}(\alpha) \to \bar{g}(\alpha)$. Therefore $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$.

Conversely, assume $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$. In particular, taking X to be $\mathbb{A}$ and $\phi(a) = \downarrow(a)$, we find $\bar{f}(\phi)(a) = f(a)$, $\bar{g}(\phi)(a) = g(a)$, and there is an element $b \in \mathbb{B}$ such that $b \in \bigcap_{a \in \mathbb{A}} f(a) \to g(a)$, proving $f \leq g$.

□

We can summarize by the following theorem:

Theorem 3.3.10 *There is a 2-functor from the opposite of the Kleisli 2-category* $\mathbf{Kl}(T)$ *to the 2-category* **RTrip** *of realizability triposes. This functor is bijective on objects and a local equivalence.*

3.4 Pseudo-algebras for T

In this section we relate properties of the category $\mathbf{PAss}(\mathbb{A})$ to monad-theoretic properties of $\mathbb{A}$. The first thing to notice is, that our monad is an instance of a KZ-monad (see chapter 2). The verification of this comes down to observing that the following hold: $T\delta_{\mathbb{A}} \leq \delta_{T\mathbb{A}}$, $\cup \circ T\delta_{\mathbb{A}} = \cup \circ \delta_{T\mathbb{A}}$ and $T\delta_{\mathbb{A}} \circ \delta_{\mathbb{A}} = \delta_{T\mathbb{A}} \circ \delta_{\mathbb{A}}$. We will use some facts about KZ-monads to simplify some of the proofs below. Recall that a *pseudo-algebra* for the monad T is a map $\phi : T\mathbb{A} \to \mathbb{A}$ such that the two diagrams below commute up to 2-isomorphism:

$$\begin{tikzcd}
 & T\mathbb{A} \arrow[d, "\phi"] \\
\mathbb{A} \arrow[ur, "\delta"] \arrow[r, "Id"'] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2\mathbb{A} \arrow[r, "T\phi"] \arrow[d, "\cup"'] & T\mathbb{A} \arrow[d, "\phi"] \\
T\mathbb{A} \arrow[r, "\phi"'] & \mathbb{A}.
\end{tikzcd}$$

Similarly, we say that a map $f : \mathbb{A} \to \mathbb{B}$ is a pseudo-T-homomorphism if the diagram

$$\begin{tikzcd}
T\mathbb{A} \arrow[r, "Tf"] \arrow[d, "\phi"'] & T\mathbb{B} \arrow[d, "\psi"] \\
\mathbb{A} \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd}$$

commutes up to 2-isomorphism (where ϕ, ψ are the pseudo-algebra structures for $\mathbb{A}$ and $\mathbb{B}$ respectively).

The facts about KZ-monads of which we will make use are:

1. A pseudo-algebra is the same as a left adjoint reflection for the unit. Hence pseudo-algebras are unique up to isomorphism.

2. If $\phi : T\mathbb{A} \to \mathbb{A}$ is a pseudo-algebra, then a left adjoint for ϕ is automatically a pseudo-T-homomorphism.

3. If $T^2\mathbb{A} \to T\mathbb{A}$ is a free algebra, then the algebra map always has a left adjoint.

As a heuristic, one can think of a pseudo-algebra $\phi : T\mathbb{A} \to \mathbb{A}$ for T as a "complete" OPCA where ϕ plays the role of supremum map. For free algebras, the multiplication is a genuine supremum map, but in general ϕ is only a supremum map up to a realizer (and the underlying poset of $\mathbb{A}$ also has non-empty suprema up to a realizer).

Also, notice that if a pseudo-algebra exists, then it is automatically a computationally dense map. This is true, because $\phi \dashv \delta$ implies that $T\phi \dashv T\delta$. So ϕ induces a geometric morphism of triposes, and must therefore be a computationally dense map.

Now we turn to the categories of partitioned assemblies. First, we show that OPCA-maps from $\mathbb{A}$ to $\mathbb{B}$ are precisely finite limit-preserving functors from **PAss**($\mathbb{A}$) to **PAss**($\mathbb{B}$) that commute with the inclusion of **Sets**.

Lemma 3.4.1

1. *An OPCA-map $h : \mathbb{A} \to \mathbb{B}$ induces a left exact functor H :* **PAss**($\mathbb{A}$) $\to$ **PAss**($\mathbb{B}$) *that commutes with the inclusion of* **Sets***.*

2. *A left exact functor H :* **PAss**($\mathbb{A}$) $\to$ **PAss**($\mathbb{B}$) *that commutes with the inclusion of* **Sets** *induces an OPCA-map $h : \mathbb{A} \to \mathbb{B}$.*

3. *The operations $h \mapsto H$ and $H \mapsto h$ are, up to 2-isomorphism, inverse to each other.*

Proof. We just remark that $h : \mathbb{A} \to \mathbb{B}$ gives H by $H(X, \epsilon_X) = (X, h \circ \epsilon_X)$. Conversely, every functor H satisfying the above property is, up to isomorphism, induced by its action on the generic object. Details of the proof are omitted, since there is a very similar theorem for the categories of assemblies in [54].

□

Remark. In fact, lemma 3.4.1 could be stated in terms of a 2-functor from **OPCA+** to the 2-category of categories of the form **PAss**($\mathbb{A}$), and lex functors that commute with the inclusion of **Sets**. This functor then is a local equivalence.

Another point worth noticing is, that it follows now that two maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ are adjoint if and only if the induced functors between **PAss**($\mathbb{A}$) and **PAss**($\mathbb{B}$) are adjoint. This fact will be exploited later on.

Theorem 3.4.2 *The following are equivalent for an ordered PCA $\mathbb{A}$:*

1. *$\mathbb{A}$ admits a pseudo-algebra structure*
2. **PAss**($\mathbb{A}$) *is regular*
3. *The embedding of* **PAss**($\mathbb{A}$) *into* **Ass**($\mathbb{A}$) *is a localization that commutes with the inclusion of* **Sets***.*

Proof. First, assume 1). As in 3.4.1, such a structure $\phi : T\mathbb{A} \to \mathbb{A}$ gives a functor ϕ : **Ass**($\mathbb{A}$) $\simeq$ **PAss**($T\mathbb{A}$) $\to$ **PAss**($\mathbb{A}$), that is left adjoint to the embedding (which corresponds to the unit of the monad at $\mathbb{A}$)). The counit of the adjunction is an isomorphism, since it is so on the level of OPCAs. This proves 3).

Now assume that a localization as in 3) exists. This gives, again by the lemma, some OPCA-map $\phi : T\mathbb{A} \to \mathbb{A}$, that is left adjoint to the unit at $\mathbb{A}$, and hence a pseudo-algebra. Thus, 3) implies 1).

Next, assume 2). Because of the universal property of **Ass**($\mathbb{A}$) w.r.t. regular categories, there is a retraction ϕ : **Ass**($\mathbb{A}$) $\to$ **PAss**($\mathbb{A}$). It is straightforward to check that this commutes with the inclusion of **Sets** and that the adjointness holds, so we have 3).

Finally, assume 3) (again, the left adjoint is called ϕ). Because any parallel pair in **PAss**($\mathbb{A}$) has a coequalizer in **Ass**($\mathbb{A}$), and because ϕ preserves coequalizers, **PAss**($\mathbb{A}$) has coequalizers. Moreover, the fact that ϕ is left exact ensures that these coequalizers are pullback-stable. So **PAss**($\mathbb{A}$) is regular.

□

If **PAss**($\mathbb{A}$) is regular, then we can give the following characterization of the regular epimorphisms:

Lemma 3.4.3 *Let* **PAss**($\mathbb{A}$) *be regular, and let ϕ be the pseudo-algebra map that exists by theorem 3.4.2. Then a surjective map $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$ is regular epi iff there is an element p with $p \bullet \epsilon_Y(y) \leq \phi(\downarrow\{\epsilon_X(x) | f(x) = y\})$.*

Proof. First, take a surjection $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$ and p with the property that $p \bullet \epsilon_Y(y) \leq \phi(\downarrow \{\epsilon_X(x) | f(x) = y\})$. Suppose that there is another map $g : (X, \epsilon_X) \to (Z, \epsilon_Z)$ such that the underlying function g can be written as $g = hf$ for some $h : Y \to Z$. In other words, $f(x) = f(x')$ implies $g(x) = g(x')$. We show that the map h has a tracking.

Let c be an element tracking g, so $c \bullet \epsilon_X(x) \leq \epsilon_Z(g(x))$ for all $x \in X$. Take any $y \in Y$ and write α_y for the set $\downarrow\{\epsilon_X(x)|f(x) = y\}$. Now c inhabits $\alpha_y \to \epsilon_Z(h(y))$, so $\downarrow(c) \bullet \alpha_y \subseteq \downarrow(\epsilon_Z(h(y)))$. By the fact that ϕ preserves the ordering and application up to a realizer, we obtain a realizer c' with the property $c' \bullet \phi(\alpha_y) \leq \epsilon_Z(h(y))$, and hence (using p) also a realizer c'' such that $c'' \bullet \epsilon_Y(y) \leq \epsilon_Z(h(y))$.

On the other hand, let $f, g : (X, \epsilon_X) \to (Y, \epsilon_Y)$ be a parallel pair. We form the coequalizer (Z, ϵ_Z) by letting $q : Y \to Z$ be the underlying coequalizer in **Sets**, and $\epsilon_Z(z) = \phi(\downarrow\{\epsilon_Y(y)|q(y) = z\})$. If $(Z', \epsilon_{Z'})$ is isomorphic to (Z, ϵ_Z), then $p \bullet \epsilon_{Z'}(z) \leq \epsilon_Z(z)$ for some p, and hence $\epsilon_{Z'}$ is of the required form.

□

Before we state the next theorem, we recall that a diagram of the form

$$\begin{tikzcd}
(X, \epsilon_X) \arrow[r, "f"] \arrow[d, "\eta_X"'] & (Y, \epsilon_Y) \arrow[d, "\eta_Y"] \\
\nabla(X) \arrow[r, "\nabla(f)"'] & \nabla(Y)
\end{tikzcd}$$

is a pullback if and only if $(X, \epsilon_X) \cong (X, \epsilon')$ with $\epsilon'(x) = \epsilon_Y(f(x))$ for all $x \in X$. Recall that such maps are called *cartesian maps.*

Theorem 3.4.4 *For any OPCA $\mathbb{A}$, the following are equivalent:*

1. **PAss**$(\mathbb{A})$ *is a regular completion;*
2. *There is a "cylinder" of adjoints $\psi \dashv \phi \dashv \delta$:* **PAss**$(\mathbb{A}) \to$ **Ass**$(\mathbb{A})$*, with $\phi \circ \delta \sim Id$, and with ψ preserving finite limits and commuting with the inclusion of* **Sets***;*
3. *$\mathbb{A}$ admits a pseudo-algebra structure, and this pseudo-algebra has a left adjoint $\psi : \mathbb{A} \to T\mathbb{A}$;*
4. *$\mathbb{A}$ is equivalent to a free T-algebra.*

Proof. The equivalence between 2) and 3) needs no explication. First assume 1). Since **PAss**$(\mathbb{A})$ is regular, we have the map $\phi : T\mathbb{A} \to \mathbb{A}$. By the characterization of completions, there are enough projectives and the projectives are closed under finite limits. We first explain why there is a *generic projective* object. Take the generic object, namely (A, Id), and cover it with a projective $e : (B, \epsilon_B) \to (A, Id)$. This means that for each $a \in \mathbb{A}$ there is a set $\beta_a = \downarrow\{\epsilon_B(b)|e(b) = a\}$. Also put $Irr = \cup_{a \in \mathbb{A}} \beta_a$. Just as the map

ϕ can be thought of as supremum mapping, we think of Irr as the set of join-irreducible elements in $\mathbb{A}$, and of β_a as the join-irreducibles that are below a. Moreover, because the covering is regular epi, we have an isomorphism $(A, Id) \cong (A, \lambda a.\phi(\beta_a))$.

The fact that (B, ϵ_B) is projective now implies that, for some realizer r, if $b \in Irr$, and $b \leq \phi(\alpha)$ for some set $\alpha \in T\mathbb{A}$, then there is some $a \in \alpha$ such that $r \bullet b \leq a$ (this is just writing out what it means that every regular epi with codomain (B, ϵ_B) has a section). Note that this is, indeed, some kind of irreducibility.

From this one deduces that if an object (Y, ϵ_Y) has $\epsilon_Y(y) \in Irr$ for every $y \in Y$, then it is also projective.

This object (B, ϵ_B) is generic projective in the following sense: if (X, ϵ_X) is any object, then we have a map $\epsilon_X : (X, \epsilon_X) \to (A, Id)$. If we form the pullback

$$\begin{tikzcd}
(Q, \epsilon_Q) \arrow[r, "h"] \arrow[d] & (B, \epsilon_B) \arrow[d] \\
(X, \epsilon_X) \arrow[r, "\epsilon_X"'] & (A, Id)
\end{tikzcd}$$

then the left-hand map is again regular epi.

The map ϵ_X is a pullback of $\nabla(\epsilon_X)$, hence the top map is also a cartesian map. This means that, for any $q \in Q$, $\epsilon_Q(q) = \epsilon_B(h(q)) \in Irr$. From this we obtain that (Q, ϵ_Q) is also projective. We refer to coverings obtained in this way by *canonical coverings*.

Moreover, if (X, ϵ_X) already happened to be projective, then the left-hand map would split, presenting (X, ϵ_X) as a (regular) subobject of (Q, ϵ_Q). But regular monos are cartesian maps in this context, so (X, ϵ_X) is pre-embedded in (B, ϵ_B). Hence every projective is a pullback of (B, ϵ_B).

Now the map $\psi : \mathbb{A} \to T\mathbb{A}$, defined by $a \mapsto \beta_a$ gives a functor $\psi : \mathbf{PAss}(\mathbb{A}) \to \mathbf{Ass}(\mathbb{A})$, by saying $\psi(X, \epsilon_X) = (X, \psi \circ \epsilon_X)$. Let us check that this is well-defined: take $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$, and consider the diagram

$$\begin{tikzcd}
(P, \epsilon_P) \arrow[r] \arrow[d] & (Q, \epsilon_Q) \arrow[d] \\
(X, \epsilon_X) \arrow[r, "f"'] & (Y, \epsilon_Y).
\end{tikzcd}$$

Here, the vertical maps are canonical projective covers, and the top map arises because of the projectivity of (P, ϵ_P). The fact that this map has a tracking is just the same as the fact that $f : (X, \psi \circ \epsilon_X) \to (Y, \psi \circ \epsilon_Y)$ does.

Next, the composite $\phi \circ \psi$ is isomorphic to the identity, since $(A, Id) \cong (A, \lambda a.\phi(\beta_a))$. Moreover, $\psi \dashv \phi$. Indeed, if $\psi(a) \to \beta$ is inhabited (uniformly in $a \in \mathbb{A}$, and in $\beta \in T\mathbb{A}$), then so is $\phi\psi(a) \to \phi(\beta)$. But then $a \to \phi(\beta)$ is also inhabited. Conversely, if $a \to \phi(\alpha_a)$ is inhabited then we have a regular epi $f : (X, \epsilon_X) \to (A, Id)$, where $X = \{(a, b)|b \in \alpha_a\}$, and $\epsilon_X(a, b) = b$. Thus there is a map $g : (B, \epsilon_B) \to (X, \epsilon_X)$, such that the composite fg equals the projection $(B, \epsilon_B) \to (A, Id)$. Now it is easily deduced that the tracking element for g sends all elements in $\psi(a)$ to elements in α_a, and the adjointness is proved.

Finally, since the projectives are closed under finite limits, we can derive that ψ preserves finite limits.

Next, we prove the converse; so assume that ϕ has a left adjoint ψ, which, by the considerations that we saw before, may be taken to be induced by a function $\psi : \mathbb{A} \to T\mathbb{A}$. Now consider the generic object (A, Id) in $\mathbf{PAss}(\mathbb{A})$, and cover this object by (B, ϵ_B), where $B = \{(a, c)|c \in \psi(a)\}$ and $\epsilon_B(a, c) = c$. The projection is regular epi since the unit of the adjunction $\phi \dashv \psi$ is an isomorphism. We show that (B, ϵ_B) is (generic) projective. The fact that ϕ is right adjoint to ψ translates into the fact that the object (B, ϵ_B) has the property that for every regular epi $f : (X, \epsilon_X) \to (A, Id)$ there is a map $(B, \epsilon_B) \to (X, \epsilon_X)$, that makes the projection factor through f:

$$\begin{tikzcd}
& (B,\epsilon_B) \arrow[ld, dashed] \arrow[d, two heads] \\
(X,\epsilon_X) \arrow[r, two heads, "f"] & (A,Id).
\end{tikzcd}$$

Indeed, f regular epi means $\downarrow(a) \to \phi(\downarrow\{\epsilon_X(x)|f(x) = a\})$ inhabited, and by the adjunction, $\psi(a) \to \{\epsilon_X(x)|f(x) = a\}$ inhabited. This says precisely that there is a tracked function from $(B, \epsilon_B) \to (X, \epsilon_X)$.

Consider the pullback

$$\begin{tikzcd}
(Q,\epsilon_Q) \arrow[r] \arrow[d, two heads] & (B,\epsilon_B) \arrow[d, two heads] \\
(B,\epsilon_B) \arrow[r, "\epsilon_B"'] & (A,Id)
\end{tikzcd}$$

where the bottom map is cartesian (and hence the top map, too). The left-hand map has a section, say m. Now if $Y \to X$ is any regular epi, and $f : (B, \epsilon_B) \to X$ any arrow, then the adjunction gives us a map as in the diagram:

$$\begin{tikzcd}
& (Q,\epsilon_Q) \arrow[ldd, dashed] \arrow[d, two heads] \\
& (B,\epsilon_B) \arrow[d, "f"] \\
Y \arrow[r, two heads] & X.
\end{tikzcd}$$

We obtain a map from (B, ϵ_B) to Y by using the section $m : (B, \epsilon_B) \to (Q, \epsilon_Q)$. Hence (B, ϵ_B) is projective. Now it is also easily established that (B, ϵ_B) is generic projective, as we in the proof of the other direction.

The implication from 4) to 3) is just the third fact about KZ-monads that we listed at the beginning of this section. It remains to show that 3) implies 4). So let ψ be left adjoint to ϕ, and consider the set $Irr = \{c \in \mathbb{A} | c \in \psi(a), a \in \mathbb{A}\}$. We endow this set with an OPCA-structure. Observe that we may assume that ψ preserves the ordering on the nose; because $T\mathbb{A}$ is free, lemma 3.3.3 is applicable. Let r be a realizer up to which ψ preserves application. Now put

$$c \bullet' c' \simeq r \bullet c \bullet c'$$

and order Irr as a subset of $T\mathbb{A}$. It is an easy exercise to verify that this is indeed an OPCA, that ϕ restricts to a map $\phi : T(Irr) \to \mathbb{A}$ and that ψ takes values in $T(Irr)$. We only have to show that these restricted maps form an equivalence of OPCAs. Since $\phi \circ \psi$ is isomorphic to the identity, it remains to show that $\psi \circ \phi$ is isomorphic to the identity on $T(Irr)$. The direction $\psi \circ \phi \leq 1$ is just the counit of the adjunction. By the second fact about KZ-monads, ψ is a pseudo-T-homomorphism, meaning that the square

$$\begin{tikzcd} T\mathbb{A} \arrow[r, "T\psi"] \arrow[d, "\phi"'] & T^2\mathbb{A} \arrow[d, "\cup"] \\ \mathbb{A} \arrow[r, "\psi"'] & T\mathbb{A} \end{tikzcd}$$

commutes up to isomorphism. Hence we can show that $1 \leq \cup \circ T\psi$. Recall that there is a realizer s that takes each $c \in Irr$ to an element in $\psi(c)$. (This is just expressing that a covering of a projective object has a section.) But $\cup \circ T\psi(\gamma) = \cup_{c\in\gamma}\psi(c)$, so s takes γ to $\cup_{c\in\gamma}\psi(c)$, uniformly in γ. This completes the proof.

□

Remark. If there exists a left adjoint to the pseudo-algebra map, then this left adjoint is automatically a computationally dense map, since it has a right adjoint.

3.5 Applications

In this section we discuss four applications of the machinery that we developed. First, we study relative realizability and local maps. This subject has been treated for ordinary PCAs in [3]; we have a look at some facts that emerge when we consider ordered PCAs. In particular, we see when an inclusion of ordered PCAs gives rise to a local map of toposes. Next we consider the Effective Monad, and show how it gives rise to a comonad on the category **OPCA**. Finally, we slightly generalize the fact that the Effective topos is not equivalent to any realizability topos obtained from a total PCA.

3.5.1 Local maps

Let $\mathbb{B}$ be some PCA and let $\mathbb{A}$ be a sub-PCA of $\mathbb{B}$, that is, $\mathbb{A}$ is a subset containing (some choice for) k and s that is closed under the partial application. In [3] the toposes $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[\mathbb{B}]$ are compared. In the previous section we saw that a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$ is, up to isomorphism, the same as a map $f : \mathbb{A} \to T\mathbb{B}$ that is computationally dense. Note, however, that for ordinary PCAs this requirement implies surjectivity of the map f, and from this it readily follows that there will never be a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$, except for the trivial case where $\mathbb{A} = \mathbb{B}$. There is, however, a topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$, called the *relative realizability topos* , that has the property that there is a local localic geometric morphism $\mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{A}]$, and a logical functor $L : \mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{B}]$. (For more on local maps we refer to [45].) In a picture:

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}] \arrow[r, bend left, "\bar{i}"] \arrow[r, bend right, "i_*"'] \arrow[r, phantom, bend left=12, "\perp"] \arrow[r, phantom, bend right=12, "\perp"'] & \mathbf{RT}[\mathbb{B}, \mathbb{A}] \arrow[l, "i^{-1}" description] \arrow[r, "L"] & \mathbf{RT}[\mathbb{B}]
\end{tikzcd}$$

The intermediate topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$ is constructed by taking the tripos $I(\mathbb{B})^{(-)}$ and taking the following preorder: $\phi \vdash' \psi$ iff $\exists a \in \mathbb{A} : a \in \bigcap_{x \in X} (\phi(x) \Rightarrow \psi(x))$. (All the other structure is exactly as in the tripos $I(\mathbb{B})^{(-)}$.) Now the maps $\bar{i}, i_*$ and i^{-1} are defined on the tripos-level, as follows (for $\phi : X \to I(\mathbb{A}), \psi : X \to \mathbb{B}$):

$$\bar{i}(\phi)(x) = {\downarrow}(\phi(x)), \qquad i^{-1}(\psi)(x) = \psi(x) \cap \mathbb{A},$$
$$i_*(\phi)(x) = \bigcup_{\alpha \in I(\mathbb{B})} (\alpha \wedge (\mathbb{A} \cap \alpha \to {\downarrow}(\phi(x)))).$$

Remarks.

1. First of all, we have given this definition in such a way, that it also applies to ordered PCAs. That is, we say that $\mathbb{A}$ is a *sub-OPCA* of $\mathbb{B}$ if it is a full sub-poset, closed under the partial application and contains (some choice of) k and s. It is completely straightforward to check that this still gives a local geometric morphism: one can copy the proof of theorem 3.1 in [3] almost literally.

2. Second, note that the functors $\bar{i}$ and i^{-1} are precisely the maps that are induced by the inclusion $\mathbb{A} \hookrightarrow \mathbb{B} \hookrightarrow T\mathbb{B}$ as in the previous section.

3. We also mention that the counit of the adjunction $i^{-1} \dashv i_*$ is an isomorphism, just as the unit of $\bar{i} \dashv i^{-1}$ is, so that $\mathbf{RT}[\mathbb{A}]$ is actually a retract of $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$.

Now for our purposes it will be interesting to know when the functor L is an equivalence.

Proposition 3.5.1 *If $\mathbb{A}$ is a sub-OPCA of $\mathbb{B}$, the functor L is an equivalence if and only if $\forall b \in \mathbb{B} \exists a \in \mathbb{A} : i(a) \leq b$.*

Proof. ($\Rightarrow$:) If L is an equivalence, then i induces a geometric morphism, and therefore is computationally dense.

($\Leftarrow$:) Take $\phi, \psi : X \to I(\mathbb{B})$, and assume that we have $b \in \mathbb{B}$ with $b \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. Pick $a \in \mathbb{A}$ with $i(a) \leq b$. Then $i(a) \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. □

Remarks.

1. In our opinion, this proposition can be taken as providing some evidence for the claim that ordered PCAs really are a useful generalization of ordinary PCAs, because it shows us that there are non-trivial inclusions of ordered PCAs that induce topos morphisms, something which is impossible for PCAs (see the first paragraph of this section).
2. If we have such a local localic map, induced by an inclusion $\mathbb{A} \hookrightarrow \mathbb{B}$ of ordered PCAs, then it follows that $\mathbb{A}$ is actually a retract of $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(T)$. The converse need not be true.
3. We said before, that an inclusion of ordinary PCAs would never yield a geometric morphism between the associated realizability toposes. It must be stressed, however, that the proof of this fact relies on classical logic, and does not remain true when we switch to an arbitrary base topos instead of **Set**. In fact, in [15] the notion of an *elementary subobject* is introduced. This definition is chosen in such a way, that if $\mathbb{B}$ is now a PCA-object in an arbitrary topos $\mathcal{S}$, and $\mathbb{A}$ is a sub-PCA of $\mathbb{B}$, then the requirement that $\mathbb{A}$ is an elementary subobject (rather than the maximal subobject) of $\mathbb{B}$ is enough to guarantee that there is a local map between the realizability toposes.

3.5.2 The Effective Monad

In Pitts' thesis [59]) it is shown that the operation

$$\mathcal{E} \mapsto \mathbf{Eff}_{\mathcal{E}}$$

which sends a topos $\mathcal{E}$ to the external Effective Topos over $\mathcal{E}$, is the object part of a monad on the category of toposes and geometric morphisms. This monad is appropriately called the Effective Monad. If we restrict to realizability toposes, then there is an explicit description, which directly generalizes to realizability toposes over ordered PCAs. This description is based on the iteration theorem for triposes: for a topos $\mathbf{RT}[\mathbb{A}]$, the result $\mathbf{Eff}_{\mathbf{RT}[\mathbb{A}]}$ is again of the form $\mathbf{RT}[\mathbb{B}]$ for some ordered PCA $\mathbb{B}$. Let us recall the construction of $\mathbb{B}$..

Starting from an ordered PCA $\mathbb{A}$, we can endow the set-theoretic product $\mathbb{N} \times \mathbb{A}$ with the coordinatewise ordering (where the order on $\mathbb{N}$ is discrete), and a partial application

$$(n, a) \bullet (m, b) \simeq (nm, a\langle \bar{m}, b\rangle)$$

Here, we have presupposed a *choice of numerals* $\bar{()} : \mathbb{N} \to \mathbb{A}$.

The OPCA-map $\epsilon : \mathbb{N} \times \mathbb{A} \to \mathbb{A}$, given as $\epsilon(n, a) = \langle \bar{n}, a \rangle$ is easily seen to be computationally dense, thus explaining why there is a geometric morphism $\mathbf{RT}[\mathbb{A}] \to \mathbf{RT}[\mathbb{N} \times \mathbb{A}] \simeq \mathbf{Eff}_{\mathbf{RT}[\mathbb{A}]}$. This geometric morphism is the unit of the Effective Monad, and the multiplication is induced by the OPCA-map $\nu : \mathbb{N} \times \mathbb{A} \to \mathbb{N} \times \mathbb{N} \times \mathbb{A}$, defined by $\nu(n, a) = (n, n, a)$. Again, this is computationally dense, and it is easily verified directly that $(\mathbb{N} \times -, \epsilon, \nu)$ is a comonad on the category **OPCA**. In fact, it is a comonad for which the coalgebra-structures are adjoint to the counits, so a co-KZ-monad.

This translation of the Effective Monad onto the "Effective Comonad" on ordered PCAs might be helpful in the search for a topos $\mathcal{E}$ for which $\mathcal{E} \simeq \mathbf{Eff}_{\mathcal{E}}$, i.e. in identifying the fixed points of the Effective Topos-construction. We do not pursue this line here, though.

3.5.3 Totality and decidability

In a very short paper [46], Johnstone and Robinson gave a categorical proof of the fact that the Effective Topos is not equivalent to a realizability topos obtained from a total PCA . Longley observed, that, for two PCAs $\mathbb{A}$ and $\mathbb{B}$, $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{RT}[\mathbb{B}]$ iff there are functions $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{A}$ such that $fg \sim 1, gf \sim 1$. Using this, he showed that $\mathbb{A}$ is decidable iff $\mathbb{B}$ is. So if we are to prove the inequivalence of two realizability toposes, then it suffices to show that one of the underlying PCAs is decidable, whereas the other is not. Now Kleene's PCA $\mathbb{N}$ is decidable, but a total PCA is never decidable.

We wish to give a variation on this proof. First, it can be shown (this is already in [54]) that if $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{RT}[\mathbb{B}]$, then there are bijective maps $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{A}$, with f and g inverse. Then we have the following:

Lemma 3.5.2 *Let $\mathbb{A}, \mathbb{B}$ be PCAs. Assume that $\mathbb{A}$ is total, and $\mathbb{B}$ has an element z such that for all $b \in \mathbb{B}$: $zb\downarrow$ and $zb \neq b$. Then $\mathbf{RT}[\mathbb{A}] \not\simeq \mathbf{RT}[\mathbb{B}]$.*

Proof. Assume that the toposes are equivalent, and take functions f, g as above and realizers $r \in \mathbb{B}$ with $rf(a)f(a') = f(aa')$, and $s \in \mathbb{B}$ with $bb'\downarrow \Rightarrow rg(b)g(b') \leq g(bb')$. Also, using the recursion theorem in $\mathbb{B}$, choose an element $e \in \mathbb{B}$ such that

$$e \bullet x \simeq z \bullet (r \bullet (r \bullet f(s) \bullet e) \bullet x).$$

Then:

$$\begin{aligned} e \bullet x &= fg(e \bullet x) \\ &= f(s \bullet g(e) \bullet g(x)) \\ &= r \bullet (r \bullet f(s) \bullet fg(e)) \bullet fg(x) \\ &= r \bullet (r \bullet f(s) \bullet e) \bullet x \end{aligned}$$

but, on the other hand, $e \bullet x \neq r \bullet (r \bullet f(s) \bullet e) \bullet x$ because of the property of the element z. Contradiction.

□

Note that this proof is properly more general in that it doesn't depend on the decidability of the PCAs involved (e.g. it also works for Kleene's PCA of functions, which is not decidable).

3.6 Iteration of T

In this section we study iteration of the endofunctor T. This gives rise to a sequence of ordered PCAs, and, as we will see, to a sequence of the corresponding realizability toposes. It was already predicted by Menni that certain chains of realizability toposes could be obtained in this fashion. Then we take some first steps in the exploration of possible colimits for such chains, which was suggested to us by Martin Hyland.

3.6.1 Hierarchies of toposes

Let us fix an ordered PCA $\mathbb{A}$. In the category **OPCA**, we have a diagram

$$\mathbb{A} \xrightarrow{\delta} T\mathbb{A} \xrightarrow{\delta} T^2\mathbb{A} \xrightarrow{\cup} T\mathbb{A}$$

This composition equals the map $\delta : \mathbb{A} \to T\mathbb{A}$ (this is one of the monad identities), so in the category $Kl(T)$, $\mathbb{A}$ is a retract of $T\mathbb{A}$. Now the inclusion of $\mathbb{A}$ in $T\mathbb{A}$ is easily seen to satisfy the condition of proposition (3.7) of the previous section. This means that there is an induced local localic geometric morphism. On the tripos level, it looks like this:

$$\begin{tikzcd} I(\mathbb{A}) \arrow[r, bend left, "D"] \arrow[r, bend right, "P"'] & I(T\mathbb{A}) \arrow[l, "U" description] \end{tikzcd}.$$

Let us give a direct description of the functors in this diagram (take $\alpha \in I(\mathbb{A})$ and $\xi \in I(T\mathbb{A})$):

$$D(\alpha) = \downarrow(\{\downarrow(a) \mid a \in \alpha\}), \qquad P(\alpha) = \downarrow(\alpha),$$
$$U(\xi) = \bigcup_{\alpha \in \xi} \{a \mid a \in \alpha\}.$$

We used the notation U, D, and P as to remind the reader of the words "union", "discrete" and "principal", respectively.

On the level of toposes, we get the following, similar picture:

$$\begin{tikzcd} \mathbf{RT}[\mathbb{A}] \arrow[r, bend left, "D"] \arrow[r, bend right, "P"'] & \mathbf{RT}[T\mathbb{A}] \arrow[l, "U" description] \end{tikzcd}.$$

We have the following:

Theorem 3.6.1 *There is an equivalence* $\mathbf{RT}[T\mathbb{A}] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}]})_{reg})_{ex}$.

Proof. We know that each $\mathbf{RT}[T\mathbb{A}]$ is the exact completion of its category of projectives, which is the same as the category of separated objects in $\mathbf{RT}[\mathbb{A}]$. But this latter category is the regular completion of the category of projectives of $\mathbf{RT}[\mathbb{A}]$.

□

Remark. In [75] it is remarked, that in some cases there is another tripos that we can associated with an ordered PCA: we can define $J(\mathbb{A}) \subseteq I(\mathbb{A})$ as those downsets in A that are closed under pushouts.

There is an inclusion map $i : J(\mathbb{A}) \hookrightarrow I(\mathbb{A})$, which induces an indexed map of preorders $i : J(\mathbb{A})^X \hookrightarrow I(\mathbb{A})^X$. Left adjoint to this map is composition with the operation Cl_p, which takes a downset to its closure under pushouts. From this it is not hard to establish that there is a geometric inclusion of triposes $J(\mathbb{A})^{(-)} \hookrightarrow I(\mathbb{A})^{(-)}$, and hence an inclusion of toposes (denote the topos represented by the tripos $J(\mathbb{A})^{(-)}$ by $\mathbf{RT}'[\mathbb{A}]$), $\mathbf{RT}'[\mathbb{A}] \hookrightarrow \mathbf{RT}[\mathbb{A}]$.

To complete the picture, we remark that the local localic map between $\mathbf{RT}[T\mathbb{A}]$ and $\mathbf{RT}[\mathbb{A}]$ restricts:

$$\begin{tikzcd}
\mathbf{RT}'[\mathbb{A}] \arrow[r, shift right=1.5, "P"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1.5, "i"'] \arrow[d, phantom, "\vdash"] & \mathbf{RT}'[T\mathbb{A}] \arrow[l, shift right=1.5, "U"'] \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1.5, "i"'] \arrow[d, phantom, "\vdash"] & \mathbf{RT}'[\mathbb{A}] \arrow[l, shift right=1.5, "D"'] \arrow[d, shift right=1.5, "i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{RT}[\mathbb{A}] \arrow[u, shift right=1.5, "Cl_p"'] \arrow[r, shift right=1.5, "P"'] \arrow[r, phantom, "\perp"] & \mathbf{RT}[T\mathbb{A}] \arrow[u, shift right=1.5, "Cl_p"'] \arrow[l, shift right=1.5, "U"'] \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{RT}[\mathbb{A}] \arrow[u, shift right=1.5, "Cl_p"'] \arrow[l, shift right=1.5, "D"']
\end{tikzcd}$$

It is easiest to see why the functors U, P and D restrict if we consider them on the tripos-level (again, we use the same notation for the functors on the tripos- and on the topos-level). Note first that $P(\alpha)$ is trivially closed under pushouts, since it is principal. Second, if $\alpha \in I(\mathbb{A})$ is closed under pushouts, then the same holds for $D(\alpha)$, since if $\downarrow\{a\}, \downarrow\{b\} \in D(\alpha)$, then $\downarrow\{a\} \cup \downarrow\{b\} \subseteq \downarrow\{a \vee b\}$. Third, the map U also preserves the property of being closed under pushouts. Now the adjointness is immediate, and so is the commutation of the diagram.

We can iterate the downset-construction: starting with an arbitrary ordered PCA $\mathbb{A} = \mathbb{A}_0$, we get a sequence $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ when we put $\mathbb{A}_{n+1} = (T\mathbb{A}_n)$.

This immediately gives us a sequence $I(\mathbb{A}_0)^{(-)}, I(\mathbb{A}_1)^{(-)}, \ldots$ of triposes, and hence a sequence $\mathbf{RT}[\mathbb{A}_0], \mathbf{RT}[\mathbb{A}_1], \ldots$ of toposes.

On the other hand, the results in [57] show that there are sequences of toposes of the form $(\mathcal{C}_{reg(n)})_{ex}$, (for appropriate categories $\mathcal{C}$). With the previous results in mind, the following theorem should not be too surprising:

Theorem 3.6.2 *For each* $n \in \mathbb{N}$*, there is an equivalence of categories* $\mathbf{RT}[\mathbb{A}_n] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}_0]})_{reg(n)})_{ex}$.

Proof. This goes by induction and is an immediate consequence of the facts that we established concerning $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[T\mathbb{A}]$.

$\square$

As a last observation, we mention the fact that there is also a chain of toposes coming from the hierarchy $J(\mathbb{A}), J(T\mathbb{A}), \ldots$. This chain is included in the one coming from $I(\mathbb{A}), I(T\mathbb{A}), \ldots$.

3.6.2 Colimits

When we look at the chain of inclusions $\mathbf{RT}[\mathbb{A}_0] \hookrightarrow \mathbf{RT}[\mathbb{A}_1] \hookrightarrow \ldots$, a natural question that arises is about the existence of a colimit of this chain. Is there a colimiting topos, and, moreover, does it come from an ordered PCA? We will not obtain a full result here, but present an approximation. This will be done in a couple of steps: first, we see that the diagram of inclusions of realizability toposes corresponds to a diagram of locales in $\mathbf{RT}[\mathbb{A}_0]$. Then, we see that we can view this diagram of locales as an internal diagram, so that we can compute the internal colimit. This gives a colimiting locale, and we can take sheaves over that locale. This topos still has a universal property, although weaker than a genuine colimit.

Let me say a word about why I think that these hierarchies of toposes are interesting, and why we would want to spend time on trying to find interesting cocones over them. Recall that the first step in the discovery of these hierarchies was made by van Oosten when he constructed the topos for extensional realizability. The passage from the Effective topos to the topos for extensional realizability involves a change in the logic of the natural number object: the logic is "extensionalized", in a way that we do not yet know how to describe precisely on a syntactical level. The hierarchy results presented here (and there will be many more in the chapter on indexed preorders) show, that this process of extensionalizing may be applied to almost every realizability topos. Hopefully, this gives rise to new, interesting notions of realizability and to a more systematic grasp on them. Furthermore, the question for the construction of a colimit is not only natural from a topos-theoretic, but also from a logical point of view; tentatively viewing the operation on OPCAs as a modification of the notion of realizability, it is natural to wonder whether this operation can be applied *ad infinitum*, and whether realizability can be made "fully extensional". In the two approximations below, we will encounter two previously unstudied triposes, the logic of which deserves investigation.

Internal locales In order to avoid any confusion, let us state here that, concerning the use of the notions *locale* and *frame*, we stick to the convention that a frame homomorphism $f : X \to Y$ is map of posets that preserves finite meets and arbitrary suprema, while a locale map $Y \to X$ is the same as a frame map $g : X \to Y$.

We first show that all the toposes in the hierarchy can be viewed as localic extensions of $\mathbf{RT}[\mathbb{A}_0]$. Then we show that the chain of toposes corresponds to a chain of locales in $\mathbf{RT}[\mathbb{A}_0]$.

Lemma 3.6.3 *Each topos* $\mathbf{RT}[\mathbb{A}_i]$ *is a localic extension of* $\mathbf{RT}[\mathbb{A}_0]$.

Proof. We do this for $i = 1$, since the proof is the same for all toposes in the chain. Consider the diagram of toposes

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_0] \arrow[r, shift right, "P"'] \arrow[dr, equal] & \mathbf{RT}[\mathbb{A}_1] \arrow[l, shift right, "U"'] \arrow[l, phantom, "\perp"] \arrow[d, shift right, "U"'] \arrow[d, phantom, "\vdash"] \\
 & \mathbf{RT}[\mathbb{A}_0]. \arrow[u, shift right, "D"']
\end{tikzcd}$$

Because the geometric morphisms in this diagram come from geometric morphisms of triposes, we know that they are localic. In particular, we know that $\mathbf{RT}[\mathbb{A}_1] \simeq Sh(U\Omega_1)$, where Ω_1 denotes the subobject classifier in $\mathbf{RT}[\mathbb{A}_1]$. Hence $\mathbf{RT}[\mathbb{A}_1]$ is a localic extension of $\mathbf{RT}[\mathbb{A}_0]$.

□

In the topos $\mathbf{RT}[\mathbb{A}_1]$, the subobject classifier can be described explicitly by $U\Omega_1 = (I\mathbb{A}_1, =)$, with $[\![\xi = \xi']\!] = (\xi \leftrightarrow \xi')$. Applying the functor U, we find that $U\Omega_1 = U(I\mathbb{A}_1, =) = (I\mathbb{A}_1, =_1)$, with $[\![\xi =_1 \xi']\!] = \cup(\xi \leftrightarrow \xi')$.

From now on, we will hold on to the following convention: $[\![. =_n .]\!]$ is the non-standard equality predicate on $U^n\Omega_n \cong (I\mathbb{A}_n, =_n)$.

By Diaconescu's theorem, we know that geometric morphisms $\mathbf{RT}[\mathbb{A}_0] \to \mathbf{RT}[\mathbb{A}_1]$ over $\mathbf{RT}[\mathbb{A}_0]$ correspond to locale maps $\Omega_0 \to U\Omega_1$. Applying this to the above diagram, we obtain a frame map $\cup : U\Omega_1 \to \Omega_0$. This arrow is the classifier of $U(true)$, as in the pullback

$$\begin{tikzcd}
U1 \arrow[r, tail, "U(true)"] \arrow[d] & U\Omega_1 \arrow[d, "\cup"] \\
1 \arrow[r, tail, "true"] & \Omega_0.
\end{tikzcd}$$

Explicitly, for $\xi \in I\mathbb{A}_1, \alpha \in I\mathbb{A}_0$:

$$\cup(\xi, \alpha) = (\cup\xi \leftrightarrow \alpha).$$

If we do the same thing for the other toposes in the chain, we get a chain of internal frames in $\mathbf{RT}[\mathbb{A}_0]$:

$$\begin{tikzcd}
\Omega_0 & U\Omega_1 \arrow[l, "\cup"'] & U^2\Omega_2 \arrow[l, "\cup"'] & \cdots \arrow[l, "\cup"']
\end{tikzcd}$$

This diagram of internal frames corresponds to a diagram of inclusions of locales, which in turn corresponds to the chain of inclusions of toposes over $\mathbf{RT}[\mathbb{A}_0]$.

Internalizing the diagram. Remember that we are interested in the colimit of the chain of toposes, or at least in a cocone over the chain, that has some interesting universal property. The external limit of the diagram of frames need of course not exist in the topos $\mathbf{RT}[\mathbb{A}_0]$, so we will internalize this diagram, because we know that every topos is internally complete. This means that we will exhibit an internal diagram $A \to \mathbb{N}^{op}$, that represents the external diagram. In a while, this statement will take a more precise meaning.

For the object A we take $(\{(\alpha, n) \mid \alpha \in I\mathbb{A}_n\}, =)$ with

$$[\![(\alpha, n) = (\beta, m)]\!] = \{m\} \cap \{n\} \wedge [\![\alpha =_n \beta]\!].$$

There is an evident projection $\pi : A \to \mathbb{N}^{op}$.

Now consider the pullback

$$\begin{tikzcd} s^*A \arrow[r, "\pi_A"] \arrow[d] & A \arrow[d, "\pi"] \\ \mathbb{N}^{op} \arrow[r, "s"] & \mathbb{N}^{op} \end{tikzcd}$$

where s denotes the successor map. We can write $s^*A = (\{(\alpha, n+1) \mid \alpha \in I\mathbb{A}_{n+1}\}, =)$ where the non-standard equality is the same as in A. There is a map from s^*A to A, given by

$$\cup\big((\alpha, n+1), (\beta, m)\big) = \{m\} \cap \{n\} \wedge [\![\cup\alpha =_n \beta]\!].$$

Let us recover the chain of frames from this internal diagram. It is not hard to verify that

$$\begin{tikzcd} U^n\Omega_n \arrow[r] \arrow[d] & A \arrow[d, "\pi"] \\ 1 \arrow[r, "\overline{n}"] & \mathbb{N}^{op} \end{tikzcd} \qquad \begin{tikzcd} U^{n+1}\Omega_{n+1} \arrow[r, "\cup"] \arrow[d] & U^n\Omega_n \arrow[d] \\ s^*A \arrow[r, "\cup"] & A \end{tikzcd}$$

are both pullback diagrams, so that we can recover the frames $U^n\Omega_n$ and the arrows $\cup : U^{n+1}\Omega_{n+1} \to U^n\Omega_n$. (This is what we mean by saying that $A \to \mathbb{N}^{op}$ *represents* the external diagram.)

Now $A \to \mathbb{N}^{op}$ is not only an internal diagram, but an internal diagram of frames:

Lemma 3.6.4 *$\pi : A \to \mathbb{N}^{op}$ is a frame object in the slice topos $\mathbf{RT}[\mathbb{A}_0]/\mathbb{N}^{op}$, and $\cup : s^*A \to A$ is a frame homomorphism.*

Proof. We know that each $U^n\Omega_n$ is a frame object in $\mathbf{RT}[\mathbb{A}_0]$, so there are realizers $e_n \in \mathbb{A}_0$ witnessing for

$$\mathbf{RT}[\mathbb{A}_0] \models U^n\Omega_n \text{ is a frame.}$$

This sentence is of course an abbreviation of a complex sentence in second order logic, expressing that all the frame axioms are satisfied. Now the point is, that the realizer e_0 works for all $n \in \mathbb{N}$. Therefore we also have

$$\mathbf{RT}[\mathbb{A}_0] \models \forall n \in \mathbb{N} : U^n\Omega_n \text{ is a frame.}$$

This means exactly that $A \to \mathbb{N}^{op}$ is a frame in the slice, and a similar argument shows that $\cup : s^*A \to A$ is a frame map over $\mathbb{N}^{op}$.

□

Calculation of the limit. We have an internal diagram, so by internal completeness we know that the limit exists. The following proposition gives an explicit description of the limiting frame.

Proposition 3.6.5 *The limit of the diagram $A \to \mathbb{N}^{op}$ is the object $(B, =)$, where the underlying set B can be written $B = \{(\alpha_0, \alpha_1, \ldots) \mid \alpha_i \in I\mathbb{A}_i\}$, while*

$$[\![(\alpha_n)_{n\in\mathbb{N}} = (\beta_n)_{n\in\mathbb{N}}]\!] = \forall n \in \mathbb{N}([\![\alpha_n =_n \beta_n]\!] \wedge [\![\cup\alpha_{n+1} =_n \alpha_n]\!])$$

Proof. Recall that the limit of a diagram may be taken to be the object of sections of the corresponding discrete opfibration. Any section of $A \to \mathbb{N}^{op}$ is easily seen to give an element of B. And two sections are isomorphic if and only if their corresponding elements $b, b' \in B$ have $[\![b = b']\!] \neq \emptyset$.

□

Remark. We may call the elements of B *recursively coherent* or simply *coherent*[1].

Let us now examine the frame structure of $(B, =)$ a bit; we just sketch the constructions without proving their correctness.

1. The ordering on $(B, =)$ is given by (for $\alpha = (\alpha_0, \alpha_1, \ldots), \beta = (\beta_0, \beta_1, \ldots)$):

   $$[\![\alpha \leq \beta]\!] = E(\alpha) \wedge E(\beta) \wedge \forall n \in \mathbb{N} : \alpha_n \leq_n \beta_n.$$

   Here $\leq_n$ denotes the ordering in the frame $U^n\Omega_n$. In other words, the ordering is coordinatewise.

2. Meets are also given coordinatewise, as:

   $$\alpha \wedge \beta = (\alpha_n \wedge \beta_n)_{n\in\mathbb{N}}.$$

3. For the supremum map, observe first that

$$\begin{tikzcd}
\Omega_0^{U^{n+1}\Omega_{n+1}} \arrow[r, "\bigvee_{n+1}"] \arrow[d, "\Omega_0^{(\cup)}"'] & U^{n+1}\Omega_n \arrow[d, "\cup"] \\
\Omega_0^{U^n\Omega_n} \arrow[r, "\bigvee_n"] & U^n\Omega_n
\end{tikzcd}$$

---

[1] It would be nice if the collection of recursive sequences could be endowed with an OPCA-structure, or if it would give rise to a tripos. But I couldn't see how...

commutes, where the maps $\bigvee_i : \Omega_0^{U^i\Omega_i} \to U^i\Omega_i$ are the supremum maps of the frames $U^i\Omega_i$. Moreover, we have this uniformely in $\mathbb{N}$ so that

$$\mathbf{RT}[\mathbb{A}_0] \models \forall n \in \mathbb{N} : \cup \circ \bigvee_{n+1} = \bigvee_n \circ \Omega_0^{(\cup)}$$

Also, we have projections $\pi_n : (B, =) \to U^n\Omega_n$, which can be represented by (again for $\alpha = (\alpha_0, \alpha_1, \ldots)$)

$$\pi_n(\alpha, \xi) = E(\alpha) \wedge [\![\alpha_n =_n \xi]\!].$$

Now we can define the supremum map $\bigvee : \Omega_0^{(B,=)} \to (B, =)$ as the map represented by the functional relation (where $\Gamma \in \Omega_0^B$ and $\alpha = (\alpha_0, \alpha_1, \ldots)$ is a coherent sequence):

$$\bigvee(\Gamma, \alpha) = E(\Gamma) \wedge E(\alpha) \wedge \forall n \in \mathbb{N}\ \exists P_n \subseteq U^n\Omega_n.\ \Omega_0^{\pi_n}(\Gamma, P_n) \wedge \bigvee_n(P_n, \alpha_n)$$

4. The implication map will be defined using the higher-order definition of implication in terms of the supremum map. The reason for this is the following: if we take two coherent sequences α, β, then the result of putting $(\alpha \to \beta)_n = (\alpha_n \to \beta_n)$ is not necessarily a coherent sequence again. (The reason for this is the fact that the union maps do not preserve implication.)

   So, first we associate with each pair of coherent sequences α, β, a predicate P on $(B, =)$:
   $$P(\gamma) = (\gamma \wedge \alpha \leq \beta).$$

   This is easily seen to be strict and relational, so we have a morphism $F : (B, =) \times (B, =) \to \Omega_0^{(B,=)}$:

   $$F(\alpha, \beta, P) = E(\alpha) \wedge E(\beta) \wedge E(P) \wedge \forall \gamma \in B\ (\gamma \wedge \alpha \leq \beta \leftrightarrow \gamma \in P).$$

   We compose this map with the supremum map to obtain the implication:

$$\begin{tikzcd}
(B,=) \times (B,=) \arrow[r, "F"'] \arrow[rr, bend left, "\to"] & \Omega_0^{(B,)} \arrow[r, "\bigvee"'] & (B,=)
\end{tikzcd}$$

The Topos Sh$(B, =)$**.** Our next focus is the topos $Sh(B, =)$. First, observe that this topos is of the form $\mathcal{P} -$ **Sets** for some tripos $\mathcal{P}$. This follows from Pitts' iteration theorem for triposes. We will use the technique displayed in [59] to sketch a calculation of the canonical presentation of this tripos.

The first step in the calculation consists of presenting our newly found locale object $(B, =)$ as a subquotient of a sheaf. For this sheaf, we take $\nabla(I\mathbb{A}_0^B)$. Then define a predicate $S : I\mathbb{A}_0^B \to I\mathbb{A}_0$ by

$$S(P) = \exists \beta \in B\ (\beta \in P \wedge \forall \beta'\ (\beta' \in P \leftrightarrow \beta = \beta'))$$

This predicate S expresses that an element P of $I\mathbb{A}_0^B$ is a singleton for $(B, =)$. Now S is strict and relational, so represents a subobject $\mid S \mid$ of the sheaf $\nabla(I\mathbb{A}_0)^B)$. Finally, an epimorphism $b : \mid S \mid \rightarrow (B, =)$ is defined by

$$b(P, \beta) = S(P) \wedge \beta \in P.$$

This gives the presentation:

$$\begin{tikzcd}
\mid S \mid \arrow[r, two heads, "b"] \arrow[d, tail, "i"'] & (B, =) \\
\nabla(I\mathbb{A}_0^B) &
\end{tikzcd}$$

By Pitts' iteration theorem, the tripos $\mathcal{P}$ assigns to a set X the collection $\mathcal{P}(X) = \mathbf{RT}[\mathbb{A}_0](\nabla X, (B, =))$. We will now calculate the map $\exists_i b : \nabla(I\mathbb{A}_0^B) \rightarrow (B, =)$ as the composition

$$\nabla(I\mathbb{A}_0^B) \xrightarrow{\psi} \Omega_0^{|S|} \xrightarrow{\Omega_0^b} \Omega_0^{(B,=)} \xrightarrow{\bigvee_0} (B, =).$$

Here, ψ is the transpose of the characteristic map of the graph of i. It can now be shown that compostion with $\exists_i b$ induces a bijection between the Hom-sets $\mathbf{RT}[\mathbb{A}_0](\nabla X, (B, =))$ and $\mathbf{RT}[\mathbb{A}_0](\nabla X, \nabla(I\mathbb{A}_0^B)) \simeq \mathbf{Sets}(X, I\mathbb{A}_0^B)$. Therefore, the canonical presentation $\mathcal{P}_C$ of the tripos $\mathcal{P}$ is given by

$$\mathcal{P}_C(X) = \{f : X \rightarrow I\mathbb{A}_0^B\}.$$

The Heyting structure on this function space is induced by the internal locale structure on $(B, =)$, via composing with $\exists_i b$.

Universal Property. We now investigate the relation between the toposes $\mathbf{RT}[\mathbb{A}_i]$ and $Sh(B, =)$, and the universal property of the topos $Sh(B, =)$.

Lemma 3.6.6 *Every* $\mathbf{RT}[\mathbb{A}_i]$ *is a retract of* $Sh(B, =)$.

Proof. We do this for $i = 1$. In $\mathbf{RT}[\mathbb{A}_0]$, we have the following maps:

$$\begin{tikzcd}
\Omega_0 \arrow[r, bend left=35, "D_\omega"] \arrow[r, bend right=35, "P_\omega"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & (B, =) \arrow[l, "U_\omega" description]
\end{tikzcd}$$

The map U_ω is the projection of the limit frame onto Ω_0, and may be defined as $U_\omega((\beta_0, \beta_1, \ldots), \alpha) = [\![\alpha =_0 \beta_0]\!]$. The map D_ω is defined $D_\omega(\alpha, (\beta_0, \beta_1, \ldots)) = \forall n\, [\![D^n(\alpha) =_n \beta_n]\!]$, and P_ω is given by $P_\omega(\alpha, (\beta_0, \beta_1, \ldots)) = \forall n\, [\![P^n(\alpha) =_n \beta_n]\!]$.

Of course, the maps U_ω and D_ω are frame maps (the map P_ω is not), so we obtain a local localic map of toposes

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_0] \arrow[r, bend left=35, "D_\omega"] \arrow[r, bend right=35, "P_\omega"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & Sh(B, =). \arrow[l, "U_\omega" description]
\end{tikzcd}$$

□

Next, we state the universal property that the topos $Sh(B,=)$ has:

Proposition 3.6.7 *Let Y be a locale in $\mathbf{RT}[\mathbb{A}_0]$, and let a cocone be given, as in the diagram*

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_0] \arrow[r, hook] \arrow[drr, "\phi_0"'] & \mathbf{RT}[\mathbb{A}_1] \arrow[r, hook] \arrow[dr, "\phi_1"] & \cdots \arrow[d, dotted, no head] \\
& & Sh(Y)
\end{tikzcd}$$

If $\mathbf{RT}[\mathbb{A}_0] \models \forall n \in \mathbb{N}$: "ϕ_n^* is a frame map", then there is a geometric morphism $\phi_\omega : Sh(B) \to Sh(Y)$ making every

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_n] \arrow[r, hook] \arrow[dr, "\phi_n"'] & Sh(B,=) \arrow[d, "\phi_\omega"'] \\
& Sh(Y)
\end{tikzcd}$$

commute; moreover, this geometric morphism is unique up to isomorphism with these properties.

Proof. Because $\mathbf{RT}[\mathbb{A}_0] \models \forall n \in \mathbb{N}$: "ϕ_n^* is a frame map", there is a unique frame homomorphism $p : Y \to (B,=)$, making the diagrams

$$\begin{tikzcd}
\cup^n \Omega_n & (B,=) \arrow[l, "U_\omega"'] \\
& Y \arrow[u, "p"'] \arrow[ul, "\phi_n^*"]
\end{tikzcd}$$

commute. This forces the associated geometric morphism to commute as well.

□

Example. Let us illustrate the universal property of the topos $Sh(B,=)$ by considering another cocone. This cocone will stem from a cocone in the category of ordered PCAs over the diagram $\mathbb{A}_0 \hookrightarrow \mathbb{A}_1 \hookrightarrow \ldots$. So we will construct an ordered PCA $\mathbb{A}_\omega$ and maps $\mathbb{A}_n \to \mathbb{A}_\omega$.

The underlying set of $\mathbb{A}_\omega$ is the disjoint union of the sets A_n, divided by the equivalence relation generated by:

$$(\alpha, n) \sim (\downarrow(\alpha), n+1).$$

So we put $A_\omega = \left(\coprod_{n\in\mathbb{N}} A_n\right)/\sim$, and we write $[\alpha]$ for its elements.

The induced ordering on this set A_ω can be described as: $[\alpha] \leq [\beta]$ iff there are representatives (α', n) of $[\alpha]$ and (β', n) of $[\beta]$, with $\alpha' \leq \beta'$. (So, the underlying partial ordering is the colimit of the chain of posets.)

Similarly we define a partial application on A_ω by putting $[\alpha] \bullet [\beta]\downarrow$ iff there are representatives (α', n) of $[\alpha]$ and (β', n) of $[\beta]$, such that $\alpha' \bullet \beta' \downarrow$. In this case, $[\alpha] \bullet [\beta] = [\alpha' \bullet \beta']$. It is not a hard exercise that all this is well-defined on equivalence classes, and that the axioms for an ordered PCA are satisfied.

We give a convenient representation of downsets in $\mathbb{A}_\omega$, so that we do not have to work with equivalence classes all the time.

Lemma 3.6.8 *Consider the set* $R = \{(\alpha_0, \alpha_1, \dots) \mid \alpha_i \in I(\mathbb{A}_i), \alpha_i = \cup\alpha_{i+1}\}$. *The elements of* R *are in bijective correspondence with those of* $I(A_\omega)$.

We call the elements of R *strictly coherent*.

Proof. Take $\alpha = (\alpha_0, \alpha_1, \dots) \in R$. Consider $\hat{\alpha} = \{[a] \mid a \in \alpha_n\}$. It is not hard to show that $\hat{\alpha}$ is downward closed. Conversely, given $\beta \in I(\mathbb{A}_\omega)$, define $\tilde{\beta} = (\sigma_0, \sigma_1, \dots)$ as $\sigma_i = \{a \in A_i \mid [a] \in S\}$. These two assignments are inverse. This representation is useful when we want a description of the tripos associated with $\mathbb{A}_\omega$: for two functions $\phi, \psi : X \to R$, put

$$\phi \vdash \psi \text{ iff } \exists a \in A_0 : \forall n \in \mathbb{N} : a \in \bigcap_{x \in X} (\phi(x))_n \to (\psi(x))_n$$

Intuitively, $\phi \vdash \psi$ iff there is a realizer that works in all triposes in the chain.

Lemma 3.6.9 *For each* $n \in \mathbb{N}$ *there is a local localic map from* $\mathbf{RT}[\mathbb{A}_\omega]$ *to* $\mathbf{RT}[\mathbb{A}_n]$.

Proof. We prove this for $n = 0$. The 0-th OPCA $\mathbb{A}_0$ is a sub-OPCA of $\mathbb{A}_\omega$; an element $a \in \mathbb{A}_0$ can be identified with $[a] \in \mathbb{A}_\omega$. Take any $[b]$ represented by $b \in A_m$. Then choose $a \in A_0$ with the property that for any $b' \in A_m$: if $bb'\downarrow$ then $i(a)b'\downarrow$ & $i(a)b' \le bb'$ (where i is the inclusion $\mathbb{A}_0 \hookrightarrow \mathbb{A}_m$). Now it easily follows that $[a]$ has the property that for each $[b'] \in \mathbb{A}_\omega$: if $[b][b']\downarrow$ then $[a][b']\downarrow$ & $[a][b'] \le [b][b']$.

We have shown that the condition of Proposition 3.6.7 is fulfilled.

□

The functors that constitute this local localic map have a simple description on the tripos-level. $D_n, P_n : I(\mathbb{A}_n) \to R, U_n : R \to I(\mathbb{A}_n)$ are defined (remember the definition of the functors P, D and U from the previous section):

$$\begin{aligned} D_n(\alpha) &= (U^n\alpha, U^{n-1}\alpha, \dots, U\alpha, \alpha, P\alpha, P^2\alpha, \dots) \\ P_n(\alpha) &= (U^n\alpha, U^{n-1}\alpha, \dots, U\alpha, \alpha, D\alpha, D^2\alpha, \dots) \\ U_n(\beta_0, \beta_1, \dots) &= \beta_n. \end{aligned}$$

Lemma 3.6.10 *For each $n \in \mathbb{N}$, there are commutative diagrams*

$$\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_n] \arrow[r, shift right, "P"'] \arrow[r, phantom, "\perp"] \arrow[dr, shift right, "P_n"'] & \mathbf{RT}[\mathbb{A}_{n+1}] \arrow[l, shift right, "U"'] \arrow[d, shift right, "P_{n+1}"'] \arrow[d, phantom, "\vdash"] \\
& \mathbf{RT}[\mathbb{A}_\omega] \arrow[u, shift right, "U_{n+1}"'] \arrow[ul, shift right, "U_n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{RT}[\mathbb{A}_{n+1}] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "D_{n+1}"'] \arrow[d, phantom, "\dashv"] & \mathbf{RT}[\mathbb{A}_n] \arrow[l, shift right, "D"'] \arrow[dl, shift right, "D_n"'] \\
\mathbf{RT}[\mathbb{A}_\omega] \arrow[u, shift right, "U_{n+1}"'] \arrow[ur, shift right, "U_n"'] &
\end{tikzcd}$$

Proof. This is easily checked on the level of the OPCA's involved: if $\alpha \in \mathbb{A}_n$ then $D(\alpha) = \downarrow\{\alpha\} \in \mathbb{A}_{n+1}$. But $[\alpha] \sim [\downarrow\{\alpha\}]$, so the inclusion of $\mathbb{A}_n$ into $\mathbb{A}_\omega$ factors through $\mathbb{A}_{n+1}$.

□

Summarizing the previous lemmas, we get:

Corollary 3.6.11 *Every* $\mathbf{RT}[\mathbb{A}_n]$ *is a retract of the topos* $\mathbf{RT}[\mathbb{A}_\omega]$*, so that* $\mathbf{RT}[\mathbb{A}_\omega]$ *is the vertex of a cocone over the diagram* $\mathbf{RT}[\mathbb{A}_0] \hookrightarrow \mathbf{RT}[\mathbb{A}_1] \hookrightarrow \cdots$*. and, at the same time, the vertex of a cone over the diagram* $\mathbf{RT}[\mathbb{A}_0] \leftarrow \mathbf{RT}[\mathbb{A}_1] \leftarrow \cdots$*.*

The cocone just described certainly satisfies the conditions of proposition 3.6.7, so we get a geometric morphism $q : Sh(B, =) \to \mathbf{RT}[\mathbb{A}_\omega]$. To describe this morphism, observe that $\mathbf{RT}[\mathbb{A}_\omega]$ is a localic extension of $\mathbf{RT}[\mathbb{A}_0]$, and that the associated locale in $\mathbf{RT}[\mathbb{A}_0]$ is the object $(R, =_R)$, where R is again the set of strict sequences, and

$$[\![(\alpha_i)_{i\in\mathbb{N}} =_R (\beta_i)_{i\in\mathbb{N}}]\!] = \{n \in \mathbb{N} | \forall i \in \mathbb{N} : n \bullet i \in [\![\alpha_i =_i \beta_i]\!]\}.$$

There is a frame map $p : (R, =_R) \to (B, =)$, which is simply the inclusion of strict sequences into recursive sequences, and the geometric morphism $q : Sh(B, =) \to \mathbf{RT}[\mathbb{A}_\omega]$ is induced by this frame map.

Open Problem. We have only sketched the structure of the topos $Sh(B, =)$, and its tripos. We couldn't see whether it is of the form $\mathbf{RT}[\mathbb{C}]$ for some ordered PCA $\mathbb{C}$. Moreover, it would be good to know some of the logical properties of the topos, in particular it would be interesting to have a description of the logic of the natural number object.

Chapter 4

Relative Completions

The material in this chapter is based on a paper that has been submitted for publication. The text below is largely the same in that paper; I have adapted terminology and typography to the rest of the thesis, some of the preliminaries have been left out, because they were already in chapter 2. Also, I have expanded the material at various places.

4.1 Introduction

Since the discovery that realizability toposes enjoy a certain universal property, a lot of work has been done on the study of regular and exact completions, especially their applications to categories that play a prominent role in realizability. The most important (and best-known) results in this area are, that, starting from a partial combinatory algebra $\mathbb{A}$, the category of Assemblies $\mathbf{Ass}(\mathbb{A})$ is the regular completion of the category of Partitioned Assemblies $\mathbf{PAss}(\mathbb{A})$, and that the realizability topos $\mathbf{RT}(\mathbb{A})$ is the exact completion of $\mathbf{PAss}(\mathbb{A})$. These results are useful, because they give a simple presentation of a realizability topos and also display some of its structure. An important restriction is, however, that they rely on an essential use of the axiom of choice in the base topos. For example, if one is to show that the Effective Topos arises as an exact completion, then one has to show that **Eff** has enough projectives. But in order to do so, one cannot avoid an appeal to choice in **Set**.

The work that we present here is intended as a first attempt at analysing what happens if we wish to refrain from using choice. Put differently, what happens when we do not work over the base topos **Set**, but over an arbitrary topos $\mathcal{E}$, in which the axiom of choice fails? Can the construction of a realizability topos then still be seen as a solution to a universal problem? Is it still some kind of completion of the category of Partitioned Assemblies?

The chapter is structured in the following manner: section 2 will contain some basic results about the categories of Partitioned Assemblies and Assemblies over an arbitrary base topos. These results are unsurprising, but not recorded before, whence their inclusion.

Section 3 contains the central definitions of the *relative regular completion* and the *relative exact completion.* The idea is, that we do not simply form the completion of a category $\mathcal{C}$, but take into account that there is a functor $F : \mathcal{E} \to \mathcal{C}$, which bears information about how $\mathcal{C}$ is related to the base topos. We get the following picture:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] & \mathcal{C}_{reg} \arrow[r, "P_\Sigma"] & \mathcal{C}_{\mathcal{E}/reg} \end{tikzcd}$$

where $\mathcal{C}_{\mathcal{E}/reg}$ denotes the relative regular completion. In fact, it will be constructed from $\mathcal{C}_{reg}$ as a category of fractions. Thus we get a quotient functor P_Σ as in the picture above. After explaining the construction, we give some simple examples, and we also show, that the construction is monadic, in a suitable 2-categorical sense.

The focus of sections 4, 5 and 6 is an analysis of the functor P_Σ: this is mainly motivated by the fact that the definition of section 3 is not very elegant, and far from convenient to work with. Therefore we give two different presentations of the relative completion: the first one (section 4) makes use of pushouts in the category of regular categories, and the second one (section 5) is based on topologies, as introduced in [57]. This enables us to identify some situations in which the relative completion of a category is somewhat better behaved than in general. In particular, we find a simple condition under which $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$. Section 6 is devoted to a more detailed analysis of the situation where the relative completion is a reflective subcategory of the ordinary completion, that is, when the functor P_Σ has a full and faithful right adjoint.

With the theory from sections 5 and 6, we have the major ingredients for our characterization of assemblies, which, together with locales, will be carried out in section 7. This will also answer the initial question that we posed, namely that the realizability topos can still be seen as a completion of the category of partitioned assemblies, namely the relative exact completion.

Finally, we present a number of open questions related to our constructions, to which we think it would be nice to have an answer.

4.2 Preliminary Results

Assemblies. Recall the definitions of the categories of *Partitioned Assemblies*, $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, and *Assemblies*, $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$, where $\mathbb{A}$ is some PCA in a base topos $\mathcal{E}$. The objects of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ are pairs (X, α), where X is an object of $\mathcal{E}$, and $\alpha : X \to \mathbb{A}$ is a map in $\mathcal{E}$ to the internal pca $\mathbb{A}$. An arrow from (X, α) to (Y, β) is a map $f : X \to Y$ in $\mathcal{E}$ such that $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X.\ a \bullet \alpha(x)\downarrow\ \wedge a \bullet \alpha(x) = \beta(fx)$. An assembly is also a pair (X, α), but now $\alpha : X \to \mathcal{P}_i(\mathbb{A})$, where $\mathcal{P}_i(\mathbb{A})$ stands for the object of inhabited subsets of $\mathbb{A}$. Similarly, a map $f : X \to Y$ is a map of assemblies if we have $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X \forall b \in \alpha(x).\ a \bullet b\downarrow\ \wedge a \bullet b \in \beta(fx)$.

As usual, we have an embedding $\nabla : \mathcal{E} \to \mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, that has a faithful left adjoint, denoted Γ. ∇ preserves regular epis, although $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ is not a regular category. We use the same notation ∇, Γ to denote the localization of $\mathcal{E}$ in $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$. Again, ∇ is an exact functor.

Lemma 4.2.1 *The category* $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ *of assemblies is equivalent to the full subcategory of* $\mathbf{RT}_{\mathcal{E}}[\mathbb{A}]$ *on the subobjects of objects of the form* $\nabla(X)$.

Proof. This is straightforward.

□

In the chapter on ordered PCAs, it is explained that there is a monad on the category of ordered partial combinatory algebras, based on the fact that the collection of non-empty downsets in an ordered PCA inherits the combinatorial structure. This generalizes to OPCAs in an arbitrary topos. Thus we get that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is equivalent to $\mathbf{PAss}_{\mathcal{E}}(I\mathbb{A})$, where I is the nonempty downset-monad.

In the classical case, one has a convenient characterization of regular epis in Assemblies; this goes through in the general setting:

Lemma 4.2.2 *In* $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$, *a map* $e' : (Y', \beta') \to (X, \alpha)$ *is regular epi if and only if it is isomorphic (over* (X, α)*) to a map* $e : (Y, \beta) \to (X, \alpha)$ *that satisfies* $\alpha(x) = \bigcup_{e(y)=x} \beta(y)$.

Proof. As usual.

□

Corollary 4.2.3 *The functor* $\nabla : \mathcal{E} \to \mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ *preserves regular epis, and hence* $\Gamma : \mathbf{Ass}_{\mathcal{E}}(\mathbb{A}) \to \mathcal{E}$ *preserves regular projectives.*

Proof. Immediate.

□

Lemma 4.2.4 *An object* (X, α) *in* $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ *is projective if and only if it is a partitioned assembly and* X *is projective in* $\mathcal{E}$.

Proof. Observe first that any assembly can be covered by a partitioned assembly, namely cover (X, α) by (Q, π), where $Q = \{(x, a) | a \in \alpha(x)\}$. Moreover, the partitioned assemblies are closed under finite limits. Now if (X, α) is projective, then this cover has a section, presenting (X, α) as a regular subobject of a partitioned assembly, hence as a partitioned assembly. Also, X is projective in $\mathcal{E}$ by the previous lemma.

Conversely, any partitioned assembly (X, α) with X projective in $\mathcal{E}$ is projective. For let $e : (Y, \beta) \to (X, \alpha)$ be regular epi. Then $e(y) = x$ implies $\beta(y) = \alpha(x)$. So take any section in $\mathcal{E}$, and it will be tracked by the identity.

□

We refer to the covering Q as in the lemma as the *canonical covering* of (X, α). From this lemma it follows that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is in general not equivalent to the regular completion of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, since in this completion, every partitioned assembly is projective.

Finally, we recall a folklore theorem [18]:

Theorem 4.2.5 *Let $\mathcal{P}$ be a tripos on a category $\mathcal{C}$, let $\mathcal{C}[\mathcal{P}]$ denote the resulting topos and write $\nabla : \mathcal{C} \to \mathcal{C}[\mathcal{E}]$ for the constant objects functor. Then $\mathcal{C}[\mathcal{P}]$ is the ex/reg-completion of its full subcategory on the subobjects of objects in the image of ∇.*

For us, the main implication of this theorem is, that the realizability topos $\mathbf{RT}[\mathbb{A}]$ is the *ex*/*reg*-completion of $\mathbf{Ass}(\mathbb{A})$.

4.3 A Universal Construction

We first introduce the relative version of the regular completion. Then we look at the 2-categorical aspects of the construction.

4.3.1 Relative regular completion

We fix a category $\mathcal{E}$ with finite limits (this is the minimum amount of structure required for the construction[1]; in most applications however, $\mathcal{E}$ will be a topos). Consider the category $\mathcal{E}/\mathbf{LEX}$. Objects are left exact functors $F : \mathcal{E} \to \mathcal{C}$ with $\mathcal{C}$ a lex category, and morphisms are commutative triangles of lex functors. Similarly, we have a category $\mathcal{E}/\mathbf{REG}$ where all categories and functors involved are regular, and $\mathcal{E}/\mathbf{EX}$, where all categories and functors are exact. The theorem that we aim for is the following:

Theorem 4.3.1 *The forgetful functor $\mathcal{E}/\mathbf{REG} \to \mathcal{E}/\mathbf{LEX}$ has a left biadjoint.*

Proof. Send $F : \mathcal{E} \to \mathcal{C}$ to the composite

$$\begin{tikzcd}\mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] & \mathcal{C}_{reg} \arrow[r, "P_\Sigma"] & \mathcal{C}_{reg}[\Sigma^{-1}].\end{tikzcd}$$

Here, $\mathcal{C}_{reg}[\Sigma^{-1}]$ refers to the category obtained from $\mathcal{C}_{reg}$ by formally inverting all arrows in a class Σ. This class of arrows Σ is defined as follows: consider a regular epi $f : X \to Y$ in $\mathcal{E}$. The functor F sends f to Ff, and the embedding y takes this to yFf. In $\mathcal{C}_{reg}$, the arrow yFf has a regular epi-mono factorization, as in the diagram:

$$\begin{tikzcd}yFX \arrow[r, two heads, "{[1]}"] & \begin{pmatrix} FX \\ Ff\downarrow \\ FY \end{pmatrix} \arrow[r, tail, "{[Ff]}"] & yFY.\end{tikzcd}$$

The reflection of $F : \mathcal{E} \to \mathcal{C}$ in $\mathcal{E}/\mathbf{REG}$ must be a regular functor, which means that the arrow $[Ff]$ has to be inverted. So define Σ_0 to be the class of all the arrows $[Ff]$ that arise as in diagrams such as the one above. Then define Σ to be the least class of maps containing Σ_0, with the properties that

- All isomorphisms are in Σ,

---

[1] Well, one can also do these kind of things with weak finite limits, but we leave that aside.

- If two out of three sides of a commutative triangle are in Σ, then so is the third,
- Σ is pullback-stable,
- If $e^*\sigma \in \Sigma$ for some regular epi e, then $\sigma \in \Sigma$.

Following Bénabou, we call a collection of arrows Σ satisfying these closure properties a *local pullback congruence*. Now it follows from the theory of categories of fractions that $\mathcal{C}_{reg}[\Sigma^{-1}]$ is a regular category, and that P_Σ is a regular functor (see [11], Theorem 2.2.2).

For the universal property, consider any left exact functor $G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a regular category, and where the composite GF is regular. Then in the diagram below:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] \arrow[rrd, "G"'] & \mathcal{C}_{reg} \arrow[r, "P_\Sigma"] \arrow[rd, dashed, "\hat{G}"] & \mathcal{C}_{reg}[\Sigma^{-1}] \arrow[d, dashed, "\tilde{G}"] \\
 & & & \mathcal{D}
\end{tikzcd}$$

the regular functor $\hat{G}$ arises because of the universal property of $\mathcal{C}_{reg}$. $\hat{G}$ inverts all arrows in Σ_0 and therefore also all arrows in Σ. Hence the universal property of the category of fractions gives us the required regular $\tilde{G}$.

□

We introduce the following terminology: given $F : \mathcal{E} \to \mathcal{C}$ left exact, we shall write $\mathcal{C}_{\mathcal{E}/reg}$ for the value (at F) of the biadjoint of theorem 4.3.1, and we call it the *relative regular completion* of $\mathcal{C}$ (relative to $\mathcal{E}$). We are aware of the deficits of this notation - it omits the functor F, and it does not cohere completely with the usual $(-)_{reg/lex}$ "fractional" notation. But in practice, this will not pose any difficulties.

One can summarize the idea behind the construction as follows: the ordinary regular completion $y : \mathcal{C} \to \mathcal{C}_{reg}$ sends regular epis to epis which are not regular (except for those that have a splitting), so it destroys the regular structure that exists in $\mathcal{C}$. The fraction construction tries to restore as much of the regular structure that comes from $\mathcal{E}$ as possible.

Although we concentrate on the relative regular completion in this paper, we mention that there is also a natural notion of a relative exact completion:

Theorem 4.3.2 *The forgetful functor from $\mathcal{E}/\mathbf{EX} \to \mathcal{E}/\mathbf{REG}$ has a left biadjoint.*

Proof. Send $F : \mathcal{E} \to \mathcal{C}$ to the composite

$$\mathcal{E} \xrightarrow{F} \mathcal{C} \xrightarrow{y} \mathcal{C}_{ex/reg}.$$

The universal property is the same as that for the ordinary ex/reg-completion.

□

Let us denote these biadjoints by $(-)_{\mathcal{E}/reg}$,$(-)_{\mathcal{E}/ex/reg}$ and the composite (the relative exact completion) $((-)_{\mathcal{E}/ex/reg})_{\mathcal{E}/reg}$ by $(-)_{\mathcal{E}/ex}$.

Our motivating examples, namely Partitioned Assemblies and locales, will appear in section 7. At this point, we will give some simpler examples, to give the reader a feel for the construction.

Examples 4.3.3

1. First of all, let's see what happens when applying this construction to the identity on $\mathcal{C}$, when $\mathcal{C}$ is an arbitrary finite limit category. If there exists a regular epi in $\mathcal{C}$ which has no section, then such a map will still be a regular epi in the relative completion $\mathcal{C}_{\mathcal{C}/reg}$, and hence $\mathcal{C}_{\mathcal{C}/reg}$ will differ from $\mathcal{C}_{reg}$. Somewhat more generally, let $\mathcal{C}' \to \mathcal{C}$ be the inclusion of a subcategory which is closed under finite limits. Then the relative completion may be seen as the closest approximation to $\mathcal{C}_{reg}$, in which the regular structure of $\mathcal{C}'$ is preserved.
2. Now let $\mathcal{C}$ be regular, and consider again the identity functor on $\mathcal{C}$. Then the relative completion of $\mathcal{C}$ is equivalent to $\mathcal{C}$ itself (and in fact, the quotient functor from $\mathcal{C}_{reg}$ to $\mathcal{C}_{\mathcal{C}/reg} \simeq \mathcal{C}$ is simply the left adjoint to the inclusion $y : \mathcal{C} \to \mathcal{C}_{reg}$).
3. On the other hand, we might take any finite limit category for $\mathcal{E}$, and the functor which has the terminal object 1 of $\mathcal{C}$ as constant value. Then the relative completion coincides with the ordinary regular completion. (For, each map in Σ is an isomorphism.) Somewhat more generally, if F sends every regular epi to an isomorphism, or even to a split epi, then $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$.
4. The ordinary regular completion will always send non-equivalent categories to non-equivalent completions. The relative version need not do so; as an example, take $F : \mathcal{E} \to \mathcal{C}$ such that $\mathcal{C}_{\mathcal{E}/reg}$ is not equivalent to $\mathcal{C}_{reg}$. Then consider $(\mathcal{C}_{\mathcal{E}/reg})_{\mathcal{E}/reg}$ and $(\mathcal{C}_{reg})_{\mathcal{E}/reg}$. These are easily seen to be equivalent, since they have the same universal property.

4.3.2 2-categorical aspects

We now turn attention to some 2-categorical aspects of our construction. In what follows, we will adopt the convention that by 2-functor, 2-monad, we actually mean pseudo-functor, pseudo-monad, and by algebra, retract, we actually mean pseudo-algebra, pseudo-retract. In short, we adhere to the terminology in [44].

First, the relative completion is 2-functorial. Given a finite limit-preserving functor $K : \mathcal{C} \to \mathcal{D}$ we get a regular extension $K_{\mathcal{E}/reg} : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{D}_{\mathcal{E}/reg}$ via the universal property of $\mathcal{C}_{\mathcal{E}/reg}$.

Next, just like the ordinary completion, the relative completion carries a 2-monad structure on the 2-category $\mathcal{E}/\mathbf{LEX}$.

- The unit η of the monad has components (at $F : \mathcal{E} \to \mathcal{C}$) given by the functor $P_\Sigma \circ y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$.

- The multiplication comes from the following diagram:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] & \mathcal{C}_{\mathcal{E}/reg} \arrow[r, "y"] \arrow[drr, "Id"'] & (\mathcal{C}_{\mathcal{E}/reg})_{reg} \arrow[r, "P_\Sigma"] & (\mathcal{C}_{\mathcal{E}/reg})_{\mathcal{E}/reg} \arrow[d, dashed, "\mu"] \\
 & & & \mathcal{C}_{\mathcal{E}/reg}.
\end{tikzcd}$$

  Since F is regular, so is $Id \circ F$, trivially. Hence we get an extension μ, which is unique up to isomorphism. Moreover, $\mu \circ P_\Sigma \circ y \cong Id$, so $\mathcal{C}_{\mathcal{E}/reg}$ is a retract of $(\mathcal{C}_{\mathcal{E}/reg})_{\mathcal{E}/reg}$.

The relation between the ordinary and the relative completion is further clarified by the following lemma.

Lemma 4.3.4 *The fraction-construction P_Σ may be viewed as a transformation from $(-)_{reg}$ to $(-)_{\mathcal{E}/reg}$ (where $(-)_{reg}$ is viewed as a 2-monad on the category $\mathcal{E}/$**LEX**, of course). The naturality squares are then pushouts.*

Proof. Given a left exact functor K in $\mathcal{E}/$**LEX**:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C} \arrow[d, "K"] \\
 & \mathcal{D}
\end{tikzcd}$$

consider the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C} \arrow[r, "y_\mathcal{C}"] \arrow[d, "K"] & \mathcal{C}_{reg} \arrow[r, "P_\Sigma(F)"] \arrow[d, "K_{reg}"] & \mathcal{C}_{\mathcal{E}/reg} \arrow[d, "K_{\mathcal{E}/reg}"] \\
 & \mathcal{D} \arrow[r, "y_\mathcal{D}"] & \mathcal{D}_{reg} \arrow[r, "P_\Sigma(G)"] & \mathcal{D}_{\mathcal{E}/reg}
\end{tikzcd}$$

in which the right hand square is the naturality square for the transformation P_Σ. (We have labelled the components $P_\Sigma(F), P_\Sigma(G)$ in order to make clear at which object of $\mathcal{E}/$**REG** we take the transformation.)

The diagram is easily seen to commute, so we show that the right-hand square is a pushout in $\mathcal{E}/$**REG**. To this end, assume that there are regular functors $M : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{K}$ and $N : \mathcal{D}_{reg} \to \mathcal{K}$, for which $M \circ P_\Sigma(F) \cong N \circ K_{reg}$. Then the composite $M \circ P_\Sigma(F) \circ y_\mathcal{C} \circ F \cong N \circ y_\mathcal{D} \circ G$ is regular and therefore there is a factorization of N through $P_\Sigma(G)$, by the universal property of $\mathcal{D}_{\mathcal{E}/reg}$. This factorization is essentially unique, again by the universal property.

□

Concerning the algebras for the monad $(-)_{\mathcal{E}/reg}$, we have the following result:

Theorem 4.3.5 *The category $\mathcal{E}/\mathbf{REG}$ is equivalent to the category of algebras for the monad $(-)_{\mathcal{E}/reg}$.*

Proof. First, it is immediate from the universal property of the relative completion that every object of $\mathcal{E}/\mathbf{REG}$ carries an algebra structure.

Next, I claim that every algebra is an object of $\mathcal{E}/\mathbf{REG}$. So, suppose an object $F : \mathcal{E} \to \mathcal{C}$ of $\mathcal{E}/\mathbf{LEX}$ has a algebra structure, i.e. a functor $a : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{C}$ for which $a \circ P_\Sigma \circ y \cong Id_{\mathcal{C}}$. This not only exhibits $\mathcal{C}$ as a retract of $\mathcal{C}_{\mathcal{E}/reg}$, but also as a retract of $\mathcal{C}_{reg}$. Now use the fact that the ordinary regular completion is a KZ-monad; this tells us that the category $\mathcal{C}$, as a retract of a free algebra, is itself a regular category. It remains to be seen that the functor $F : \mathcal{E} \to \mathcal{C}$ is also regular. To this end, take a regular epi $e : X \to Y$ in $\mathcal{E}$. Since $\mathcal{C}$ is regular, we get a diagram

$$\begin{tikzcd}
FX \arrow[rr, "Fe"] \arrow[dr, two heads, "q"'] & & FY \\
& Q \arrow[ur, tail] &
\end{tikzcd}$$

where $Fe = mq$ is the regular epi-mono factorization of Fe. Now it suffices to show that m is an isomorphism, because then F sends regular epis to regular epis. Now in $\mathcal{C}_{reg}$ we get a diagram

$$\begin{tikzcd}
y(FX) \arrow[r, two heads, "{[1]}"] \arrow[dr, two heads, "y(q)"'] & \begin{pmatrix} FX \\ Fe\downarrow \\ FY \end{pmatrix} \arrow[r, tail, "Fe"] \arrow[d, "{[q]}"] & y(FY). \\
& y(Q) \arrow[ur, tail, "y(m)"'] &
\end{tikzcd}$$

Here, the top row is the new regular epi-mono factorization of the map Fe. By definition of the class Σ in $\mathcal{C}_{reg}$, the comparison map $[Fe]$ is in Σ. Because the map $y(m)$ is mono and the right-hand triangle commutes, this forces that $y(m)$ is in Σ as well, so $P_\Sigma(y(m))$ is an isomorphism in $\mathcal{C}_{\mathcal{E}/reg}$. But since $\mathcal{C}$ is a retraction of $\mathcal{C}_{\mathcal{E}/reg}$, the map $P_\Sigma \circ y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ will reflect isomorphisms. Thus, m is already an isomorphism, as required.

Now, by the universal property of $\mathcal{C}_{\mathcal{E}/reg}$, such an algebra map is unique up to isomorphism, so that (isomorphism classes of) objects of $\mathcal{E}/\mathbf{REG}$ correspond to (isomorphism classes of) algebras.

Finally, algebra morphisms $\mathcal{C} \to \mathcal{D}$ are precisely functors in $\mathcal{E}/\mathbf{REG}$; this gives the desired equivalence.

□

4.4 Algebraic Presentation

In the case that $\mathcal{E}$ is itself a regular category, we can give an alternative characterization of the category $\mathcal{C}_{\mathcal{E}/reg}$. First, we show that $\mathcal{C}_{\mathcal{E}/reg}$ can be constructed as a pseudo-pushout in the category of regular categories. Note that, because the base category $\mathcal{E}$ is regular, the embedding $\mathcal{E} \to \mathcal{E}_{reg}$ has a regular left adjoint r.

Proposition 4.4.1 *Let $F : \mathcal{E} \to \mathcal{C}$ preserve finite limits. The following square is a pseudo-pushout in* **REG***:*

$$\begin{tikzcd}
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] & \mathcal{C}_{\mathcal{E}/reg}.
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{C} \arrow[d, "y"] \\
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] & \mathcal{C}_{\mathcal{E}/reg}.
\end{tikzcd}$$

First, the large square commutes since $r \circ y \cong Id$. Also, the top square commutes, so we have $P_\Sigma \circ F_{reg} \circ y \cong (P_\Sigma \circ y \circ F) \circ r \circ y : \mathcal{E} \to \mathcal{C}_{\mathcal{E}/reg}$. Both $P_\Sigma \circ F_{reg}$ and $(P_\Sigma \circ y \circ F) \circ r$ are regular functors from $\mathcal{E}_{reg}$ to $\mathcal{C}_{\mathcal{E}/reg}$, and hence determined up to isomorphism by their composites with $y : \mathcal{E} \to \mathcal{E}_{reg}$. These are isomorphic, so it follows that $P_\Sigma \circ F_{reg} \cong (P_\Sigma \circ y \circ F) \circ r$, and the lower square commutes.

For the universal property we take regular functors $G : \mathcal{C}_{reg} \to \mathcal{D}$ and $H : \mathcal{E} \to \mathcal{D}$, such that $H \circ r \cong G \circ F_{reg}$. Then because $H \cong H \circ r \circ y \cong G \circ F_{reg} \circ y$ and H is regular, $G \circ y \circ F : \mathcal{E} \to \mathcal{D}$ is also regular. By the universal property of $\mathcal{C}_{\mathcal{E}/reg}$, we obtain a factorization $G \cong K \circ P_\Sigma$. It only remains to be checked that $H \cong K \circ (P_\Sigma \circ y \circ F)$. But

$$\begin{aligned}
H &\cong H \circ r \circ y \\
&\cong G \circ F_{reg} \circ y \\
&\cong (K \circ P_\Sigma) \circ F_{reg} \circ y \\
&\cong K \circ (P_\Sigma \circ y \circ F),
\end{aligned}$$

which completes the proof.

□

We can also easily show the analogous statement for the relative exact completion (for this to make sense, assume $\mathcal{E}$ to be exact):

Proposition 4.4.2 *Let $\mathcal{E}$ be exact, $\mathcal{C}$ have finite limits and let $F : \mathcal{E} \to \mathcal{C}$ preserve finite limits. The following square is a pseudo-pushout in* **EX***:*

$$\begin{tikzcd}
\mathcal{E}_{ex} \arrow[r, "F_{ex}"] \arrow[d, "r"'] & \mathcal{C}_{ex} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{C}_{\mathcal{E}/ex}.
\end{tikzcd}$$

Proof. Apply the *ex/reg*-construction to the pushout of proposition 4.4.1. The *ex/reg*-construction is a left bi-adjoint, and therefore preserves pseudo-pushouts.

□

Before we have a look at some of the consequences of these presentations, we show that the situation is surprisingly similar to some constructions in algebra. For instance, let R be a ring, M a monoid and $f : R \to M$ a map of monoids. If we write $F(R)$ and $F(M)$ for the free rings on R and M, we construct a ring $F_R(M)$ by forming the pushout

$$\begin{tikzcd}
F(R) \arrow[r, "F(f)"] \arrow[d] & F(M) \arrow[d] \\
R \arrow[r] & F_R(M).
\end{tikzcd}$$

The ring $F_R(M)$ is the free ring on M such that $R \to F_R(M)$ is a ring homomorphism, i.e. for any ring N and any map of monoids $k : M \to N$ such that $kf : R \to N$ is a ringhomomorphism, there is a unique ring homomorphism $\hat{k} : F_R(M) \to N$ through which k factors.

Observe that it now follows that the relative exact completion can also be obtained as a category of fractions; this follows from the fact that for any functor $P : \mathcal{C} \to \mathcal{D}$ in **LEX**, and any class of maps Ξ in $\mathcal{C}$, the the following square is a pushout, where $P\Xi$ denotes the image of Ξ under P:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "P"] \arrow[d, "P_{\Xi}"'] & \mathcal{D} \arrow[d, "P_{P\Xi}"] \\
\mathcal{D}[\Xi^{-1}] \arrow[r] & \mathcal{D}[P\Xi^{-1}].
\end{tikzcd}$$

Combined with the fact that $\mathcal{E}$ is a localization of $\mathcal{E}_{ex}$ (and may therefore be seen as a category of fractions), we see that $\mathcal{C}_{ex} \to \mathcal{C}_{\mathcal{E}/ex}$, being a pushout of $\mathcal{E}_{ex} \to \mathcal{E}$, is itself of this form.

As a simple corollary of proposition 4.4.1, we get that the ordinary regular completion coincides with the relative completion when the base category $\mathcal{E}$ satisfies the axiom of choice (meaning that every regular epi splits):

Corollary 4.4.3 *If $\mathcal{E}$ is regular and $\mathcal{E} \models AC$, then $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$.*

Proof. If every regular epi splits in $\mathcal{E}$, then $\mathcal{E} \simeq \mathcal{E}_{reg}$ (see, for instance [57]). So, in the pushout square of proposition 4.4.1 the left-hand map is an equivalence, and therefore the right-hand map is an equivalence, too.

□

A converse to this corollary holds if we assume the functor $F : \mathcal{E} \to \mathcal{C}$ to be fully faithful:

Proposition 4.4.4 *If $F : \mathcal{E} \to \mathcal{C}$ is fully faithful, and $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$, then $\mathcal{E} \models AC$.*

Proof. Consider a regular epi $e : X \to Y$ in $\mathcal{E}$. This is sent to $yF(e) : yF(X) \to yF(Y)$ in $\mathcal{C}_{reg}$. This map is again regular epi, because the composite yF is now a regular functor. This in turn means that the mono part of the reg-epi/mono factorization of $yF(e)$ is an isomorphism. Thus it has an inverse

$$yF(Y) \xrightarrow{[k]} \begin{pmatrix} FX \\ Fe\downarrow \\ FY \end{pmatrix}$$

and the underlying arrow $k : FY \to FX$ is easily seen to be a splitting for Fe. Now F is full, so k is in the image of F, say $k = Fh$, and h is a splitting for e.

□

Similar statements hold when $\mathcal{E}$ is an exact category; since the proofs are the same as for the regular case, we omit them.

Proposition 4.4.5 *Let $\mathcal{E}$ be exact, and $F : \mathcal{E} \to \mathcal{C}$ be a finite limit-preserving functor. If $\mathcal{E} \models AC$, then $\mathcal{C}_{ex} \simeq \mathcal{C}_{\mathcal{E}/ex}$. If F is fully faithful, then the converse holds.*

Chaotic Situations. Finally, we show that relative completions, just like ordinary completions, inherit chaotic stiuations. The notion of a chaotic situation was formulated in [57]:

Definition 4.4.6 A left exact category $\mathcal{C}$ has $\mathcal{E}$ as a chaotic situation if $\mathcal{E}$ is a topos, and if there is an embedding $F : \mathcal{E} \to \mathcal{C}$ which has a faithful left exact left adjoint G.

Lemma 4.4.7 *If $F : \mathcal{E} \to \mathcal{C}$ is a chaotic situation, then $\mathcal{C}_{\mathcal{E}/reg}$ also has a chaotic situation.*

Proof. Denoting the left adjoint of F by G, the universal property of $\mathcal{C}_{reg}$ gives an extension of G to $\hat{G} : \mathcal{C}_{reg} \to \mathcal{E}$, which is left adjoint to the composite

$y \circ F : \mathcal{E} \to \mathcal{C}_{reg}$. Thus we get

$$\begin{tikzcd}
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \arrow[ddr, bend left, "\hat{G}"] & \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] \arrow[drr, bend right] & \mathcal{C}_{\mathcal{E}/reg} & \\
 & & \mathcal{E}
\end{tikzcd}$$

and there is a factorization through the pushout $\tilde{G} : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{E}$. It is easily verified that this map is again faithful, and left adjoint to the embedding of $\mathcal{E}$ in $\mathcal{C}_{\mathcal{E}/reg}$.

□

4.5 Sheaves

Next, we concentrate on a presentation in terms of sheaves. We make use of the notion of a quasi-topology and of a topology on $\mathcal{C}$. These were introduced in [57], but we provide a short recapitulation.

Definition 4.5.1 Let $\mathcal{C}$ be a finite limit category. A *quasi-topology* on $\mathcal{C}$ is a family $J(X)$ for each object X of $\mathcal{C}$, of maps with codomain X, subject to the following conditions:

- $1_X \in J(X)$
- for $f : Y \to X$, if $g \in J(X)$ then $f^*g \in J(Y)$ (where f^*g denotes the pullback of g along f)
- if $g \circ h \in J(X)$, then $g \in J(X)$
- if $f : Y \to X \in J(X)$ and $g \in J(Y)$ then $f \circ g \in J(X)$.

Definition 4.5.2 A map $h : Z \to X$ is *closed* for a quasi-topology J if for every $f : Y \to X$, $f^*h \in J(Y)$ implies that f factors through h.

Definition 4.5.3 A quasi-topology J is a *topology* if for every map $f : Y \to X$ there is a $g : V \to W \in J(W)$ and a closed $h : W \to X$ such that f factors through $h \circ g$ and vice versa.

The point of these definitions is, that topologies on $\mathcal{C}$ correspond to universal closure operators on $\mathcal{C}_{reg}$ (and on $\mathcal{C}_{ex}$). A (quasi-)topology J is called *subcanonical* if every map in J is regular epi.

Construction 4.5.4 Consider again a functor $F : \mathcal{E} \to \mathcal{C}$. We will construct a quasi-topology on $\mathcal{C}$ by defining:

- $f \in K(C)$ if and only if there is a diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] & C \arrow[d, "\alpha"'] \\
F(E') \arrow[r, "Fe"] & F(E)
\end{tikzcd}$$

  where e is a regular epi in $\mathcal{E}$, and the square is a pullback.

- J is the closure of K under composition and under right-halves, i.e. if $hk \in J$ then so is h.

The verification that J is a quasi-topology on $\mathcal{C}$ is straightforward. Now there is a technical lemma to be proved:

Lemma 4.5.5 *Let $f : X \to Y$ be a map in $\mathcal{C}$, inducing a mono $[f] : f \to yY$ in $\mathcal{C}_{reg}$. Then $f \in J(Y)$ implies $[f] \in \Sigma$.*

Proof. We show this by induction on the structure of J. Observe, for the basic case, that if f is of the form $F(e)$ with e regular epi in $\mathcal{E}$, then $[f] \in \Sigma_0$ (and vice versa). Next, if f arises as the pullback of such a map $F(e)$ as in the left diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\alpha"'] \\
FE' \arrow[r, "Fe"] & FE
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} M \\ p\downarrow \\ N \end{pmatrix} \arrow[dr, dashed, "{[u]}"] \arrow[drr, "{[r]}"] \arrow[ddr, "{[s]}"'] & & \\
& \begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix} \arrow[r] \arrow[d, "{[f]}"] & \begin{pmatrix} FE' \\ Fe\downarrow \\ FE \end{pmatrix} \arrow[d, "{[Fe]}"] \\
& y(Y) \arrow[r, "y(\alpha)"] & y(FE)
\end{tikzcd}$$

then we can show that the square in the diagram on the right is a pullback in $\mathcal{C}_{reg}$: for consider another object $(p : M \to N)$ in $\mathcal{C}_{reg}$, and maps $[r] : p \to Fe, [s] : p \to y(Y)$, such that $[Fe] \circ [r] = y(\alpha) \circ [s]$, i.e., $Fe \circ r = \alpha \circ s$. Since the left diagram is a pullback in $\mathcal{C}$, there is a unique $u : M \to X$, such that $\pi_Y \circ u = s, \pi_{FE} \circ u = r$. No in $\mathcal{C}_{reg}$, u induces a map $[u] : p \to f$, because (writing p_0, p_1 for the kernel pair of p), $f \circ u \circ p_0 = s \circ p_0 = s \circ p_1 = f \circ u \circ p_1$. This map $[u]$ is the unique map that makes $[r]$ and $[s]$ factor through the object f. Hence the square is a pullback. Since Σ was closed under pullbacks and the right-hand map was in it, so is the left-hand map.

Then, suppose that h is in $J(Y)$ because $f = hg$ is in $J(Y)$. By induction hypothesis, $[f] = [hg]$ is in Σ. We need to verify that $[h] \in \Sigma$. But $[h]$ is mono, and Σ is a pullback congruence, so if $[hg]$ will be inverted, so will $[h]$, and therefore $[h] \in \Sigma$.

Finally, consider a composite of such arrows (it suffices to look only at a binary composite): suppose $h \in J(Y), g \in J(Y')$, so that, by induction hypothesis, $[h] : h \to y(Y), [g] : g \to y(Y') \in \Sigma$. We must show that $[hg] : hg \to y(Y) \in \Sigma$. First, consider the following pullbacks, where the first one is in $\mathcal{C}$, and the second one in $\mathcal{C}_{reg}$:

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, "1"] \arrow[dddr, "g"'] & & \\
& X \times_Y Y' \arrow[r] \arrow[dd] & X \arrow[d, "g"] \\
& & Y' \arrow[d, "h"] \\
& Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{pmatrix} X \\ g\downarrow \\ Y' \end{pmatrix} \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& \begin{pmatrix} X \times_Y Y' \\ \downarrow \\ Y' \end{pmatrix} \arrow[r, two heads] \arrow[d, tail] & \begin{pmatrix} X \\ g\downarrow \\ Y' \\ h\downarrow \\ Y \end{pmatrix} \arrow[d, tail, "{[g]}"] \\
& yY' \arrow[r, two heads, "{[1]}"] & \begin{pmatrix} Y' \\ h\downarrow \\ Y \end{pmatrix}
\end{tikzcd}$$

In the second diagram, the object $X \times_Y Y' \to Y'$ is the projection as in the first pullback. It is easily verified that the outer square of the second diagram commutes, so there is a factorization through the pullback. Now, since $[g] : g \to yY'$ in Σ, so is $[\pi_{Y'}] : \pi_{Y'} \to yY'$. Because pullbacks along regular epimorphisms reflect Σ-maps, we obtain that $[g] : hg \to h$ is also in Σ. Using that Σ is closed under composition, we find that $[hg] : hg \to h \to yY \in \Sigma$, and our induction is complete.

□

Theorem 4.5.6 *The quasi-topology J is a topology if and only if $\mathcal{C}_{\mathcal{E}/reg}$ is a reflective subcategory of $\mathcal{C}_{reg}$ (in which case it is of the form: sheaves for the induced closure operator on $\mathcal{C}_{reg}$).*

Proof. If J is a topology, then there is an induced universal closure operator on $\mathcal{C}_{reg}$, with the property that for any arrow $f : C' \to C$, $[f] : f \to yC$ is dense and only if $f \in J$. Using the previous lemma, we get that $[f]$ dense implies $[f] \in \Sigma$. From this, it follows that all dense monos are in Σ.

On the other hand, all maps in Σ_0 are dense, and hence are all monos in Σ. We conclude that the class of dense maps coincides with the class Σ.

□

This theorem shows that in some cases, the relative completion may be seen as a category of sheaves for a universal closure operator; the next section studies this situation in some more detail, and we will see that this gives a more manageable presentation than one in terms of categories of fractions.

It is clear that, in general, $\mathcal{C}$ is not a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$, and also, that the image of $F : \mathcal{E} \to \mathcal{C}$ need not be so. The following is an obvious criterion:

Lemma 4.5.7 *1. $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ if and only if every map in J is regular epi;*

2. $Im(F)$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ if and only if objects in the image of F think that all maps in J are regular epi. By this, we mean that for every map $f : X \to Y$ in $J(X)$ with kernel f_0, f_1, and every map $m : X \to F(W)$ for which $mf_0 = mf_1$, there is a unique extension of m along f.

Proof. For 1), clearly, every map in J is regular epi if and only if J is subcanonical, see [57]. But then we find that $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$.

2) is treated similarly.

□

As example 3.3.4. showed, non-equivalent categories may yield the same completion. The following lemma provides some insight:

Lemma 4.5.8 *Let $F : \mathcal{E} \to \mathcal{C}$ be given and consider $y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$. Define $\mathcal{D}$ to be the full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ on the objects in the image of y. Then $\mathcal{C}_{\mathcal{E}/reg} \simeq \mathcal{D}_{\mathcal{E}/reg}$.*

Proof. We have a factorization of y as

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] \arrow[d, "G"'] & \mathcal{C}_{\mathcal{E}/reg} \\
 & \mathcal{D} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Consider $y' : \mathcal{D} \to \mathcal{D}_{\mathcal{E}/reg}$. By the universal property of $\mathcal{C}_{\mathcal{E}/reg}$, the map $y \circ G : \mathcal{C} \to \mathcal{D}_{\mathcal{E}/reg}$ can be extended along y to give a map $\hat{G} : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{D}_{\mathcal{E}/reg}$. On the other hand, the universal property of $\mathcal{D}_{\mathcal{E}/reg}$ gives an extension $\hat{\mu}$ of μ along y'. Then $\hat{\mu}$ and $\hat{G}$ are pseudo-inverses of each other.

□

It would be good to know when two objects of $\mathcal{E}/\mathbf{REG}$ give rise to an equivalent completion ("Morita equivalence") but I haven't found it.

4.6 Minimal covers and sheaves

We further analyse the situation of the previous section, in which the relative completion was reflective in the ordinary completion. To this end, we first introduce a technical notion, called a minimal cover.

Let $\mathcal{C}$ be a lex category, $\mathcal{D}$ a regular category and let $F : \mathcal{C} \to \mathcal{D}$ be a left exact full and faithful functor. First we recall that a map $k : FC \to D$ is called *$\mathcal{C}$-projecting* [57] if every other map $FC' \to D$ factors through e. Then we define the following.

Definition 4.6.1 $F : \mathcal{C} \to \mathcal{D}$ is called a *minimal cover* if and only if for every D in $\mathcal{D}$ there is a C in $\mathcal{C}$ and a $\mathcal{C}$-projecting regular epi $e : FC \to D$.

We can now formulate the connection between minimal covers, topologies and regular completions.

Theorem 4.6.2 *Let $F : \mathcal{C} \to \mathcal{D}$ be a lex, full and faithful functor, with $\mathcal{D}$ regular. Consider the extension $\hat{F} : \mathcal{C}_{reg} \to \mathcal{D}$. Then the following are equivalent:*

1. *The functor $\hat{F}$ has a right adjoint G with $GF \cong y$ and $\hat{F}G \cong Id$;*
2. *F is a minimal cover, and every object in D embeds into an object in the image of F;*
3. *There is a subcanonical topology on $\mathcal{C}$ such that $\mathcal{D}$ is equivalent to the category of sheaves for the induced universal closure operator on $\mathcal{C}_{reg}$.*

Proof. First assume 2. We define $G : \mathcal{D} \to \mathcal{C}_{reg}$ as follows. An object D gives a composite map

$$\begin{tikzcd} FC \arrow[r, two heads, "e"] & D \arrow[r, tail, "m"] & FC' \end{tikzcd}$$

with e a regular epi. Since F is full, there is a map $f : C \to C'$ in $\mathcal{C}$ with $Ff = me$. This map f is the value of G on D. This is well-defined, because any other cover e' will factor through e and vice versa. (note in particular that

$G(FC) = C$.) For arrows, consider the diagram

$$\begin{tikzcd}
FC \arrow[r, dashed, "\bar{f}"] \arrow[d, two heads, "e_1"'] & FB \arrow[d, two heads, "e_2"] \\
D \arrow[r, "f"] \arrow[d, tail, "m_1"'] & E \arrow[d, tail, "m_2"] \\
FC' & FB'.
\end{tikzcd}$$

The lifting $\bar{f}$ exists because fe_1 factors through e_2. Since F is a full embedding, $\bar{f}$ is of the form $Fh : FB \to FC$, and h, in turn, represents an arrow in $\mathcal{C}_{reg}$ from GD to GE. The adjointness is easily verified, just as the facts $\hat{F}G \cong Id$ and $GF \cong y$. This proves 1).

For the converse, if a right adjoint G exists with $GF = y$ and counit iso, then cover an object D in $\mathcal{D}$ as follows: G sends D to some map $k : C \to C'$. This gives $Fk : FC \to FC'$. The image of Fk is D, so the factorization of $Fk = me$ gives a cover of D. Also, D embeds into FC'. If $p : FB \to D$ is any arrow, then Gp is a map in $\mathcal{C}_{reg}$ from $GFB = yB$ to k. Thus it has a representative $h : B \to C$. This shows that p factors through e. Therefore e is a cover with the required properties.

For the equivalence between 1) and 3), we start from the correspondence of topologies on $\mathcal{C}$ and universal closure operators on $\mathcal{C}_{reg}$. Thus, any topology gives a category $\mathcal{D}$ of sheaves, and the condition $GF \cong y$ corresponds to this topology being subcanonical, i.e. to the condition that $\mathcal{C}$ is full in $\mathcal{D}$.

□

Let us remark that if the right adjoint is regular, then it is automatically an equivalence, since $\mathcal{D}$ then has the same universal property as $\mathcal{C}_{reg}$.

For the remainder of this section we assume that $F : \mathcal{E} \to \mathcal{C}$ is such, that the induced class J is a subcanonical topology. By the above lemma, this means that $\mathcal{C}_{\mathcal{E}/reg}$ is reflective in $\mathcal{C}_{reg}$. In this case, we make the following easy observations:

Lemma 4.6.3 *If J is a subcanonical topology, then:*

1. *$\mathcal{C}_{\mathcal{E}/reg}$ is the full subcategory of $\mathcal{C}_{reg}$ on the objects $\begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix}$ for which f is closed w.r.t J;*
2. *the functor $F : \mathcal{E} \to \mathcal{C}$ is regular;*
3. *the functor $P_\Sigma \circ y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is regular and is a minimal cover;*
4. *an object is projective in $\mathcal{C}_{\mathcal{E}/reg}$ if it is isomorphic to an object of the form $P_\Sigma y(X)$ for which X is projective w.r.t. all regular maps in J in $\mathcal{C}$. Thus $P_\Sigma y$ preserves projectives.*

Proof. Item 1) is direct from the correspondence between topologies on $\mathcal{C}$ and closure operators on $\mathcal{C}_{reg}$. 2) follows from the definition of J, 3) follows from theorem 4.6.2 and 4) follows from the observation that the regular epis in $\mathcal{C}_{\mathcal{E}/reg}$ are the maps for which the underlying arrow is a map in J.

□

This lemma gives a good description of the properties of $\mathcal{C}_{\mathcal{E}/reg}$ as a subcategory of $\mathcal{C}_{reg}$. Moreover, we show that $\mathcal{C}_{\mathcal{E}/reg}$ is in fact the largest such subcategory:

Theorem 4.6.4 *Let J be a topology. Then $\mathcal{C}_{\mathcal{E}/reg}$ is characterized as the largest category with the following properties:*

- *$\mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is a minimal cover;*
- *the composite $\mathcal{E} \to \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is regular.*

*This means, that any other category of $\mathcal{E}/$**REG** that satisfies these properties will be a full reflective subcategory of $\mathcal{C}_{\mathcal{E}/reg}$.*

Proof. The previous lemma showed that $\mathcal{C}_{\mathcal{E}/reg}$ indeed has these properties. If some category D also has them, then this implies that there is a topology H on $\mathcal{C}$ such that D is sheaves for the induced closure operator on $\mathcal{C}_{reg}$. Moreover, from the fact that $\mathcal{E} \to \mathcal{C} \to \mathcal{D}$ is regular, we find that this topology H is *larger* than J, because maps in Σ are dense for it. Therefore, any map f in $\mathcal{C}$ that is closed for H is automatically closed for J. Now $\mathcal{D}$ is the full subcategory of $\mathcal{C}_{reg}$ on the H-closed maps, whereas $\mathcal{C}_{\mathcal{E}/reg}$ is the full subcategory on the J-closed maps. Hence $\mathcal{D}$ is contained in $\mathcal{C}_{\mathcal{E}/reg}$. A reflection is obtained via the universal property of $\mathcal{C}_{\mathcal{E}/reg}$.

□

It would be desirable to know what the role of the objects of $\mathcal{C}$ inside the category $\mathcal{C}_{\mathcal{E}/reg}$ is. It is clear that they are not, in general, the projective objects. Also, given a minimal cover as in theorem 6.2, how can we, categorically, distinguish the objects of $\mathcal{C}$ inside $\mathcal{D}$? The following is worth noticing: the objects of $\mathcal{C}$ are precisely the objects which are projective with respect to a certain class of regular epimorphisms. This class can be described in various ways; for instance, it is the class of regular epis that are preserved by the inclusion $\mathcal{D} \to \mathcal{C}_{reg}$. Unfortunately, we could not find a description of this class that makes no reference to the category $\mathcal{C}$.

We conclude this section with a remark about the topology J. The smallest possible topology is that of the split epis: this is obtained by taking a functor $F : \mathcal{E} \to \mathcal{C}$ that sends every regular epi of $\mathcal{E}$ to a split epi (cf. example 3.3.3). On the other hand, the largest topology that we can obtain is the topology consisting of all regular epis in $\mathcal{C}$. The localic examples that we will deal with in section 7 will be instances of this. It might be good to know necessary and sufficient conditions on $F : \mathcal{E} \to \mathcal{C}$ under which J is a topology consisting of all regular epis.

4.7 Assemblies and Locales

Let $\mathbb{A}$ be a PCA in $\mathcal{E}$.

Definition 4.7.1 (Choice of Functions (CF)) The PCA $\mathbb{A}$ is said to have a *Choice of Functions*, (CF) for short, if every inhabited subset $B \subset \mathbb{A}$ has a global element $1 \to B$.

Under the assumption of this condition, we can now give a categorical characterization of the category of assemblies $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$.

Theorem 4.7.2 *Let the PCA $\mathbb{A}$ satisfy CF. The categories $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/reg}$ and $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ are equivalent.*

Proof. First, it is easily seen that the inclusion $i : \mathbf{PAss}_{\mathcal{E}}(\mathbb{A}) \to \mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is a minimal cover: we already described how to cover an assembly (X, α) with a partitioned assembly (Q, π). If $m : (Y, \beta) \to (X, \alpha)$ is any map with (Y, β) partitioned, then we have $\mathcal{E} \models \exists r : r \bullet \beta(y) \in \alpha(m(y))$. Now we use our assumption and pick a global element $r : 1 \to \mathbb{A}$. Now put $\bar{m}(y) = (m(y), r \bullet \beta(y))$. Thus we have a lifting $\bar{m} : (Y, \beta) \to (Q, \pi)$, which shows that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is a reflective subcategory of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{reg}$. Denote the associated topology on $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ by M.

The corresponding universal closure operator may now be described in the following manner: given $f : (X, \alpha) \to (Y, \beta)$, we define an equivalence relation on $X : x \sim x' \Leftrightarrow f(x) = f(x') \wedge \alpha(x) = \alpha(x')$. This induces an object $(X/{\sim}, \hat{\alpha})$ and a factorization of f through $(X/{\sim}, \hat{\alpha})$, where $(X, \alpha) \to (X/{\sim}, \hat{\alpha})$ is regular epi.

Now it is easily derived that if a map is in the corresponding topology, then it must be the right-half of a cartesian map, hence in the topology induced by the regular epis in the image of ∇. Conversely, for such a regular epi $\nabla(e)$, we see that it is a sheaf (considered as an object of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{reg}$). So the two topologies coincide.

$\square$

As a side remark, we mention that the topology M in the proof does not consist of *all* regular epis in the category **PAss**, even over **Set**. In the case of **Set**, the topology consists of the split epis. But the example presented in chapter 2 (page 27) shows that not all regular epis in **PAss** are split.

Another point worth observing is that the construction in the proof gives us a *factorization system* on the category of partitioned assemblies. Intuitively, the construction removes "redundancy in the fibres"; a map $f : (X, \alpha) \to (Y, \beta)$ can be said to have no redundancy in the fibres if $f(x) = f(x'), \alpha(x) = \alpha(x')$ implies $x = x'$. So, different elements in the same fibre of f have different realizers.

The above theorem has the following consequence:

Theorem 4.7.3 *Let $\mathbb{A}$ satisfy CF. Then $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/ex} \simeq \mathbf{RT}[\mathbb{A}]$.*

Proof. This is a straightforward consequence of Theorem 4.2.5 and the previous theorem, because we have $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/ex} \simeq (\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/reg})_{ex/reg} \simeq \mathbf{Ass}_{\mathcal{E}}(\mathbb{A})_{ex/reg} \simeq \mathbf{RT}[\mathbb{A}]$.

□

So, we have made precise the idea that even over a base topos where choice fails, the realizability topos may still be seen as a solution to a universal problem.

Locales. Let H be a locale in $\mathcal{E}$. The category of elements for H, denoted $\mathcal{G}_{\mathcal{E}}H$ has pairs (X, α) as objects, where X is an object of $\mathcal{E}$, and $\alpha : X \to H$ a map into the locale H; maps are arrows $f : X \to X'$ for which $\alpha(x) \leq \alpha'(f(x))$ for all $x \in X$. In case that $\mathcal{E} = \mathbf{Set}$, this is the usual category H_+, the coproduct completion of H, viewed as a small category. $\mathcal{G}_{\mathcal{E}}H$ is a regular category.

Given H, form a new locale by taking non-empty downsets in H, denoted I^*H, ordered by inclusion. There is an embedding $H \to I^*H$ (which is given by $a \mapsto \downarrow(a)$), that induces an embedding $\mathcal{G}_{\mathcal{E}}H \to \mathcal{G}_{\mathcal{E}}(I^*H)$.

Theorem 4.7.4 (Menni) *Let H be a locale, and let I^*H denote the non-empty downsets in H. Then $(H_+)_{reg} \simeq (I^*H)_+$.*

We will generalize this to an arbitrary locale in an arbitrary topos. So let $\mathcal{E}$ be such a topos, and let H be a locale in $\mathcal{E}$. Then, with notation as in section 2, we get:

Theorem 4.7.5 *The categories $(\mathcal{G}_{\mathcal{E}}H)_{\mathcal{E}/reg}$ and $\mathcal{G}_{\mathcal{E}}(I^*H)$ are equivalent.*

Proof. This is virtually the same construction as for assemblies. There is an embedding of $\mathcal{G}_{\mathcal{E}}H$ into $\mathcal{G}_{\mathcal{E}}(I^*H)$, via $(X, \alpha) \mapsto (X, \alpha')$ with $\alpha'(x) = \downarrow(\alpha(x))$. We cover an object (Y, β) of $\mathcal{G}_{\mathcal{E}}(I^*H)$ with (Q, π), with $Q = \{(y, a) | a \in \beta(y)\}$, and $\pi(y, a) = \downarrow(a)$. Then one shows that maps $f : (Y, \beta) \to (Y', \beta')$ lift to these covers. Also, one embeds (Y, β) in $(Y, \top)$, where $\top(y) = H$. For any functor $G : \mathcal{G}_{\mathcal{E}}H \to \mathcal{D}$ in $\mathcal{E}/\mathbf{REG}$, the extension $\hat{G} : \mathcal{G}_{\mathcal{E}}(I^*H) \to \mathcal{D}$ is defined by sending an object (Y, β) to the image of the map $G(Q, \pi) \to G(Y, \top)$. This gives the universal property.

□

4.8 Discussion and Open Questions

There are a lot of interesting open questions, to which we have not provided any answers. The typical type of theorems that are proved about completions are of the form: the regular/exact completion of $\mathcal{C}$ has property X if and only if $\mathcal{C}$ has property Y, where Y is usually some weakened version of X. For example, $\mathcal{C}_{ex}$ is locally cartesian closed if and only if $\mathcal{C}$ has weak dependent products [20]. Or: $\mathcal{C}_{ex}$ is a topos if and only if $\mathcal{C}$ has weak dependent products and a generic proof [57]. For the relativised version, the same questions can be asked: in general, these seem very difficult. But in the case of minimal covers, we can obtain some results for free:

Proposition 4.8.1 *Let $\mathcal{C} \to \mathcal{D}$ be a minimal cover. Then $\mathcal{D}$ is locally cartesian closed if and only if $\mathcal{C}$ has weak dependent products.*

Proof. One side is immediate, since if $\mathcal{C}$ has weak dependent products, then $\mathcal{C}_{ex}$ is locally cartesian closed and $\mathcal{D}$ inherits this, being a reflective subcategory of $\mathcal{C}_{ex}$. For the other direction, we observe that the technique that is used to show that if $\mathcal{C}_{ex}$ is lccc then $\mathcal{C}$ has weak dependent products does not make full use of projectivity, but only of the fact that objects from $\mathcal{C}$ are projective with respect to a certain class of regular epis.

□

Proposition 4.8.2 *Let $\mathcal{C} \to \mathcal{D}$ be a minimal cover. Then $\mathcal{D}$ is a topos if $\mathcal{C}$ has weak dependent products and a generic proof.*

Proof. If the conditions are satisfied, then by Menni's result, $\mathcal{C}_{ex}$ is a topos. Hence $\mathcal{D}$ is a localization of a topos, and therefore itself a topos.

□

The converse of this proposition is false, as exemplified by taking a topos $\mathcal{E}$ without a generic proof, and $\mathcal{C}$ to be partitioned assemblies over $\mathcal{E}$. Now $\mathcal{C}$ does not have a generic proof, otherwise $\mathcal{E}$, being a localization of $\mathcal{C}$, would inherit it. But the relative exact completion of $\mathcal{C}$ is a topos (namely, the Effective Topos over $\mathcal{E}$).

Another point concerning minimal covers is the following: if $F : \mathcal{C} \to \mathcal{D}$ is a minimal cover, then the objects of $\mathcal{D}$ in the image of F are precisely the objects that are projective with respect to a certain class of regular epis. This prompts a general question: given a category $\mathcal{D}$, one can associate two posets with $\mathcal{D}$. The first poset, **Epi**($\mathcal{D}$) consists of classes of regular epis in $\mathcal{D}$, ordered by subset inclusion. The second poset, **Sub**($\mathcal{D}$) consists of collections of objects $\mathcal{D}$, also ordered by inclusion. There is a Galois connection between these posets: given a collection $\mathcal{C}$ of objects, consider the class of regular epis for which every $C \in \mathcal{C}$ is projective (i.e. the class of $\mathcal{C}-$projecting epis). Conversely, if we have a class of epis, take the collection of objects which are projective with respect to all these epis. What are the classes of objects for which the composite operation is the identity? Evidently, there are some simple closure properties (coproducts, retractions) necessary. But what more do we need?

Recall that a *projective cover* of a category $\mathcal{D}$ is a full subcategory $\mathcal{C}$ such that every object in $\mathcal{C}$ is projective in $\mathcal{D}$, and such that every object in $\mathcal{D}$ can be covered by a $\mathcal{C}$-object. Hence a projective cover is a special instance of a minimal cover. Now it is known (see [21]) that if $\mathcal{C}, \mathcal{C}'$ are both projective covers, then their idempotent splittings are equivalent. Can we say something comparable (but evidently weaker) about minimal covers? And what can we say in general about the (poset) of minimal covers of a certain category?

For the ordinary completions, we characterize the objects in the image of the inclusion $\mathcal{C} \to \mathcal{C}_{reg}$ as the projectives. But how can we characterize those objects of $\mathcal{C}_{\mathcal{E}/reg}$ that are in the image of $y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$?

This characterization is then used, to say which regular categories are free regular categories. How can we extend this to the relative case?

Another interesting point is, that our main examples were tripos-theoretic in nature. This suggests that a uniform treatment should be possible. Is there an operation on indexed pre-orders that corresponds, on the level of their categories of elements, to the relative regular completion? In fact, we can give an affirmative answer here, but this will be the subject of the next chapter.

Furthermore, in our treatment of assemblies, we used the assumption that the underlying PCA had enough global elements. Although this is not a severe limitation (it is certainly satisfied if the terminal object 1 is projective, for instance), we feel obliged to say a word about what would happen if we omitted it. From the constructions of the canonical covers, it is clear that this approach makes essential use of the assumption, so the theorem $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A}) \simeq \mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/reg}$ would be simply false if we drop it. However, one might try something along the following line: simply redefine the categories $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ and $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$, by taking the same objects, but as morphisms $(X, \alpha) \to (Y, \beta)$ total relations $R \subset X \times Y$ for which $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X \exists y : Y (R(x, y) \wedge a \bullet \alpha(x) \downarrow \wedge \beta(y) = a \bullet \alpha(x))$. This certainly circumvents the need for global elements, but now the relationship of these newly defined categories with the realizability topos is less clear....

Another problem concerning partitioned assemblies is the following: if $\mathcal{E}$, the underlying topos, does not have choice, then $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{ex}$ differs from $\mathbf{RT}[\mathbb{A}]$ (as we have seen, the latter is a reflective subcategory of the former). But can it happen that $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{ex}$ is still a topos? This is equivalent to asking whether $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ has a generic proof, which in turn implies that $\mathcal{E}$ has a generic proof. Now there are toposes that do not satisfy AC, but have a generic proof (see chapter 5). But we could not answer the question whether $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ having a generic proof[2] implies that $\mathcal{E} \models AC$, although it can be shown that if the classical construction of a generic proof in $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ still works, that this implies choice for $\mathcal{E}$; in other words, if $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ has a generic proof and $\mathcal{E}$ does not have choice, then this generic proof is not the usual one!

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[2] To make things even worse, we do not know whether this depends on the PCA in question.

Chapter 5

Indexed Preorders and Completions

In this chapter we present an analysis of realizability triposes by showing that they can be obtained as the result of applying a free construction to a simpler indexed preorder. As a corollary, we find a generalization of a theorem by Carboni, Freyd and Scedrov. We also exhibit a free construction that gives rise to previously unknown hierarchies of realizability triposes and toposes.

5.1 Introduction

In the previous chapter on relative completions, we have seen that the presentation of realizability toposes using the exact completion needs refinement when working over an arbitrary base topos. The refinement we proposed consisted of relativising the notions of regular and exact completion to the base topos, trying to keep the flavour of the classical completions as much as possible.

In this chapter, we try to approach matters from a tripos-theoretic point of view. The main strategy will be to investigate in which sense the realizability *tripos* that we associate to a partial combinatory algebra can be seen as the result of applying a free construction to a simpler indexed preorder. To this end, we first single out a class of well-behaved indexed preorders. Then, using the internal logic of the base topos, we define a free construction on those indexed preorders, which, loosely speaking, adds left adjoints to reindexing functors. The central result is a characterization of those indexed preorders of which the free completion is a tripos. It turns out that this characterization, which is phrased in terms of ordered PCAs, is in fact a generalization of a theorem by Carboni, Freyd and Scedrov [18]. We also give a closely related variation on the construction which only adds left adjoints to reindexing functors along regular epimorphisms, and it is proved that the latter construction preserves the property of being a tripos. Applying this construction to the effective tripos, for instance, yields the tripos for extensional realizability. But we will also show that there are hierarchies of toposes starting with the Modified realizability topos, the Dialectica topos and the Lifschitz topos.

The contents of this chapter are the following: section 2 describes the setting in which we will work, namely indexed preorders that arise from a poset together with a class of partial monotone endofunctions satisfying some conditions. We discuss the various indexed preorders and triposes that locales and partial combinatory algebras give rise to and we explain how all of them can be comprised by the concept of an ordered PCA equipped with a filter. The main point that we wish to convey here, is that there is not a sharp distinction between triposes from ordered PCAs and from locales, but rather a continuum.

In section 3 we describe a free construction on indexed preorders and a close variation. These constructions will turn out to be monadic, in a suitable 2-categorical sense. Then, the constructions will be applied to the motivating examples and we will see that the non-empty downset-constructions on locales and ordered PCAs can be described in this unified fashion, thus providing an explanation for the formal similarity between those constructions. Moreover, we will be able to state in which sense the realizability tripos that we associate to a PCA is a free tripos.

Section 4 contains the central results of the chapter, and is concerned with the characterization of properties of the free completion of an indexed preorder $\mathbb{P}$ in terms of properties of $\mathbb{P}$ itself, analogous to the characterization of properties of the exact completion of a category in terms of that category itself. We will find sufficient and necessary conditions under which the free completion is a tripos. This, in turn, will imply that the topos obtained from the tripos is an exact completion (relative to the base topos). Roughly speaking, these conditions say that the original indexed preorder has an ordered PCA structure. And surprisingly, some triposes coming from locales also fall under this heading (we will characterize those locales).

Also, we show that the property of being a tripos is preserved by the completions. This gives rise to hierarchies of previously unknown triposes.

In section 5 we make a comparison with related work. We compare our result to that of Carboni, Freyd and Scedrov, and then we look at the work by Aczel [1] on frames. Furthermore, the $\mathcal{F}$-construction, devised by Birkedal and further investigated by Robinson and Rosolini is considered, and we end our discussion with a short note on conditional PCAs, the tripos-theoretic role of which has been subject of recent debate[1].

Finally, there is an appendix in which a possible widening of our setting is discussed. It is explained, how we can define the downset-monad on a much larger class of indexed preorders (encompassing all **Set**-triposes), using the concept of an *admissible relation.*

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[1] This debate was conducted electronically, and took place on the categories mailing list in Februari, 2003.

5.2 Setting and basic results

In this section we first demarcate our area of investigation and lay down the basic definitions. Next, we introduce some of the motivating examples that will appear throughout the paper.

5.2.1 Indexed preorders from posets

Throughout the whole paper, $\mathcal{E}$ will be an elementary topos, about which no assumptions are made. We will often reason informally using the internal logic of the topos. Our study will concern a specific class of canonically presented $\mathcal{E}$-indexed preorders (see chapter 2, definition 2.5 ff.), which will arise from the following data and requirements:

1. Σ is an internal poset in $\mathcal{E}$, and the order is denoted by $\leq$.
2. $\mathcal{F}$ is a subobject of the object of partial endofunctions on Σ, i.e. $\mathcal{F} \hookrightarrow \tilde{\Sigma}^{\Sigma}$. (Here, $\tilde{\Sigma}$ is the partial map classifier of Σ.)
3. $\mathcal{E} \models \forall f \in \mathcal{F}\, \forall a \in \mathit{dom}(f)\, \forall b [b \leq a \Rightarrow b \in \mathit{dom}(f)\ \&\ f(b) \leq f(a)]$.
   So, the maps in $\mathcal{F}$ are monotone and have downwards closed domain.
4. $\mathcal{E} \models \exists i \in \mathcal{F}\, \forall a \in \Sigma . i(a) \leq a$.
   (There is a "weak identity".)
5. $\mathcal{E} \models \forall f, g \in \mathcal{F}\, \exists h \in \mathcal{F}\, \forall a \in \mathit{dom}(f) : f(a) \in \mathit{dom}(g)$
   $\Rightarrow a \in \mathit{dom}(h)\ \&\ h(a) \leq g(f(a))$.
   ("Weak composition")

Given these data, we make the assignment $X \mapsto \mathcal{E}(X, \Sigma)$ into an indexed preorder by defining:

Definition 5.2.1 For $\alpha, \beta : X \to \Sigma$ put

$$\alpha \vdash_X \beta \Leftrightarrow \mathcal{E} \models \exists f \in \mathcal{F}\, \forall x \in X : f(\alpha(x)) \leq \beta(x).$$

We stress, that this definition is in general different from the standard way of associating an indexed preorder to a poset, because we do not preorder $\mathcal{E}(X, \Sigma)$ by $\alpha \vdash_X \beta \Leftrightarrow \forall x \in X : \alpha(x) \leq \beta(x)$, but take the functions in the class $\mathcal{F}$ into account.

Functions in $\mathcal{F}$ will sometimes be called *admissible*, and if $\alpha \vdash_X \beta$ via f as in the above definition, we say that f is a *tracking* for $\alpha \vdash_X \beta$. If there is no danger of confusion, we will usually denote an indexed preorder with generic element Σ by $\mathcal{E}(-, \Sigma)$, leaving out the preorder.

Saturating $\mathcal{F}$. The demands on the class $\mathcal{F}$ are formulated in such a way, that definition 5.2.1 yields an indexed preorder. Sometimes it is inconvenient, however, that $\mathcal{F}$ only has weak closure properties. Therefore we define the following: $\mathit{Sat}(\mathcal{F})$ is the subobject of the object of partial monotone endofunctions on Σ defined by $f \in \mathit{Sat}(\mathcal{F}) \Leftrightarrow \exists g \in \mathcal{F} : \forall a \in \mathit{dom}(f) . g(a) \leq f(a)$. We call $\mathit{Sat}(\mathcal{F})$ the *saturation* of $\mathcal{F}$. Then we have:

Lemma 5.2.2 *With the definition of $Sat(\mathcal{F})$ as above:*

1. *The class $Sat(\mathcal{F})$ contains $\mathcal{F}$, and is closed under restricting the domain, i.e. $\mathcal{E} \models f \in Sat(\mathcal{F}), A \subseteq dom(f) \Rightarrow f|_A \in Sat(\mathcal{F})$.*
2. *The class $Sat(\mathcal{F})$ is closed under composition and contains the identity.*
3. *For $\alpha, \beta : X \to \Sigma$,*
   $\alpha \vdash_X \beta$ if and only if $\mathcal{E} \models \exists f \in Sat(\mathcal{F})\, \forall x \in X : f(\alpha(x)) \leq \beta(x)$.
4. *$Sat(\mathcal{F})$ is the largest extension of $\mathcal{F}$ with the above properties.*

Proof. Easy exercise.

□

The third item of the lemma states that we can replace $\mathcal{F}$ by its saturation without changing the indexed preorder. Therefore we shall assume from now on that $\mathcal{F}$ is saturated.

The Category PRE($\mathcal{E}$). We define the 2-category **PRE**($\mathcal{E}$):

- **Objects:** indexed preorders of the above form;
- **Morphisms:** a morphism $\mathcal{E}(-, \Sigma) \to \mathcal{E}(-, \Theta)$ is a map $\phi : \Sigma \to \Theta$ preserving the preorder; Thus for every $X \in \mathcal{E}$ and elements $\alpha, \beta : X \to \Sigma$, we require that $\alpha \vdash_X \beta$ implies $\phi \circ \alpha \vdash_X \phi \circ \beta$;
- **2-cells:** for $\phi, \psi : \Sigma \to \Theta$, we have $\phi \leq \psi$ if and only if $\phi \vdash_\Theta \psi$.

It should be noted, that this definition of morphism agrees with the usual notion of a morphism between indexed categories, when restricted to our class of indexed preorders.

The category **PRE**($\mathcal{E}$) has products: $\mathcal{E}(-, \Sigma) \times \mathcal{E}(-, \Theta) = \mathcal{E}(-, \Sigma \times \Theta)$, with the preorder $\langle \alpha, \alpha' \rangle \vdash_X \langle \beta, \beta' \rangle \Leftrightarrow \alpha \vdash_X \beta \& \alpha' \vdash_X \beta'$. The trivial indexed preorder $\mathcal{E}(-, 1)$ is the terminal object in $IP(\mathcal{E})$.

A useful characterization of maps in the category **PRE**($\mathcal{E}$) is the following:

Lemma 5.2.3 *Let $\mathcal{E}(-, \Sigma)$ and $\mathcal{E}(-, \Theta)$ be indexed preorders with classes of admissible endofunctions $\mathcal{F}_\Sigma, \mathcal{F}_\Theta$, respectively. Let $\phi : \Sigma \to \Theta$ be any map. Then ϕ is a morphism of indexed preorders if and only if the following two conditions hold:*

- *$\mathcal{E} \models \forall f \in \mathcal{F}_\Sigma\, \exists g \in \mathcal{F}_\Theta . g(\phi(a)) \leq \phi(f(a))$;*
- *$\mathcal{E} \models \exists h \in \mathcal{F}_\Theta\, \forall a \leq b . h(\phi(a)) \leq \phi(b)$.*

Proof. Suppose first that ϕ is a morphism of indexed preorders. Take $f \in \mathcal{F}_\Sigma$, and consider the object $K = \{(a, f(a)) | a \in dom(f)\}$, together with the projections $\pi_1, \pi_2 : K \to \Sigma$. Evidently, $\pi_1 \vdash_K \pi_2$, hence $\phi \circ \pi_1 \vdash_K \phi \circ \pi_2$, i.e. $\mathcal{E} \models \exists g \in \mathcal{F}_\Theta\, \forall a \in dom(f) . g(\phi(a)) \leq \phi(f(a))$. Similarly for the second condition.

Conversely, for $\alpha, \beta : X \to \Sigma$,$\alpha \vdash_X \beta$ means

$$\mathcal{E} \models \exists f \in \mathcal{F}_\Sigma \, \forall x \in X. f(\alpha(x) \leq \beta(x).$$

Thus we get

$$\mathcal{E} \models \exists g \in \mathcal{F}_\Theta \, \forall x \in X. g(\phi(\alpha(x)) \leq \phi(f(\alpha(x)))\& f(\alpha(x)) \leq \beta(x),$$

and, using the second condition,

$$\mathcal{E} \models \exists g, h \in \mathcal{F}_\Theta \, \forall x \in X. hg(\phi(\alpha(x))) \leq \phi(\beta(x)),$$

hence $\phi \circ \alpha \vdash_X \phi \circ \beta$.

□

Indexed Finite Limits. Recall from chapter 2 that we say that an indexed preorder $\mathcal{E}(-, \Sigma)$ has ($\mathcal{E}$-indexed) binary meets if every $\mathcal{E}(X, \Sigma)$ has binary meets, written $\wedge_X : \mathcal{E}(X, \Sigma) \times \mathcal{E}(X, \Sigma) \to \mathcal{E}(X, \Sigma)$, and if reindexing preserves these meets up to isomorphism. We can transfer this structure onto the generic element Σ, to obtain a map $\wedge : \Sigma \times \Sigma \to \Sigma$. Meets in $\mathcal{E}(X, \Sigma)$ are then given by (for $\alpha, \beta : X \to \Sigma$): $\alpha \wedge_X \beta = \wedge \circ \langle \alpha, \beta \rangle : X \to \Sigma$, i.e. $(\alpha \wedge_X \beta)(x) = \alpha(x) \wedge \beta(x)$.

Similarly, we say that $\mathcal{E}(-, \Sigma)$ has a top element if each $\mathcal{E}(X, \Sigma)$ has a top element $\top_X$ and if reindexing preserves top elements up to isomorphism. Again, we may transfer this to the generic element, by taking $\top : 1 \to \Sigma$. Then $\top_X = \top \circ ! : X \to 1 \to \Sigma$.

An indexed preorder is said to have finite limits if it has both binary meets and a top element. A word of caution is in order here: the fact that $\mathcal{E}(-, \Sigma)$ does or does not have binary meets is *not* necessarily related to the existence of binary meets in the poset Σ. Similarly, $\mathcal{E}(-, \Sigma)$ can have a top element while the poset Σ does not.

Whenever we encounter indexed preorders with finite limits, we will tacitly assume that these limits are given by the two maps $\top : 1 \to \Sigma$ and $\wedge : \Sigma \times \Sigma \to \Sigma$. Note that the meet map $\wedge$ need not preserve the order on Σ on the nose; this is only the case "up to a realizer", in the sense that

$$\mathcal{E} \models \exists r \in \mathcal{F} \, \forall a' \leq a, b' \leq b \in \Sigma. r(a' \wedge b') \leq a \wedge b.$$

In some situations we will need to consider meet maps which are strictly order-preserving; when this is the case, we will point this out.

Indexed Heyting Implication. We say that an indexed preorder with finite limits has *(Heyting) implication* if each fibre has implication (i.e. has a binary operation $\Rightarrow$ such that $\phi \Rightarrow -$ is right adjoint to $\phi \wedge -$, and if reindexing preserves this structure. For indexed preorders of the form $\mathcal{E}(-, \Sigma)$, we can transfer this structure to $\Rightarrow : \Sigma \times \Sigma \to \Sigma$, just as for binary meets.

Choice of Functions. In some of the theorems that we will prove, we need to use an extra assumption on our indexed preorders, namely: if $\mathcal{E} \models \exists f \in \mathcal{F}. \phi(f)$ (where ϕ is some subobject of $\mathcal{F}$) then there is a global element $f : 1 \to \mathcal{F}$

such that $\mathcal{E} \models \phi(f)$. In other words, we require that $\mathcal{F}$ has enough global elements. This condition will be refered to as *choice of functions* (CF), and was already used in chapter 4, section 7. Now if this condition is satisfied, then the indexed preorder is determined completely by the set $\mathcal{E}(1, \mathcal{F})$, via $\alpha \vdash_X \beta \Leftrightarrow \exists f \in \mathcal{E}(1, \mathcal{F}).\mathcal{E} \models \forall x \in X : f(\alpha(x)) \leq \beta(x)$. When, in a theorem, we make essential use of (CF), we will explicitly mention this.

Fibrations. We recall that if $\mathcal{E}(-, \Sigma)$ is an indexed preorder, then we have a total category (which we will denote here by $\mathcal{G}(\Sigma)$) with objects of the form (X, α), with $X \in \mathcal{E}$, $\alpha : X \to \Sigma$, and where a map $f : (X, \alpha) \to (Y, \beta)$ is just a map $f : X \to Y$ in $\mathcal{E}$ with the property that $\alpha \vdash_X \beta \circ f$. There is a faithful forgetful functor $\Gamma : \mathcal{G}(\Sigma) \to \mathcal{E}$. Also note that (Σ, id) is a generic object of $\mathcal{G}(\Sigma)$. (Again, the notation $\mathcal{G}(\Sigma)$ is potentially ambiguous.)

5.2.2 Motivating examples: locales and ordered PCAs

First we discuss how ordered PCAs give rise to indexed preorders, and then we do the same for locales. After that, we show how both can be seen as instances of one and the same phenomenon, by considering applicative filters in ordered PCAs.

Indexed Preorders from Ordered PCAs. Given an ordered PCA $\mathbb{A}$, there are three different (but very related) indexed preorders that we can construct. In what follows, we will use the following notation: if L is some partial order, then $\mathcal{I}(L)$ stands for the collection of downward closed subsets of L. Similarly, $\mathcal{I}_i(L)$ denotes the collection of *nonempty* downsets in L.

1. We can preorder the sets $\mathcal{E}(X, \mathbb{A})$ by putting (for $\phi, \psi : X \to \mathbb{A}$): $\phi \vdash_X \psi$ iff $\exists a \in \mathbb{A}\, \forall x \in X. a \bullet \phi(x)\downarrow$ & $a \bullet \phi(x) \leq \psi(x)$. We might call this the *simple indexed preorder* associated with $\mathbb{A}$. Of course, the collection of admissible endofunctions on $\mathbb{A}$ is $\mathcal{F} = \{a \bullet - | a \in \mathbb{A}\}$.
2. The *canonical tripos* associated with $\mathbb{A}$ will assign $\mathcal{E}(X, \mathcal{I}(\mathbb{A}))$ to X, preordered by (for $\phi, \psi : X \to \mathcal{I}(\mathbb{A})$): $\phi \vdash_X \psi$ iff $\exists a \in \mathbb{A}\, \forall x \in X\, \forall b \in \phi(x). a \bullet b\downarrow$ & $a \bullet b \in \psi(x)$.
3. Finally, one can replace $\mathcal{I}(\mathbb{A})$ by $\mathcal{I}_i(\mathbb{A})$ in the above clause. Strictly speaking, this falls under the first definition, since $\mathcal{I}_i(\mathbb{A})$ may itself be viewed as an ordered PCA.

Before moving to locales, let us first say a word about the role that these indexed preorders play in the study of realizability. The tripos $\mathcal{E}(-, \mathcal{I}(\mathbb{A}))$ is very well-known, because it is used to build the realizability topos associated to $\mathbb{A}$, denoted $\mathbf{RT}[\mathbb{A}]$. The interest for the simple preorder $\mathcal{E}(-, \mathbb{A})$ will be twofold: first, under the equivalence between indexed preorders and fibrations, the simple preorder corresponds to the category of Partitioned Assemblies $\mathbf{PAss}(\mathbb{A})$. Second, we will see that the tripos can be viewed as a kind of completion of the simple preorder, analogous to the fact that $\mathbf{RT}[\mathbb{A}]$ can be seen as an exact completion (relative to the base topos $\mathcal{E}$) of **PAss**($\mathbb{A}$). Similarly, the preorder $\mathcal{E}(-, \mathcal{I}_i(\mathbb{A}))$ corresponds to the category of Assemblies **Ass**($\mathbb{A}$), and this preorder will also turn out to be a completion of $\mathcal{E}(-, \mathbb{A})$, parallel to the fact that Assemblies is the regular completion (relative to $\mathcal{E}$, again) of Partitioned Assemblies.

Indexed preorders from locales. For a locale H, there will be only one indexed preorder that we will associate to H, and that is the one sending X to $\mathcal{E}(X, H)$, ordered pointwise. (This means that the class of admissible endofunctions only contains the identity function.) This is the canonical tripos for H; its interest lies in the fact that the tripos-to-topos construction yields the topos of H-valued sets, or sheaves on H. Let us also remark that, in case the base topos is **Sets**, the total category of the fibration corresponding to $\mathcal{E}(-, H)$ is simply H_+, the *free coproduct completion* of H, viewed as a small category.

Unified treatment using filters. We now wish to capture all the structures described above in one definition. To this end, we need the notion of an applicative filter in an ordered PCA. (For ordinary PCAs, this was called a *subalgebra* in [59].)

Definition 5.2.4 Let $\mathbb{A}$ be an ordered PCA. A subset $\Phi \subset \mathbb{A}$ is called an *applicative filter* in $\mathbb{A}$ if:

1. Φ contains (some choice of) k, s;
2. Φ is closed under the partial application, i.e. $a, b \in \Phi$ and $ab\!\downarrow$ imply $ab \in \Phi$.

Such a filter is itself an ordered PCA, in fact it is a sub-ordered PCA of $\mathbb{A}$, in a very strict sense. When we have an ordered PCA together with such a filter, we may define:

Definition 5.2.5 Let Φ be an applicative filter in the ordered PCA $\mathbb{A}$. We make the assignment $X \mapsto \mathcal{E}(X, \mathbb{A})$ into an indexed preorder by putting:

$$\phi \vdash_X \psi \Leftrightarrow \exists a \in \Phi\, \forall x \in X. a \bullet \phi(x)\!\downarrow \ \& \ a \bullet \phi(x) \in \psi(x).$$

Note that, from the point of view of definition 5.2.5, it makes no difference whether we use Φ or the upwards closure of Φ, since this gives the same preorder. Because we may take Φ to be upwards closed, we feel justified in calling it a filter.

It is useful to think of the filter Φ as the set of designated truth-values (see chapter 2, page 30), or as those elements that can serve as a realizer. We will have more to say about this in the part on tripos characterizations.

The definition above comprises almost all indexed preorders that we will be interested in: to obtain the tripos associated to a locale H, observe that H is itself an ordered PCA (application is meet, $k = s = \top$). For the filter Φ, we then have to take $\{\top\}$. To recapture the preorders coming from an ordered PCA $\mathbb{A}$, note first that the simple preorder $\mathcal{E}(-, \mathbb{A})$ is the same as the one in definition 5.2.5 when we take Φ to be all of $\mathbb{A}$. The tripos $\mathcal{E}(-, \mathcal{I}(\mathbb{A}))$ is dealt with by observing that $\mathcal{I}(\mathbb{A})$ is an ordered PCA, and that we may take the *nonempty* downsets of $\mathbb{A}$ as our filter Φ (again, this was in [59], but the point we wish to convey is, that the generic element is itself an ordered PCA).

We end this section by mentioning that there was a much earlier attempt to unify triposes coming from PCAs and triposes coming from locales, by Peter Aczel [1]. He also uses the idea of a filter, but his definition is narrower than ours, since it does not encompass indexed preorders that are not triposes. We shall say a bit more about this in the final section, where we compare our work with other approaches.

5.3 Two free constructions

In this section we introduce two constructions that add existential quantification to an indexed preorder. The constructions generalize the non-empty downset-constructions on locales and on ordered partial combinatory algebras that have been discussed in chapters 2 and 3. In subsection 1 we explain the construction and prove some basic properties. Subsection 2 deals with the construction from a monad-theoretic point of view and studies the algebras. And in the last subsection we will illustrate what is going on by applying it to some examples.

5.3.1 Adding left adjoints

We start with an indexed preorder $\mathcal{E}(-, \Sigma)$, coming from $\Sigma, \mathcal{F}$.

Construction 5.3.1 ($\mathcal{I}$-construction) From $\mathcal{E}(-, \Sigma)$ we construct a new indexed preorder $\mathcal{E}(-, \mathcal{I}\Sigma)$. $\mathcal{I}\Sigma$ is the object of downward closed subsets of Σ, and for an object X the preorder $\vdash^*_X$ is defined as:

$$\alpha \vdash^*_X \beta \Leftrightarrow \mathcal{E} \models \exists f \in \mathcal{F}\, \forall x \in X \forall a \in \alpha(x). f(a) \in \beta(x).$$

In this situation, we will call f a *tracking* for $\alpha \vdash^*_X \beta$. It is trivial to verify that $\vdash^*$ is indeed a preorder. The class $\mathcal{I}\mathcal{F}$ of admissible endofunctions of $\mathcal{I}\Sigma$ is the object

$$\begin{aligned} \mathcal{F}^* \;=\; & \{f : \mathcal{I}\Sigma \rightharpoonup \mathcal{I}\Sigma \mid f \text{ is monotone, has downwards closed domain, and} \\ & \exists g \in \mathcal{F}. \forall U \in dom(f), a \in U : g(a) \in f(U)\}. \end{aligned}$$

Lemma 5.3.2 *The $\mathcal{I}$-construction on indexed preorders is the object part of a 2-functor $\mathcal{I} : \mathbf{PRE}(\mathcal{E}) \to \mathbf{PRE}(\mathcal{E})$.*

Proof. Take a morphism between two indexed preorders $\phi : \mathcal{E}(-, \Sigma) \to \mathcal{E}(-, \Theta)$. That is, ϕ is a map $\phi : \Sigma \to \Theta$, such that composition with ϕ preserves the preorder. We define $\mathcal{I}(\phi)$ to be the map sending $U \subseteq \Sigma$ to the downward closure of $\phi[U]$. We want to see that this $\mathcal{I}(\phi) : \mathcal{I}\Sigma \to \mathcal{I}\Theta$ is preorder preserving.

To this end, suppose that, for $\alpha, \beta : X \to \mathcal{I}\Sigma$, we have $\alpha \vdash^*_X \beta$, say via a tracking f. Because ϕ is preorder-preserving, there is some $g \in \mathcal{F}_\Theta$ with

$$\forall a \in dom(f) : g(\phi(a)) \leq f(\phi(a)).$$

This implies that, for all $x \in X, b \in \mathcal{I}(\phi) \circ \alpha(x) : g(b) \leq \phi(f(a)) \in \mathcal{I}(\phi) \circ \beta(x)$, i.e. $\mathcal{I}(\phi) \circ \alpha \vdash^*_X \mathcal{I}(\phi) \circ \beta$. So $\mathcal{I}(-)$ is a functor.

Now if $\phi, \psi : \Sigma \to \Theta$ are two such maps with $\phi \leq \psi$, we must show that $\mathcal{I}(\phi) \leq \mathcal{I}(\psi)$. But this is immediate, since $\phi \leq \psi$ means by definition that $\phi \vdash_\Sigma \psi$, so we get $\mathcal{I}(\phi) \vdash^* \mathcal{I}(\psi)$, i.e. $\mathcal{I}(\phi) \leq \mathcal{I}(\psi)$ because $\mathcal{I}$ is preorder-preserving.

□

Before we further explore the construction, we give an equally important variation: instead of taking all downward closed subsets we may also take the *inhabited* downsets of Σ. This operation is denoted $\mathcal{I}_i$, and the preorder is defined in exactly the same way.

Proposition 5.3.3 *The assignment $\mathcal{E}(-, \Sigma) \mapsto \mathcal{E}(-, \mathcal{I}\Sigma)$ is the functor part of a 2-monad on the category of indexed preorders on $\mathcal{E}$. Similarly for $\mathcal{I}_i(-)$.*

Proof. We do this only for $\mathcal{I}(-)$, since the proof for $\mathcal{I}_i$ is the same. First of all, the unit is given by composition with the principal downset map $P : \Sigma \to \mathcal{I}\Sigma$ which sends $a \in \Sigma$ to $\downarrow(a)$. For $\alpha, \beta : X \to \Sigma$ we find

$$\begin{aligned} P(\alpha) \vdash^*_X P(\beta) &\Leftrightarrow \exists f \in \mathcal{F}\, \forall x \in X\, \forall a \in P(\alpha)(x). f(a) \in P(\beta)(x) \\ &\Leftrightarrow \exists f \in \mathcal{F}\, \forall x \in X\, \forall a \leq \alpha(x). f(a) \leq \beta(x) \\ &\Leftrightarrow \exists f \in \mathcal{F}\, \forall x \in X. f(\alpha(x)) \leq \beta(x) \\ &\Leftrightarrow \alpha \vdash_X \beta \end{aligned}$$

The multiplication is induced by the union map $\cup : \mathcal{I}\mathcal{I}\Sigma \to \mathcal{I}\Sigma$. To see that this is order preserving, we calculate, for $\alpha, \beta : X \to \mathcal{I}\mathcal{I}\Sigma$, that

$$\alpha \vdash^{**}_X \beta \Leftrightarrow \exists f \in \mathcal{F}\, \forall x \in X\, \forall U \in \alpha(x) : f[U] \in \beta(x). \tag{5.1}$$

We have to show that $\cup\alpha \vdash^*_X \cup\beta$. Take the function f as above. Then for $a \in \cup\alpha(x)$ there is some $U \in \alpha(x)$ with $a \in U$. Hence, by 5.1, $f(a) \in f[U] \in \beta(x)$. Thus we have $\forall x \in X\, \forall a \in \cup\alpha(x). f(a) \in \cup\beta(x)$ and we are done.

□

The following lemma will be used in the next section.

Lemma 5.3.4 *The operations $\mathcal{I}(-)$ and $\mathcal{I}_i(-)$ preserve the property of having indexed binary meets.*

Proof. Suppose meets for $\mathcal{E}(-,\Sigma)$ are induced by $\wedge : \Sigma \times \Sigma \to \Sigma$. We can simply define the map $\wedge^* : \mathcal{I}\Sigma \times \mathcal{I}\Sigma \to \mathcal{I}\Sigma$ as

$$\alpha \wedge^* \beta = \downarrow\{a \wedge b | a \in \alpha, b \in \beta\}.$$

□

By this construction of meets, we also see that the inclusion $\mathcal{E}(-,\Sigma) \hookrightarrow \mathcal{E}(-,\mathcal{I}\Sigma)$ preserves meets. Similarly, it preserves the top element.

5.3.2 Algebras

We now turn to the universal property of indexed preorders of the form $\mathcal{E}(-,\mathcal{I}\Sigma)$ and of the form $\mathcal{E}(-,\mathcal{I}_i\Sigma)$. First we examine the properties that algebras for the monad have:

Proposition 5.3.5 *Algebra structures for the monad $\mathcal{I}(-)$ are unique up to isomorphism (if they exist) and are adjoint to the unit. Hence $\mathcal{I}(-)$ is a KZ-monad.*

Proof. Let $\bigvee : \mathcal{I}\Sigma \to \Sigma$ be the algebra map. In order to establish the claims, it suffices to show that for an arbitrary $\alpha : X \to \mathcal{I}\Sigma$, it holds that $\alpha \vdash^*_X \eta \bigvee \alpha$, where η is the unit. To this end, we show the existence of a tracking f, such that for all $U \subseteq \Sigma$ and for all $a \in U$, $f(a) \leq \bigvee U$. Consider the set $M = \{(a,U) | a \in U, U \subseteq \Sigma\}$, and the two functions $\eta \circ \pi_1, \pi_2 : M \to \mathcal{I}\Sigma$. Then $\eta \circ \pi_1 \vdash^*_M \pi_2$, via the identity. Compose with the algebra map to obtain $\bigvee \circ \eta \circ \pi_1 \vdash \bigvee \circ \pi_2$. But the map $\bigvee \circ \eta \circ \pi_1$ is isomorphic to the first projection, so we find that $\pi_1 \vdash \bigvee \circ \pi_2$. This says precisely that there is a tracking f with the required property.

□

Next we investigate the key feature of algebras. One direction of the following proposition makes use of condition (CF).

Proposition 5.3.6 *An indexed preorder $\mathcal{E}(-,\Sigma)$ carries a pseudo-algebra structure $\bigvee : \mathcal{I}\Sigma \to \Sigma$ if and only if $\mathcal{E}(-,\Sigma)$ has left adjoints for all reindexing functors, satisfying the Beck-Chevalley condition.*

Proof. Assume first that $\bigvee$ is the pseudo-algebra map. Given a map $f : X \to Y$, and $\alpha : X \to \Sigma$, define $\exists_f \alpha : Y \to \Sigma$ as

$$\exists_f \alpha(y) = \bigvee_{f(x)=y} \alpha(x).$$

To show that this is indeed a left adjoint, denote by $\alpha_f : Y \to \mathcal{I}\Sigma$ the map sending y to $\{\alpha(x) | f(x) = y\}$. Thus $\exists_f \alpha = \bigvee \circ \alpha_f$. The adjointness $\exists_f \dashv - \circ f$ now follows directly form the fact that $\bigvee$ is adjoint to the unit.

Since we have defined this quantification fibrewise, it follows that the Beck-Chevalley condition holds on the nose.

For the converse, take M to be the object $M = \{(a, U) | a \in U \in \mathcal{I}\Sigma\}$. This gives two projections $\pi_1 : M \to \Sigma, \pi_2 : M \to \mathcal{I}\Sigma$, so we can form $\bigvee = \exists_{\pi_2}(\pi_1) : \mathcal{I}\Sigma \to \Sigma$. This is the underlying function of the algebra map. We must show that it is a morphism of indexed preorders. To this end, suppose that there are $\alpha, \beta : X \to \mathcal{I}\Sigma$ with $\alpha \vdash^*_X \beta$, via some $f \in \mathcal{F}$. Pick such an f (using the (CF)-condition). Then take the object $P = \{(x, a, f(a), \alpha(x), f[\alpha(x)]| x \in X, a \in \alpha(x)\}$, and the object $Q = \{(x, \alpha(x), f[\alpha(x)]| x \in X\}$. There is an obvious map $g : P \to Q$, and we have two projections $\pi_2, \pi_3 : P \to \Sigma$, for which $\pi_2 \vdash_P \pi_3$, via f. Hence $\exists_g(\pi_2) \vdash_Q \exists_g(\pi_3)$, so there is some $h \in \mathcal{F}$ such that for all $x \in X : h(\bigvee \alpha(x)) \leq \bigvee f[\alpha(x)]$. But from this it follows that $\bigvee \circ \alpha \vdash \bigvee \circ \beta$, so composing with the supremum map is preorder-preserving. Via similar reasoning we find that $\bigvee \circ \downarrow(-) \dashv\vdash 1$, so that we indeed have Σ as a retraction of $\mathcal{I}\Sigma$.

□

To summarize, the operations $\mathcal{I}(-)$ and $\mathcal{I}_i(-)$ freely add left adjoints to all reindexing functors (resp. to reindexing along regular epis). This is done fibrewise, so Beck-Chevalley automatically holds. In order to avoid confusion, we should make explicit here, that our construction of adding left adjoints to reindexing functors is quite different from the well-known construction to make an $\mathcal{E}$-indexed category $\mathcal{E}$-cocomplete (this construction is an exercise in [41]). For one thing, the result of applying the latter to an indexed preorder need not be a preorder again!

Universal Quantification. As an aside, we show that the operation $\mathcal{I}(-)$ also adds universal quantification along surjections:

Proposition 5.3.7 *Let $\mathcal{E}(-, \Sigma)$ be given. Then $\bigcap : \mathcal{I}\mathcal{I}\Sigma \to \mathcal{I}\Sigma$ defines universal quantification along epimorphisms.*

Proof. Consider

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
\mathcal{I}\Sigma & \mathcal{I}\Sigma.
\end{tikzcd}$$

$\beta \vdash^*_Y \forall_e \alpha$ means $\exists f \in \mathcal{F}\, \forall y\, \forall a \in \beta(y). f(a) \in \bigcap_{e(x)=y} \alpha(x)$. From this it follows immediately that for the same f: $\forall x \forall a \in \beta(e(x)). f(a) \in \alpha(x)$, so f is a tracking for $\beta e \vdash^*_X \alpha$.

The converse is similar.

□

5.3.3 Examples

First of all, it is easily seen that two similar monads, namely the nonempty downset monad on locales, and the nonempty downset monad on ordered PCAs can both be seen as instances of the above, general construction.

For the case of locales, we can say something more general: if M is a meet-semilattice, and $\mathcal{E}(-, M)$ is the indexed preorder where all $\mathcal{E}(X, M)$ are ordered pointwise, then $\mathcal{I}(M)$ is a locale, and $\mathcal{E}(-, \mathcal{I}(M))$ is the canonical tripos associated to this locale.

Even more generally, the downset monad is an extension of the covariant powerset monad on $\mathcal{E}$, and the category of algebras contains the category of complete sup-lattices. To see this, observe that there is an embedding $\mathcal{E} \hookrightarrow \mathbf{PRE}(\mathcal{E})$, sending an object A to the preorder $\mathcal{E}(-, A)$, where each $\mathcal{E}(X, A)$ is discretely ordered. Then applying $\mathcal{I}$ to this object gives the full powerset, and the associated preorder is the tripos for the locale $\mathcal{P}(A)$.

We also see, that when we apply the full downset construction $\mathcal{I}(-)$ to a simple indexed preorder for an ordered PCA, we get the realizability tripos. This makes precise the idea, that these triposes are *free* triposes.

More applications will be presented in the next section when we have our tripos-characterization at hand.

5.4 Triposes and ordered PCAs

In their 1988 paper "A categorical approach to realizability and polymorphic types", [18], Carboni, Freyd and Scedrov prove the following theorem: start with the natural numbers $\mathbb{N}$, and a collection $\mathcal{F}$ of partial endofunctions on $\mathbb{N}$, subject to certain closure conditions. Then build the category of Assemblies with respect to $(\mathbb{N}, \mathcal{F})$; objects are pairs (X, α), with X a set, and $\alpha : X \to \mathcal{P}_i\mathbb{N}$, and a map $p : (X, \alpha) \to (Y, \beta)$ is a function $p : X \to Y$ which is tracked, in the sense that there is some $f \in \mathcal{F}$ such that $\forall x \in X\ \forall a \in \alpha(x) : f(a) \in \beta(x)$. The *realizability universe* is now defined to be the ex/reg-completion of this category of assemblies. Now the realizability universe is a topos if and only if the class $\mathcal{F}$ contains (extensions of) all partial recursive functions.

In this section we will obtain a generalization of this theorem, using ordered PCAs. The data $\mathbb{N}, \mathcal{F}$ is replaced by an indexed preorder $\mathcal{E}(-, \Sigma)$; we will assume that this indexed preorder has indexed finite limits. Then we characterize when the operations $\mathcal{I}$ and $\mathcal{I}_i$, applied to $\mathcal{E}(-, \Sigma)$, yield a tripos. These characterizations will be presented in the first subsection. Along the way, we will see that the requirement that $\mathcal{E}(-, \mathcal{I}\Sigma)$ is a tripos forces the poset Σ to be an ordered PCA. This does not mean, however, that the ambient tripos is automatically the canonical tripos associated to this ordered PCA. Rather, we prove that all triposes of this form are given by the data of an ordered PCA together with a collection of designated truth-values, as in definition 5.2.5.

Another useful result is that the operation $\mathcal{I}_i$ preserves the property of being a tripos; this enables us to show that certain indexed preorders are triposes without having to calculate all tripos structure.

In subsection 2 we apply the results and calculate some interesting examples, for which it is not immediately clear that they fit into our framework. Modified realizability is examined, and we touch upon the complicated dialectica tripos.

Subsection 3 is concerned with topologies; we show that the operation $\mathcal{I}_i$ commutes with taking subtriposes. As an application, we look at Lifschitz realizability.

We deal with relative completions in subsection 4. We prove that the operation $\mathcal{I}_i$ corresponds, on the level of the categories of elements, to the relative regular completion. Formally:

$$\mathcal{G}(\mathcal{I}_i\Sigma) \simeq \mathcal{G}(\Sigma)_{\mathcal{E}/reg}.$$

Finally, we look at the tripos-to-topos construction in subsection 5. It is folklore that any topos that comes from a tripos is the ex/reg completion of its full subcategory on the subobjects of the constant objects. We generalize this slightly, and, for triposes of the form $\mathcal{E}(-, \mathcal{I}\Sigma)$, we show that this implies that the corresponding topos is a relative exact completion of the category $\mathcal{G}(\Sigma)$. Thus we see that if we have a free tripos, then the corresponding topos is also free. The converse is, however false; there are toposes that are exact completions and that, at the same time, come from a tripos that is not a free algebra for the $\mathcal{I}$-monad.

5.4.1 Tripos characterizations

We wish to emphasize that the core idea of this section, namely the translation from implication to applicative structure, is taken from [18], although adapted to our setting. In this section we will assume that our indexed preorders have finite limits, induced by $\wedge : \Sigma \times \Sigma \to \Sigma$.

We will also employ the internal logic of the indexed preorder (as explained in chapter 2); for example, when we write $a \wedge b \vdash_{a,b} b$, we mean that $\pi_1 \wedge \pi_2 \vdash \pi_1$, where $\pi_1, \pi_2 : \Sigma \times \Sigma \to \Sigma$ are the projections. This is the same as saying that there is some $f \in \mathcal{F}$ with the property $f(a \wedge b) \leq a$, for all $a, b \in \Sigma$.

Finally, we introduce some notation borrowed from [18], and prove a few technical lemmas to facilitate the oncoming proofs. Let $\mathcal{F}_2$ be the set of partial binary functions $g : \Sigma \times \Sigma \to \Sigma$ such that there exists an $f \in \mathcal{F}$ with $f(a \wedge b) \leq g(a, b)$ for all $(a, b) \in dom(g)$. Then we can derive the following closure properties for $\mathcal{F}, \mathcal{F}_2$:

Lemma 5.4.1 *$\mathcal{F}_2$ contains both projections $\pi_1, \pi_2 : \Sigma \times \Sigma \to \Sigma$ and the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$.*

Proof. We have $a \wedge b \vdash_{a,b} a$ (uniformly in a, b), which says precisely that the map $a \wedge b \mapsto a$ is admissible. So the projection $(a, b) \mapsto a$ is in $\mathcal{F}_2$. Because $(a \wedge b) \mapsto (a \wedge b)$ is in $\mathcal{F}$, the meet map is in $\mathcal{F}_2$.

□

Lemma 5.4.2 *Let $f, g \in \mathcal{F}$. Then the map $(a \wedge b) \mapsto (f(a) \wedge g(b))$ is in $\mathcal{F}$. For any map $h \in \mathcal{F}_2$, the composite $h \circ (f \times g)$ is in $\mathcal{F}_2$.*

Proof. We have $a \vdash_a f(a)$ and $b \vdash_b g(b)$, uniformly in a, b. Thus we have $a \wedge b \vdash_{a,b} f(a) \wedge g(b)$, also uniformly in a, b. This says that $(a \wedge b) \mapsto (f(a) \wedge g(b))$ is in $\mathcal{F}$. Now given $h \in \mathcal{F}_2$, we know that there is a map $\hat{h}$ with $(f(a) \wedge g(b)) \mapsto \hat{h}(f(a) \wedge g(b))$ admissible. So the composite $a \wedge b \mapsto f(a) \wedge g(b) \mapsto \hat{h}(f(a) \wedge g(b))$ is admissible, which says that $h \circ (f \times g)$ is in $\mathcal{F}_2$.

$\square$

Note that maps in $\mathcal{F}_2$ need not preserve the order in both variables separately; asking that this is the case, amounts to requiring that the meet map $\wedge$ does so.

Now we can embark on the characterization of triposes:

Theorem 5.4.3 (CF) *The following are equivalent:*

1. $\mathcal{E}(-, \mathcal{I}\Sigma)$ *has Heyting implication;*
2. *There is a map* $App \in \mathcal{F}_2$ *such that for each* $h \in \mathcal{F}_2$ *there is an* $\hat{h} \in \mathcal{F}$ *with* $(a, b) \in dom(h)$ *implies* $App(\hat{h}(a), b))$ *defined and* $App(\hat{h}(a), b) \leq h(a, b)$.

Proof. First assume (1). As the map *App*, we take a tracking m for $(A \Rightarrow B) \wedge A \vdash B$ and put $App(x, y) = m(x \wedge y)$. So *App* is in $\mathcal{F}_2$, and has the property that for each $c \in A \Rightarrow B$ and $a \in A$, $App(c, a) \in B$. In particular:

$$c \in \downarrow(a) \Rightarrow \downarrow(b) \text{ implies } App(c, a) \leq b.$$

Now take $h \in \mathcal{F}_2$; then $a \wedge b \vdash_{a,b} h(a, b)$. Therefore $\downarrow(a) \wedge \downarrow(b) \vdash^* \downarrow(h(a, b))$, so $\downarrow(a) \vdash^* \downarrow(b) \Rightarrow \downarrow(h(a, b))$. Thus we have a $\hat{h}$ with $\hat{h}(a) \in \downarrow(b) \Rightarrow \downarrow(h(a, b))$ for all $(a, b) \in dom(h)$. Hence $App(\hat{h}(a), b) \leq h(a, b)$. This proves (2).

Next assume (2). We define, for $A, B \in \mathcal{I}\Sigma$,

$$A \Rightarrow B = \downarrow\{c \in \Sigma | \forall a \in A : App(c, a) \in B\}.$$

First suppose that $\alpha \wedge \beta \vdash^*_X \gamma$. So, $\exists g \in \mathcal{F}\, \forall x\, \forall a \in \alpha(x)\, \forall b \in \beta(x) : g(a \wedge b) \in \gamma(x)$. By the closure property (2), we find a $\hat{g}$, such that for all $a \in \alpha(x), \forall b \in \beta(x) : App(\hat{g}(a), b) \leq g(a \wedge b)$. Hence $\hat{g}$ is a tracking for $\alpha \vdash^*_X \beta \Rightarrow \gamma$.

Conversely if $\alpha \vdash^*_X \beta \Rightarrow \gamma$, then we have a map $h \in \mathcal{F}$ sending each $a \in \alpha(x)$ to an element of $\beta(x) \Rightarrow \gamma(x) = \{c | \forall b \in \beta(x) : App(c, b) \in \gamma(x)\}$. Now the map $a \wedge b \mapsto App(h(a), b)$ is in $\mathcal{F}$ since both h and *App* are, and is a tracking for $\alpha \wedge \beta \vdash^*_X \gamma$. This proves (1).

$\square$

We may call, if we are in the situation of the above theorem, a map of the form $App(a, -)$ *representable*. Then the next lemma states that representable maps are given by designated truth-values (recall from chapter 2 that designated truth-values are those $a \in \Sigma$ for which $\top \vdash_1 a$):

Lemma 5.4.4 *An element* a *of* Σ *is a designated truth-value of* Σ *if and only if* $App(a, -)$ *is in* $\mathcal{F}$.

Proof. If a is a designated truth-value, so $\top \vdash_1 a$. Then, uniformly in b, we have $b \vdash_b \top \wedge b \vdash_b a \wedge b$. Also, the map m in the proof of theorem 5.4.3 gives $a \wedge b \vdash_{a,b} m(a \wedge b)$. By transitivity we get $b \vdash_b m(a \wedge b) = App(a, b)$, which means that $App(a, -)$ is in $\mathcal{F}$.

For the converse, suppose that $App(a, -) = m(a \wedge -)$ is in $\mathcal{F}$. Thus $b \vdash_b m(a \wedge b)$, uniformly in b. Take a tracking t for $\downarrow(a) \vdash^* \downarrow(\top) \Rightarrow \downarrow(a)$; so $t(a) \in \downarrow(\top) \Rightarrow \downarrow(a)$, and hence $m(t(a) \wedge \top) \leq a$ (by definition of m). Taking all this together, we find (using lemma 5.4.2) that $\top \vdash m(a \wedge \top) \vdash m(t(a) \wedge \top) \vdash a$, and a is a designated truth-value.

□

Next we derive some further properties of the map App that we obtained from implication. For readability we write $a \bullet b$ or simply ab for $App(a, b)$.

Proposition 5.4.5 *Suppose that $\mathcal{E}(-, \Sigma)$ satisfies the equivalent conditions of theorem 5.4.3. Suppose also that the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$ preserves the order in both variables. Then there are combinators $k, s \in \Sigma$ that make $(\Sigma, \leq, \bullet, k, s)$ into an ordered PCA. Moreover, k, s are designated truth-values of Σ.*

Proof. Consider the propositional scheme

$$A \Rightarrow (B \Rightarrow A).$$

There is an element k (which is a designated truth-value) with the property $\forall a \in A \, \forall b \in B : k \bullet a \bullet b \in A$. Thus for $a, b \in \Sigma : k \bullet a \bullet b \leq a$. For s, take a designated truth-value for the scheme

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C).$$

Take any $x, y \in \Sigma$, and put $A = \emptyset$. Thus $x \in A \Rightarrow (B \Rightarrow C)$, and $y \in A \Rightarrow B$. Hence $s \bullet x \bullet y\!\downarrow$. If, in addition $xz(yz)\!\downarrow$, then put $A = \downarrow(z), B = \downarrow(yz), C = \downarrow(xz(yz))$. Then $y \in A \Rightarrow B = \downarrow(z) \Rightarrow (yz)$, and $x \in A \Rightarrow (B \Rightarrow C)$. Hence $sxyz\!\downarrow$ and $sxyz \leq xz(yz)$.

The condition that $\wedge$ is strictly order-preserving in both variables implies that the application map also strictly preserves the order, i.e. that

$$ab\!\downarrow \; \& a' \leq a, b' \leq b \Rightarrow a'b'\!\downarrow \; \& a'b' \leq ab,$$

which is the last axiom for ordered PCAs that we had to verify.

□

If we have this ordered PCA-structure, we can formulate an important closure property of $\mathcal{F}$ ("restricted combinatorial completeness"): Let $t[x]$ be any term with free variable x, built from elements of Σ by means of application. Then the map $a \mapsto t[a/x]$ is in $\mathcal{F}$ provided every constant in $t[x]$ is a designated truth-value. The proof is immediate, since we know that the combinators k, s are designated truth-values, and because the maps in $\mathcal{F}$ preserve designated truth-values. (A better formulation would involve a *sequence* of free variables.)

Denoting the set of designated truth-values by Φ, we can now prove

Lemma 5.4.6 *Let $f \in \mathcal{F}$. Then there is some $a \in \Phi$ such that for all $b \in dom(f)$: $ab \leq f(a)$.*

Proof. If $f \in \mathcal{F}$ then, uniformly in all $b \in dom(f)$, $b \vdash_b f(b)$, whence $\downarrow\!(b) \vdash^* \downarrow\!(f(b))$. Then $\top \vdash^* \downarrow\!(b) \Rightarrow \downarrow\!(f(b))$, so there is some $p \in \downarrow\!(b) \Rightarrow \downarrow\!(f(b))$, which is a designated truth-value. This p satisfies $App(p, b) \leq f(b)$ for all b, so p represents f.

□

Corollary 5.4.7 *Let $\mathcal{E}(-, \Sigma)$ satisfy the conditions of theorem 5.4.3. Then the preorder on $\mathcal{E}(-, \Sigma)$ is given by:*

$$\alpha \vdash_X \beta \Leftrightarrow \exists a \in \Phi \, \forall x \in X : a \bullet \alpha(x) \leq \beta(x).$$

Consequently, the preorder on $\mathcal{E}(-, \mathcal{I}\Sigma)$ is given by

$$\alpha \vdash_X^* \beta \Leftrightarrow \exists a \in \Phi \, \forall x \in X \, \forall b \in \alpha(x) : a \bullet b \in \beta(x).$$

Proof. This follows immediately from the previous lemma identifying maps in $\mathcal{F}$ with representable functions.

□

Putting all of this together, we can characterize when $\mathcal{E}(-, \mathcal{I}\Sigma)$ is a tripos:

Theorem 5.4.8 *Let $\mathcal{E}(-, \Sigma)$ be an indexed preorder with indexed finite meets, such that the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$ is strictly order-preserving. Then the following are equivalent:*

1. *Σ carries an ordered PCA-structure together with a filter Φ of designated truth-values, and the preorder is given as in corollary 5.4.7;*
2. *$\mathcal{E}(-, \mathcal{I}\Sigma)$ is a tripos.*

Proof. One direction is immediate from theorem 5.4.3, since every tripos has implication. For the other direction we can also be brief, since all the tripos structure can be defined exactly as for an ordinary tripos from an ordered PCA, with the only difference that we restrict the collection of realizers to the designated truth-values. All constructions go through, because of the combinatorial completeness of Φ.

□

It is important to observe that the tripos that can be defined from the ordered PCA Σ together with the filter Φ is nothing but the *relative realizability tripos*, because the filter Φ may be viewed as a sub-ordered PCA of $\mathbb{A}$. Thus we see that *relative realizability* also fits into our framework in a natural fashion (for more on relative realizability, see [15]).

Another remark concerns the use of the condition on the meet map; in all the indexed preorders that arise from ordered PCAs with a filter, this condition is satisfied. This raises the question whether we can have a tripos $\mathcal{E}(-, \mathcal{I}\Sigma)$

without this condition. If so, then it might be the case that the axiom on ordered PCAs stating that application is order-preserving is slightly too strong. I couldn't see, however, how to obtain a weakening of the axiom while retaining a suitable form of combinatory completeness...

Next, we wish to characterize when $\mathcal{E}(-, \mathcal{I}_i\Sigma)$ is a tripos. The main difference with $\mathcal{E}(-, \mathcal{I}\Sigma)$ is, of course, that arbitrary intersections need not exist (and empty unions), so that we cannot always define quantification. The missing part is, that Σ needs to have a bottom element.

Theorem 5.4.9 *Let $\mathcal{E}(-, \Sigma)$ be an indexed preorder with indexed finite meets, such that the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$ is strictly order-preserving. If Σ has a least element $\perp$ then the following are equivalent:*

1. *Σ carries an ordered PCA-structure together with a filter Φ of designated truth-values, and the preorder is given as in corollary 5.4.7*
2. *$\mathcal{E}(-, \mathcal{I}_i\Sigma)$ is a tripos.*

Proof. The proof that $\mathcal{E}(-, \mathcal{I}_i\Sigma)$ has implication is completely the same as for theorem 5.4.8. For universal quantification we need only observe that the intersection of an arbitrary family of downsets always contains the bottom element, so that the usual definition works. But we know from [59] that implication and universal quantification already give us a tripos.

The other direction is the same as for theorem 5.4.8

□

As a corollary, we get the result that the operation $\mathcal{I}_i$ preserves triposes.

Corollary 5.4.10 *If $\mathcal{E}(-, \Sigma)$ is a tripos, then so is $\mathcal{E}(-, \mathcal{I}_i\Sigma)$.*

Proof. $\mathcal{E}(-, \Sigma)$ has implication, and therefore has $\mathcal{E}(-, \mathcal{I}_i\Sigma)$ has implication, too, given by

$$\alpha \Rightarrow^* \beta = \bigcap_{a \in \alpha} \bigcup_{b \in \beta} \downarrow(a \Rightarrow b),$$

where $\Rightarrow$ is the implication in $\mathcal{E}(-, \Sigma)$. Hence, Σ has an ordered PCA-structure. Furthermore, $\mathcal{E}(-, \Sigma)$ has an indexed least element, which may be taken to be induced by a global element $\perp_1 : 1 \to \Sigma$, so the conditions of theorem 5.4.9 are met.

Finally, we may take the bottom element of $\mathcal{E}(-, \mathcal{I}_i\Sigma)$ to be $\{\perp\}$, where $\perp$ is the bottom element of $\mathcal{E}(-, \Sigma)$.

□

5.4.2 Examples

Extensional Realizability. Take any ordered PCA $\mathbb{A}$, and form the realizability tripos $\mathcal{E}(-, \mathcal{I}\mathbb{A})$. We may apply the $\mathcal{I}_i(-)$-construction in order to obtain $\mathcal{E}(-,\mathcal{I}_i\mathcal{I}\mathbb{A})$, which is, by corollary 5.4.10, again a tripos. On the other hand, one can look at the realizability tripos for the (free) ordered PCA $\mathcal{I}_i\mathbb{A}$, so $\mathcal{E}(-,\mathcal{I}\mathcal{I}_i\mathbb{A})$. It is not hard to show that these triposes are equivalent. In fact, under mild assumptions concerning the behaviour of admissible functions with respect to the bottom element of a tripos, one can show that the operations $\mathcal{I}$ and $\mathcal{I}_i$ commute with eachother. For the case $\mathbb{A} = \mathbb{N}$, Kleene's PCA, we thus see that there are two ways of building the tripos for extensional realizability (for a detailed acccount of extensional realizability, see [75]). Consequently, the hierarchy of realizability triposes that has been described in chapter 3 also admits an alternative presentation by means of iterated application of the $\mathcal{I}_i$-construction on triposes, instead of on ordered PCAs.

Modified Realizability. Let us show that the modified realizability tripos fits into our framework. The generic element of the tripos is

$$\Sigma = \{(U_a, U_p) | U_a \subseteq U_p \subseteq \mathbb{N}, 0 \in U_p\}.$$

Here, a coding of partial recursive functions is chosen in such a way that $0 \bullet x = x$ for all $x \in \mathbb{N}$. The entailment in the fibre over 1 is given by:

$$(U_a, U_p) \vdash_1 (V_a, V_p) \Leftrightarrow \exists n \in \mathbb{N} : n \in (U_a \Rightarrow V_a) \cap (U_p \Rightarrow V_p).$$

where the $\Rightarrow$ on the right-hand side of the equation is the ordinary $A \Rightarrow B = \{n | \forall a \in A : na \in V\}$. In the fibre over an arbitrary X, we require that the realizer n works uniformly in all $x \in X$. The generic element may be endowed with an ordered PCA-structure. The ordering is defined pairwise:

$$(U_a, U_p) \leq (V_a, V_p) \text{ if and only if } U_a \subseteq V_a \text{ and } U_p \subseteq V_p$$

and we define

$$(U_a, U_p) \bullet (V_a, V_p) \simeq (U_a V_a, U_p V_p)$$

where the juxtaposition of sets on the righthand side is shorthand for $U_p V_p = \{ab | a \in U_p, b \in V_p\}$; so this is just pairwise application in the ordered PCA $\mathcal{P}\mathbb{N}$.

Next, there are the combinators k, s, which may be taken to be $(\{k\}, \{0, k\})$ and $(\{s\}, \{0, s\})$.

Note that, as an ordered PCA, Σ is trivial, since it has a least element $(\emptyset, \{0\})$. But the designated truth-values $\Phi \subseteq \Sigma$ are those (U_a, U_b) for which $U_a \neq \emptyset$.

The implication $\Rightarrow : \Sigma \times \Sigma \to \Sigma$ is given by

$$(U_a, U_p) \Rightarrow (V_a, V_p) = ((U_a \Rightarrow V_a) \cap (U_p \Rightarrow V_p), U_p \Rightarrow V_p)$$

Now we derive that $(U_a, U_p) \bullet (V_a, V_p) \leq (W_a, W_p)$ if and only if $(U_a, U_p) \leq$

$((V_a, V_p) \Rightarrow (W_a, W_p))$.

$$\begin{array}{rll} (U_a, U_p) \bullet & (V_a, V_p) & \leq (W_a, W_p) \\ & \Leftrightarrow & (U_a V_a, U_p V_p) \leq (W_a, W_p) \\ & \Leftrightarrow & U_a V_a \subseteq W_a \,\&\, U_p V_p \subseteq W_p \\ & \Leftrightarrow & U_a \subseteq V_a \Rightarrow W_a \,\&\, U_p \subseteq V_p \Rightarrow W_p \\ & \Leftrightarrow & U_a \subseteq V_a \Rightarrow W_a \,\&\, U_p \subseteq V_p \Rightarrow W_p \,\&\, U_a \subseteq V_p \Rightarrow W_p \\ & \Leftrightarrow & U_a \subseteq ((V_a \Rightarrow W_a) \cap (V_p \Rightarrow W_p)) \,\&\, U_p \subseteq V_p \Rightarrow W_p \\ & \Leftrightarrow & (U_a, U_p) \leq ((V_a \Rightarrow W_a) \cap (V_p \Rightarrow W_p), V_p \Rightarrow W_p) \\ & \Leftrightarrow & (U_a, U_p) \leq ((V_a, V_p) \Rightarrow (W_a, W_p)) \end{array}$$

From this, it follows that the tripos may be recaptured as

$$\alpha \vdash_X \beta \Leftrightarrow \exists (U_a, U_p) \in \Phi \, \forall x \in X : (U_a, U_p) \bullet \alpha(x) \subseteq \beta(x).$$

We remark, that application of the $\mathcal{I}_i$-operation to this tripos gives a different result than taking modified realizability over the ordered PCA $\mathcal{P}_i\mathbb{N}$.

Dialectica Tripos. We show that the dialectica tripos also fits into our framework. For a description of this tripos we refer to [14].

The dialectica tripos has a generic object

$$\Sigma = \{(X, Y, A) | X, Y \subseteq \mathbb{N}, A \subseteq X \times Y, 0 \in A \cap Y\}$$

and the preorder in the fibre over 1 is given by

$$\begin{array}{rcl} (X, Y, A) \vdash (X', Y', A') & \Leftrightarrow & \exists f, F \in \mathbb{N} : & f \in (X \Rightarrow X'), \\ & & & F \in (X \times Y' \Rightarrow Y), \\ & & & A(x, F(x, y)) \text{ implies } A'(fx, y) \end{array}$$

and in the fibre over M we require this uniformly in all $m \in M$. We order the generic element by putting

$$(X, Y, A) \leq (X', Y', A') \Leftrightarrow X \subseteq X', Y' \subseteq Y, A \subseteq A'.$$

So, $(\Sigma, \leq)$ is the underlying poset. If ϕ is a partial endofunction on Σ, then say $\phi \in \mathcal{F}$ iff there exist f, F such that if $\phi(X, Y, A) = (P, Q, B)$, then $f[X] = P$, $F[X \times Q] = Y$, and $A(x, F(x, q))$ implies $B(fx, q)$. Clearly such ϕ satisfies $(X, Y, A) \vdash \phi(X, Y, A)$, uniformly in (X, Y, A). Conversely, if we have $(X_i, Y_i, A_i) \vdash_I (P_i, Q_i, B_i)$, then we can find a $\phi \in \mathcal{F}$ such that $\phi(X_i, Y_i, A_i) \leq (P_i, Q_i, B_i)$, for all i. Namely, let f, F be the required realizers for $(X_i, Y_i, A_i) \vdash_I (P_i, Q_i, B_i)$, and put $\phi(X_i, Y_i, A_i) = (X_i', Y_i', A_i')$, where $X_i' = \{f\}X_i, Y_i' = \{a \in \mathbb{N} | \forall x \in X : F(x, a) \in Y_i\}, A_i' = \{(fx, a) | A(x, F(x, a))\}$. It is evident that the maps in $\mathcal{F}$ are monotone. From the results of section 4.1 it now follows that

$\mathbf{Set}(-,\mathcal{I}_i\Sigma)$ is again a tripos. It seems quite hard, however to give a detailed calculation of this tripos; in particular the ordered PCA-structure on Σ is difficult to compute...

Iteration. We can use corollary 5.4.10 repeatedly: this gives a sequence of triposes, and hence a sequence of toposes. The first instance of such a chain of realizability toposes was demonstrated in [57], and analysed, using the nonempty-downset monad on ordered PCAs, in chapter 3. As the two above examples show, we also have chains of toposes that begin with the Modified Realizability Topos, and with the Dialectica Topos. The logical properties of those toposes need further investigation.

5.4.3 Topologies

If we have a tripos of the form $\mathcal{E}(-,\Sigma)$, then we may consider subtriposes of this tripos; by the results in [59], these correspond to topologies on $\mathcal{E}(-,\Sigma)$, and these are, in turn, in correspondence with maps $J:\Sigma\to\Sigma$ satisfying:

1. $\top\vdash J(\top)$;
2. $JJ(a)\vdash_a J(a)$ and
3. $a\Rightarrow b\vdash_{a,b} J(a)\Rightarrow J(b)$.

The subtripos has the same generic object Σ, but the preorder is given by

$$\alpha\vdash^J_X\beta\Leftrightarrow\alpha\vdash_X J\circ\beta.$$

First we show that if $J:\Sigma\to\Sigma$ is a topology on $\mathcal{E}(-,\Sigma)$. then $\mathcal{I}_i(J):\mathcal{I}_i\Sigma\to\mathcal{I}_i\Sigma$ is a topology on $\mathcal{E}(-,\mathcal{I}_i\Sigma)$. To prove this, we need to verify the three axioms for a topology.

1. $\top\vdash J(\top)$, so $\downarrow(\top)\vdash^*\mathcal{I}_i(J)(\downarrow(\top))$.
2. take $A\subseteq\Sigma$. Then $\mathcal{I}_i(J)\mathcal{I}_i(J)(A)=\downarrow\{JJ(a)|a\in A\}$. But J is idempotent, so there is an f with $f(JJ(a))\leq J(a)$ for all $a\in\Sigma$. Hence for all $b\in\mathcal{I}_i(J)\mathcal{I}_i(J)(A)$, $f(b)\in\mathcal{I}_i(J)(A)$.
3. We need a function g such that for all $A,B\in\mathcal{I}_j\Sigma$, for all $m\in A\Rightarrow^* B$, $g(m)\in\mathcal{I}_i(J)(A)\Rightarrow^*\mathcal{I}_i(J)(B)$. Now $m\in A\Rightarrow^* B$ just means that $\forall a\in A\exists b\in B: m\leq a\Rightarrow b$. But since J is a topology, we have f with $f(a\Rightarrow b)\leq J(a)\Rightarrow J(b)$. Hence this f also satisfies $\forall a\in A\exists b\in B: f(m)\leq J(a)\Rightarrow J(b)$. But then $f(m)$ is an element of $\mathcal{I}_i(J)(A)\Rightarrow^*\mathcal{I}_i(J)(B)$.

So, the non-empty downset monad preserves topologies. We will exploit this in the following manner: taking a subtripos of a tripos in our framework does not necessarily give a tripos of the required form, so strictly speaking we cannot apply the downset-construction to subtriposes. Still, for an indexed preorder

$\mathcal{E}(-,\Sigma)$ and a topology J, the following "closest approximation" makes sense (for $\alpha, \beta : X \to \mathcal{I}_i\Sigma$):

$$\alpha(\vdash^J)^*\beta \Leftrightarrow \exists f \in \mathcal{F}\, \forall x \in X\, \forall a \in \alpha(x).f(a) \in J[\beta(x)] \qquad (\dagger)$$

The following proposition now tells us, that this is indeed the right definition (and that the non-empty downset-construction commutes with taking subtriposes).

Proposition 5.4.11 *Let J be a topology on $\mathcal{E}(-,\Sigma)$. Then the sub-indexed preorder of $\mathcal{E}(-,\mathcal{I}_i\Sigma)$ induced by $\mathcal{I}_i(J)$ has preorder in the fibre over X given by ($\dagger$).*

Proof. This is simply a matter of spelling out the definitions:

$$\begin{array}{rcl} \alpha(\vdash^*)^{\mathcal{I}_i(J)}\beta & \Leftrightarrow & \alpha \vdash^* \mathcal{I}_i(J) \circ \beta \\ & \Leftrightarrow & \exists f\, \forall x\, \forall a \in \alpha(x) : f(a) \in \mathcal{I}_i(J)(\beta(x)) \\ & \Leftrightarrow & \exists f\, \forall x\, \forall a \in \alpha(x) : f(a) \in \downarrow\{J(b) | b \in \beta(x)\} \\ & \Leftrightarrow & \exists f\, \forall x\, \forall a \in \alpha(x)\, \exists b \in \beta(x) : f(a) \leq J(b) \\ & \Leftrightarrow & \alpha(\vdash^J)^*\beta \end{array}$$

□

An application of this proposition is provided by Lifschitz Realizability: the Lifschitz tripos is a subtripos of the effective tripos. It is not a realizability for a PCA (see [73]), so we cannot directly apply the downset-construction. But we can use definition † to obtain a subtripos of the tripos for extensional realizability. This might be viewed as a kind of "extensional Lifschitz realizability".

5.4.4 Relative completions

In this section we prove the correspondence between the inhabited downset-monad on indexed preorders on the one hand, and the relative regular completion monad on the other hand. The proof is a straightforward generalization of the proof of the fact that the category of Assemblies is the relative regular completion of the category of Partitioned Assemblies (relative to the base topos), and, just as in that special case, we need the condition (CF).

Theorem 5.4.12 (CF) *Let $\mathcal{E}(-,\Sigma)$ be an indexed preorder. There is an equivalence of categories*

$$\mathcal{G}(\mathcal{I}_i\Sigma) \simeq \mathcal{G}(\Sigma)_{\mathcal{E}/reg}.$$

Proof. The proof goes in two stages: first we show that there is a minimal cover $\mathcal{G}(\Sigma) \to \mathcal{G}(\mathcal{I}_i\Sigma)$. This implies that $\mathcal{G}(\mathcal{I}_i\Sigma)$ is a reflective subcategory of $\mathcal{G}(\Sigma)_{reg}$, and may therefore be described as sheaves for a universal closure operator on $\mathcal{G}(\Sigma)_{reg}$. Step two compares the corresponding topology on $\mathcal{G}(\Sigma)$ with the topology induced by the regular epis in the image of $\nabla : \mathcal{E} \to \mathcal{G}(\Sigma)$. The observation that these two topologies coincide proves the theorem.

So we start by investigating the functor $i : \mathcal{G}(\Sigma) \to \mathcal{G}(\mathcal{I}_i\Sigma)$, which is induced by the map (also denoted by i) that sends an element $a \in \Sigma$ to $\downarrow(a) \in \mathcal{I}_i\Sigma$. This functor is full and faithful. Next, we show that every object in $\mathcal{G}(\mathcal{I}_i\Sigma)$ can be covered by an object in the image of i. So take (X, α) in $\mathcal{G}(\mathcal{I}_i\Sigma)$. The canonical cover of this object is (Q, π), where $Q = \{(x, a) | a \in \alpha(x)\}$, and $\pi(x, a) = \downarrow(a)$. Clearly, (Q, π) is in the image of i.

Along the way, we mention that regular epis in $\mathcal{G}(\mathcal{I}_i\Sigma)$ are those maps $e : (Y, \beta) \to (Z, \gamma)$ for which e is epi, and (up to isomorphism) $\gamma(z) = \bigcup_{e(y)=z} \beta(y)$. From this description it is also evident that the projection $q : (Q, \pi) \to (X, \alpha)$ is a cover.

It remains to be shown that this cover has the required projectivity property: so let (M, μ) be another object in the image of i and $k : (M, \mu) \to (X, \alpha)$ any map. Then $\exists f \in \mathcal{F}\, \forall m \in M : f(\mu(m)) \in \alpha(k(m))$. Using condition CF we pick such a function f, and define a map $M \to Q$ by $m \mapsto (k(m), f(\mu(m))$. This defines a factorization of k through q, and we have established that $i : \mathcal{G}(\Sigma) \to \mathcal{G}(\mathcal{I}_i\Sigma)$ is a minimal cover.

For the second part of the proof, we observe that the associated universal closure operator on $\mathcal{G}(\Sigma)_{reg}$ may be described as follows: any map $h : (X, \alpha) \to (Y, \beta)$ gives rise to an equivalence relation on X, via the definition

$$x \sim x' \Leftrightarrow h(x) = h(x') \wedge \alpha(x) = \alpha(x').$$

Then α factors through the quotient map $p : X \to X/\sim$, say as $\alpha' : X/\sim \to \Sigma$, and $p : (X, \alpha) \to (X/\sim, \alpha')$ is in fact a morphism in $\mathcal{G}(\Sigma)$. We have a diagram

$$\begin{tikzcd}
(X,\alpha) \arrow[r, "p"] \arrow[dr, "h"'] & (X/\sim, \alpha') \arrow[d, "h'"] \\
 & (Y,\beta).
\end{tikzcd}$$

The universal closure operator sends h (considered as a subobject of (Y, β)) to h'. The intuition is therefore, that an object h is a sheaf iff there is no "redundancy in the fibres", i.e. $h(x) = h(x')$ implies $\alpha(x) \neq \alpha(x')$.

Denoting the corresponding topology on $\mathcal{G}(\Sigma)$ by J, we find that a map $h : (X, \alpha) \to (Y, \beta)$ is in J iff (in $\mathcal{G}(\Sigma)_{reg}$) the mono $[h] : \begin{pmatrix} (X,\alpha) \\ h\downarrow \\ (Y,\beta) \end{pmatrix} \to (Y, \beta)$ is dense, that is, $h(x) = h(x')$ implies $\alpha(x) = \alpha(x')$.

Finally we compare this with the topology K generated by the regular epis in the image of $\nabla : \mathcal{E} \to \mathcal{G}(\Sigma)$. If we have $h \in J$, then h is cartesian, i.e. a pullback of $\nabla(h) : \nabla(X) \to \nabla(Y)$. So $h \in K$. Conversely, if e is regular epi in $\mathcal{E}$, then $\nabla(e)$ is again regular epi. Evidently, such a regular epi is in J, and therefore $K \subseteq J$, and we have proved the theorem.

□

5.4.5 Tripos-to-topos construction

Next, we look at some unsurprising facts concerning the tripos-to-topos construction. If $\mathcal{E}(-, \Sigma)$ is an indexed preorder with meets and existential quantification, then we denote by $\mathcal{E}[\Sigma]$ the result of this procedure. In particular, we will apply this to indexed preorders of the form $\mathcal{E}(-, \mathcal{I}\Sigma)$.

Lemma 5.4.13 *In the category $\mathcal{E}[\mathcal{I}\Sigma]$, a subobject of an object ∇X in the image of the constant objects functor ∇ may be represented by an element $\alpha : X' \to \mathcal{I}_i\Sigma$, with $X' \subseteq X$. Conversely, every such element represents a subobject of ∇X.*

Proof. It is well-known that subobjects of ∇X may be represented by elements $\alpha : X \to \mathcal{I}\Sigma$. For such α, define $\bar{X} = \{x \in X | \alpha(x) \neq \emptyset\}$, and $\bar{\alpha}$ to be the restriction of α to $\bar{X}$. Then the $(\bar{X}, \bar{\alpha})$ and (X, α) are isomorphic as subobjects of ∇X via the functional relation $F : X \times \bar{X} \to \mathcal{I}\Sigma$ given as $F(x, x') = \{a \in \alpha(x) | x = x'\}$.

□

Lemma 5.4.14 *The full subcategory of $\mathcal{E}[\mathcal{I}\Sigma]$ on the subobjects of ∇'s is equivalent to $\mathcal{G}(\mathcal{I}_i\Sigma)$.*

Proof. The previous lemma established that there is an essentially surjective map from the objects of $\mathcal{G}(\mathcal{I}_i\Sigma)$ onto the subobjects of ∇'s. So we need to verify that this is functorial and fully faithful. If $f : (X, \alpha) \to (Y, \beta)$ is a map in $\mathcal{G}(\mathcal{I}_i\Sigma)$, then this gives rise to a functional relation F from (X, α) to (Y, β) by defining $F(x, y) = \{a \in \alpha(x) | f(x) = y\}$. This construction is clearly injective on maps. If $G : X \times Y \to \mathcal{I}_i\Sigma$ is functional from (X, α) to (Y, β), then in particular $\alpha(x) \vdash^*_x \bigcup_{y \in Y} F(x, y)$ and $F(x, y) \wedge F(x, y') \vdash^*_{x,y,y'} y = y'$. We see that $\alpha(x) \neq \emptyset$ implies that there is a unique $y \in Y$ with $F(x, y) \neq \emptyset$. Therefore F represents a function from X to Y.

□

Proposition 5.4.15 *The categories $\mathcal{G}(\mathcal{I}_i\Sigma)_{ex/reg}$ and $\mathcal{E}[\mathcal{I}\Sigma]$ are equivalent.*

Proof. This follows from the previous lemma and the fact that $\mathcal{E}[\mathcal{I}\Sigma]$ is the ex/reg completion of the full subcategory on the subobjects of ∇'s.

□

Corollary 5.4.16 *Let $\mathcal{E}(-\Sigma)$ be an indexed preorder for which (CF) holds. Then $\mathcal{E}[\mathcal{I}\Sigma]$ is a relative exact completion (relative to $\mathcal{E}$), namely of the category $\mathcal{G}(\Sigma)$.*

Proof. Since $\mathcal{G}(\Sigma)_{\mathcal{E}/ex} \simeq (\mathcal{G}(\Sigma)_{\mathcal{E}/reg})_{ex/reg} \simeq \mathcal{G}(\mathcal{I}_i\Sigma)_{ex/reg}$, the result follows from the previous proposition.

□

This may be summarized by the slogan that if the tripos is free, then so is the topos.

We conclude with the observation that a category of the form $\mathcal{E}[\Sigma]$ may well be an exact completion without the indexed preorder $\mathcal{E}(-,\Sigma)$ being a free algebra for the $\mathcal{I}(-)$-monad. In fact, if Σ is a locale and $\mathcal{E} = \mathbf{Sets}$, then $\mathcal{E}[\Sigma]$ is an exact completion if and only if every $x \in \Sigma$ can be written as $x = \bigvee_{i\in I} v_i$, where all v_i have the property that any cover can be refined into a partition. But there are many locales with this property that are not of the form $\mathcal{I}M$ for some meet-semilattice M. (See next chapter for more on localic toposes that are an exact completion.)

5.5 Comparison with related work

Our framework of indexed preorders bears more than just a superficial resemblence with other settings that have been investigated in the literature. We have already said that our proof of theorem 5.4.8 is a direct generalization and adaptation of work by Carboni, Freyd and Scedrov. Our setting is obviously much more general, since we replace $\mathbb{N}$ by an arbitrary poset; therefore we capture many interesting examples, such as locales, but also modified realizability and the dialectica tripos. A priori the question "When is $\mathcal{E}(-,\mathcal{I}\Sigma)$ a tripos?" seems to be stronger than the question "When is the realizability universe $\mathcal{E}[\mathcal{I}\Sigma]$ a topos," because Pitts has shown that we don't need $\mathcal{E}(-,\mathcal{I}\Sigma)$ to be a tripos but only a first order hyperdoctrine with comprehension for $\mathcal{E}[\mathcal{I}\Sigma]$ to be a topos. But, as it turns out, there is no room for this subtlety within our framework: if $\mathcal{E}(-,\mathcal{I}\Sigma)$ has implication, then it is automatically a tripos.

Another strongly related approach can be found in Aczel's handwritten note "An interpretation of higher-order intuitionistic logic". Here, a class of structures called frames is defined; these frames give rise to triposes, and encompass both triposes coming from (ordered) PCAs and from locales. The interesting part of this work lies in the fact that it states the interdefinability of implication and application. (Although the definitions in the paper are not fully accurate and we could not produce proofs for all the claims that are made.) Again, this relation between implication and application is precisely the subject of our theorem 5.4.8, but this theorem gives a more detailed analysis because the relation with the downset-monad is brought into the picture. This enables us to analyse triposes in terms of simpler structures, whereas for Aczel, all indexed preorders under consideration are triposes. Contemplating the role of the filter of truth-values in Aczel's frames, we realized that many interesting triposes simply consist of a (trivial) complete ordered PCA together with a filter of truth-values.

A comparison with the $\mathcal{F}$-construction is also in order. This construction was discovered by Birkedal and submitted to detailed analysis by De Marchi, Robinson and Rosolini. The starting point of their two-step enterprise is a small category $\mathbb{C}$ together with a (forgetful) functor to the category of sets and partial functions. Using a variation on the comma construction, $\mathbb{C}$ is glued along this forgetful functor, as to obtain a category $\mathcal{F}(\mathbb{C})$. Then one takes the exact completion $\mathcal{F}(\mathbb{C})_{ex}$. The main result is now, that $\mathcal{F}(\mathbb{C})_{ex}$ is a topos precisely when $\mathbb{C}$ has a universal object V carrying a PCA-structure.

The category $\mathbb{C}$ plays the same role as our poset Σ. Therefore, it may be argued, the approach via the $\mathcal{F}$-construction is more general. This is true, but only to a certain extent: all the "impredicativity entails untypedness" results ([53]) show that, as long as we are interested in realizability *toposes*, we can make do with a one-object category $\mathbb{C}$. (Of course, when one is interested in topological settings, such as equilogical spaces, then the generalization is essential. See [13].) Another limitation of the $\mathcal{F}$-construction is, that one only obtains realizability toposes of the form $RT[\mathbb{A}]$, for some PCA $\mathbb{A}$. Toposes for modified realizability and the dialectica interpretation fall outside the scope of the construction.

Finally, we wish to add a comment about a recent discussion on conditional PCAs (c-PCAs for short) as well. In this discussion, it was asked by Peter Johnstone whether c-PCAs give rise to a tripos. John Longley explicitly showed that, by redefining the application, every c-PCA is indeed equivalent to a PCA and therefore gives a tripos. The question which applicative structures give rise to a tripos is still open, but our results show, that any such applicative structure must be equivalent to a PCA.

5.6 Appendix: relations instead of functions

Admittedly, the class of indexed preorders that we have discussed is fairly narrow. Therefore, this appendix is devoted to a rough sketch of how one can extend the downset-monad to a much wider class. We will first describe this class, and then give an equivalent formulation which allows the extension of the downset-monad to this class. Then we will indicate which results carry over, and which don't.

5.6.1 Admissible Relations

We will now consider $\mathcal{E}$-indexed preorders that have the following two properties:

- they are canonically presentable;
- reindexing along regular epis should reflect the ordering.

As we have seen, the first condition means that the indexed preorder is given as $X \mapsto \mathcal{E}(X, \Sigma)$, for some object Σ. The second condition says, that if $e : X \to Y$ is a regular epi, then for any $\alpha, \beta : Y \to \Sigma$ we have

$$\alpha \circ e \vdash_X \beta \circ e \text{ implies } \alpha \vdash_Y \beta.$$

(This condition is not vacuous; for triposes, for example, it is equivalent to asking that quantification can be taken fibrewise. Over the category of sets it always holds, though.)

Indexed preorders coming from posets, as defined in section 2, are easily seen to satisfy these two requirements, so the class we are considering here is indeed a generalization.

We will now give an alternative description of this class: given $\mathcal{E}(-, \Sigma)$ satisfying these two properties, define a collection $\mathcal{R}$ of relations on Σ by saying that

$$R \in \mathcal{R} \Leftrightarrow \text{ for some } \alpha, \beta : X \to \Sigma, \alpha \vdash_X \beta \;\&\; R = Im\langle\alpha, \beta\rangle.$$

We call the relations in $\mathcal{R}$ *admissible*. It is easily seen that $\mathcal{R}$ is closed under taking subrelations and under composition of relations, and that $\mathcal{R}$ contains the diagonal.

Now suppose that $\alpha \vdash_X \beta$. Factor $\langle\alpha, \beta\rangle$ as

$$\begin{tikzcd} X \arrow[r, two heads, "e"] & Im\langle\alpha, \beta\rangle \arrow[r, tail] & \Sigma \times \Sigma \end{tikzcd}$$

Consider the two projections $\pi_1, \pi_2 : Im\langle\alpha, \beta\rangle \to \Sigma$; since reindexing along e reflects the preorder then we get $\pi_1 \vdash_{Im\langle\alpha,\beta\rangle} \pi_2$ if and only if $\alpha \vdash_X \beta$. Therefore we find $\alpha \vdash_X \beta$ precisely if $Im\langle\alpha, \beta\rangle \in \mathcal{R}$; it follows that the preorder is completely determined by $\mathcal{R}$ and that we may reformulate by saying

$$\alpha \vdash_X \beta \Leftrightarrow \exists R \in \mathcal{R}\, \forall x \in X. R(\alpha(x), \beta(x)).$$

From now on, we may therefore assume that our indexed preorder is given by an object Σ and a collection $\mathcal{R}$ of relations on Σ, such that R contains the diagonal, is closed under subrelations and under composition.

It is not hard to see that if we have a poset Σ, and a class of admissible endofunctions $\mathcal{F}$, that the admissible relations are then given by: $R \in \mathcal{R}$ iff $\exists f \in \mathcal{F}\, \forall (a, b) \in R : f(a) \leq b$.

We define a 2-category $\mathbf{pre}(\mathcal{E})$

- **Objects:** indexed preorders of the above form;
- **Morphisms:** a morphism $\mathcal{E}(-, \Sigma) \to \mathcal{E}(-, \Theta)$ is a map $\phi : \Sigma \to \Theta$ preserving the preorder;
- **2-cells:** for $\phi, \psi : \Sigma \to \Theta$, we have $\phi \leq \psi$ if and only if $\phi \vdash_\Theta \psi$.

Thus we have a full embedding $\mathbf{PRE}(\mathcal{E}) \hookrightarrow \mathbf{pre}(\mathcal{E})$.

The following lemma is a useful characterization of maps in the category $\mathbf{pre}(\mathcal{E})$.

Lemma 5.6.1 *Let $\mathcal{E}(-,\Sigma),\mathcal{E}(-,\Theta)$ be indexed preorders with classes of admissible relations $\mathcal{R}_\Sigma,\mathcal{R}_\Theta$, respectively. Then a map $\phi:\Sigma\to\Theta$ is a morphism in $\mathbf{pre}(\mathcal{E})$ if and only if ϕ preserves admissible relations, i.e. $R\in\mathcal{R}_\Sigma$ implies $\phi[R]=\{(\phi a,\phi b)|(a,b)\in R\}\in\mathcal{R}_\Theta$.*

Proof. Suppose first that ϕ is a morphism in $\mathbf{pre}(\mathcal{E})$, so that composing with ϕ preserves the preorder. Take $R\in\mathcal{R}_\Sigma$. Then the two projections $\pi_1,\pi_2:R\to\Sigma$ satisfy $\pi_1\vdash_R\pi_2$, so we get $\phi\pi_1\vdash_R\phi\pi_2$ (the first entailment takes place in $\mathcal{E}(R,\Sigma)$, the second in $\mathcal{E}(R,\Theta)$). But this means that the image of $\langle\phi\pi_1,\phi\pi_2\rangle$ is admissible for Θ. But $Im\langle\phi\pi_1,\phi\pi_2\rangle=\phi[R]$, and so $\phi[R]$ is admissible, as required.

For the converse, assume that ϕ preserves admissible relations, and consider $\alpha,\beta:X\to\Sigma$ for which $\alpha\vdash_X\beta$. This means that $Im\langle\alpha,\beta\rangle$ is admissible for Σ. Thus $\phi[Im\langle\alpha,\beta\rangle]=\{(\phi\alpha(x),\phi\beta(x))|x\in X\}=Im\langle\phi\alpha,\phi\beta\rangle$ is admissible for Θ. But that says precisely that $\phi\alpha\vdash_X\phi\beta$.

□

5.6.2 Downsets

We can now extend the downset-monad (construction 5.3.1 to this wider class of indexed preorders.

Construction 5.6.2 ($\mathcal{P}$-construction, extended) Starting from an indexed preorder with generic object Σ and admissible relations $\mathcal{R}$, we define a new indexed preorder with generic object $\mathcal{P}\Sigma$, the full powerset of Σ. For $\alpha,\beta:X\to\mathcal{P}\Sigma$, put

$$\alpha\vdash^*_X\beta\Leftrightarrow\text{ for some }R\in\mathcal{R}.\mathcal{E}\models\forall x\in X\,\forall a\in\alpha(x)\exists b\in\beta(x).R(a,b)$$

It is easy to check that this construction indeed gives an indexed preorder, and that the collection of admissible relations on $\mathcal{P}\Sigma$ is given by:

$$K\subseteq\mathcal{P}\Sigma\times\mathcal{P}\Sigma\text{ admissible }\Leftrightarrow$$
$$\text{for some }R\in\mathcal{R}.\mathcal{E}\models\exists R\in\mathcal{R}\,\forall(U,V)\in K\,\forall u\in U\,\exists v\in V:R(u,v).$$

Proposition 5.6.3 *The construction 5.6.2 sending $\mathcal{E}(-,\Sigma)$ to $\mathcal{E}(-,\mathcal{P}\Sigma)$ is object part of a 2-functor $\mathcal{P}:\mathbf{pre}(\mathcal{E})\to\mathbf{pre}(\mathcal{E})$.*

Proof. Suppose that $\phi:\Sigma\to\Theta$ is preorder-preserving, then define $\mathcal{P}\phi$ to be the map sending $U\subseteq\Sigma$ to $\{\phi(u)|u\in U\}$. By lemma 5.6.1 we need only verify that this preserves admissible relations. So take admissible K on $\mathcal{P}\Sigma$. Then $\mathcal{P}\phi[K]=\{(\phi[U],\phi[V])|(U,V)\in K\}$. By the description of admissible relations on $\mathcal{P}\Sigma$, there is some $R\in\mathcal{R}$ such that $(U,V)\in K,u\in U$ implies $\exists v\in V:R(u,v)$. Because $\phi[R]$ is admissible, we see that for all $(\phi[U],\phi[V])\in\mathcal{P}\phi[K],\phi(u)\in\phi[U]$ there is some $v\in V$ with $\phi(v)\in\phi[V]$ and $\phi[R](\phi(u),\phi(v))$. Thus $\mathcal{P}\phi[K]$ is admissible.

The fact that $\mathcal{P}$ also works on 2-cells is easy, because these were defined in terms of 1-cells.

□

Next, the functor $\mathcal{P}$ also carries a 2-monad structure.

Proposition 5.6.4 *The 2-functor* $\mathcal{P} : \mathbf{pre}(\mathcal{E}) \to \mathbf{pre}(\mathcal{E})$ *is part of a 2-monad.*

Proof. The unit is induced by the singleton map $S : \Sigma \to \mathcal{P}\Sigma$. This map is easily seen to preserve admissible relations.

The multiplication comes from the union map $\cup : \mathcal{P}\mathcal{P}\Sigma \to \mathcal{P}\Sigma$. It is a straightforward unwinding of definitions to show that this map preserves admissible relations.

□

Recall that in order to characterize the algebras for the $\mathcal{I}$-monad (proposition 5.3.6), we needed the condition (CF). This condition needs to be extended to have the same effect for preorders from our wider class $\mathbf{pre}(\mathcal{E})$. The formulation becomes (for an indexed preorder $\mathcal{E}(-\Sigma)$ with admissible relations $\mathcal{R}$):

Every admissible relation R contains a partial function $f : \Sigma \rightharpoonup \Sigma$ such that $Dom(R) = dom(f)$ (CF')

Now we can use the same characterization for algebras as for the $\mathcal{I}$-monad:

Proposition 5.6.5 *Let* $\mathcal{E}(-, \Sigma)$ *satisfy (CF'). Then* $\mathcal{E}(-, \Sigma)$ *carries an algebra structure (which is necessarily unique up to isomorphism) if and only if* $\mathcal{E}(-, \Sigma)$ *has left adjoints for reindexing functors, satisfying Beck-Chevalley.*

Proof. Straightforward adaptation of the proof of proposition 5.3.6.

□

Non-empty Version. Just as for the downset-monad, there is a variation, obtained by taking non-empty (or equivalently, inhabited) subsets. We get the same results for this variation: it is also a 2-monad, and the algebras correspond to indexed preorders that have left adjoints for reindexing along regular epis, satisfying Beck-Chevalley.

5.6.3 Triposes?

It would be desirable to have a characterization of when $\mathcal{E}(-, \mathcal{P}\Sigma)$ is a tripos. But I couldn't find it (actually, this was the main reason for restricting to indexed preorders from posets, where we only have admissible functions, not relations).

Let me briefly indicate why this is more difficult: take some preorder $\mathcal{E}(-, \Sigma)$ from the narrow class $\mathbf{PRE}(\mathcal{E})$, and suppose it has implication. Then it is straightforward to define implication on $\mathcal{E}(-, \mathcal{I}\Sigma)$ by putting

$$U \Rightarrow^* V = \bigcap_{u \in U} \bigcup_{v \in V} u \Rightarrow v.$$

It is not even possible, using the same definition, to show that $\mathcal{E}(-,\mathcal{P}\Sigma)$ inherits implication from $\mathcal{E}(-,\Sigma)$. This indicates that a different approach to characterizing when $\mathcal{E}(-,\mathcal{P}\Sigma)$ has implication is needed, and that it is not at all evident, why ordered PCAs would play a role there[2].

---

[2] Maybe it is worth considering a generalization of ordered PCAs where the application is *many-valued*, mimicking the passage from functions to relations.

Chapter 6

Exact Completions and Toposes

In this chapter we collect some results concerning completions and toposes. In 1995, Aurelio Carboni posed the question for which toposes the exact completion is again a topos (see [17]). By Menni's characterization of exact completions that are toposes [56], we know that these are precisely the ones which have a generic proof. This doesn't make the question any easier, though, because it is hard to derive any topos-theoretic properties from the existence of a generic proof.

We were not able to present a solution to Carboni's question, but obtain some related results.

In the first section we collect a few observations about the situation where a topos is an exact completion of another topos.

Next, we characterize those toposes which arise as a coproduct completion of a small category. Not very surprisingly, these turn out to be the atomic toposes.

Then we restrict Carboni's problem to Grothendieck toposes and characterize the Grothendieck toposes that have a Grothendieck topos for their exact completion in terms of the geometric morphism to **Set**.

Finally we give a site characterization of those Grothendieck toposes which arise as an exact completion. This is pretty straightforward, but we didn't find this in the literature.

6.1 Basic Observations

In this section we first present some stability properties. These are quite trivial, but do not seem to have been recorded before. Then we make some observations about the relation between a topos and its exact completion.

6.1.1 Stability Properties

First, we observe that the class of toposes with a generic proof is stable under slicing. (Recall from chapter 2, section 4 that a proof is an element of the poset reflection of the slice of a category, and that a generic proof is a map such that each proof can be represented as a pullback of that map.)

Lemma 6.1.1 *If $\mathcal{E}$ has a generic proof, then so does $\mathcal{E}/E$ for any object E.*

Proof. We indicate the constructions: If $k : X \to Y$ is generic in $\mathcal{E}$ then $k \times E : X \times E \to Y \times E$ is generic in $\mathcal{E}/E$. Then if we have an arrow $t : M \to N$ from $M \to E$ to $p : N \to E$, we find a characteristic map χ_t of t. Now $\langle \chi_t, p\rangle : N \to Y \times E$ is a characteristic map of t in $\mathcal{E}/E$.

□

From this lemma and the proof it is evident that pullback functors preserve generic proofs.

The following is also straightforward:

Lemma 6.1.2 *If $\mathcal{E}$ has a generic proof, then so does any sheaf subtopos of $\mathcal{E}$.*

Proof. If the associated sheaf functor is denoted by a, then it is easy to verify that application of a to the generic proof in $\mathcal{E}$ gives a generic proof in the sheaf subtopos.

□

Via the same idea one also obtains that if $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism with f^* full and faithful (i.e. when f is connected) and $\mathcal{F}$ has a generic proof, then so does $\mathcal{E}$.

Finally, if $\mathcal{E}$ has a generic proof, then so does every filter-quotient of $\mathcal{E}$.

6.1.2 The relation between $\mathcal{E}$ and $\mathcal{E}_{ex}$

We start by observing why $\mathcal{E}$ is a sheaf subtopos of $\mathcal{E}_{ex}$.

Lemma 6.1.3 *Let $\mathcal{C}$ be any exact category. Then $\mathcal{C}$ is a full reflective subcategory of $\mathcal{C}_{ex}$ and the reflector preserves finite limits, i.e. $\mathcal{C}$ is a localization of $\mathcal{C}_{ex}$.*

Proof. This follows from the general theory of KZ-monads.

□

Corollary 6.1.4 *Let $\mathcal{E}$ be a topos with a generic proof. Then $\mathcal{E}$ is a sheaf subtopos of $\mathcal{E}_{ex}$.*

Proof. $\mathcal{E}$ is a localization of a topos, and this is the same as a subtopos.

□

So we have topology on $\mathcal{E}_{ex}$ for which the sheaves are exactly the projective objects. This implies, for instance, that the sheaves are closed under coproducts and that the projectives are an exponential ideal. Moreover, every object can be covered by a sheaf. Also, the topology is dense, since the inclusion preserves the initial object.

As a minor application of this, we have the following lemma:

Lemma 6.1.5 *1. If $\mathcal{E}$ has a generic proof, then $\mathcal{E}_{ex}$ is not Boolean, unless $\mathcal{E} \models AC$.*

2. $\mathcal{E}$ satisfies de Morgan if and only if $\mathcal{E}_{ex}$ does.

Recall that a topos satisfies de Morgan (see [44]) if the de Morgan laws

$$\neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b) \text{ and } \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$$

hold in the internal logic. The second is always valid, but the first is not. There is a wide variety of equivalent conditions, but the one which we will use here is $\Omega_{\neg\neg} \cong 1 + 1$. Recall also that a topos is Boolean if $a \vee \neg a$, the law of the excluded middle is valid; this is equivalent to saying that $\Omega \cong 1 + 1$.

Proof. As for 1), Booleaness means that $\Omega \cong 1 + 1$; the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}_{ex}$ preserves the terminal object and all existing sums, so the subobject classifier of $\mathcal{E}_{ex}$ is in the image of the inclusion. But then $\mathcal{E} \simeq \mathcal{E}_{ex}$, and hence the axiom of choice must hold.

For 2), de Morgan means that $\Omega_{\neg\neg} \cong 1 + 1$. Now $\mathcal{E}_{\neg\neg} \simeq (\mathcal{E}_{ex})_{\neg\neg}$, because $y : \mathcal{E} \to \mathcal{E}_{ex}$ is dense. Thus if $\Omega_{\neg\neg} \cong 1 + 1$ holds in one of the toposes, then also in the other, because both the inclusion i and the associated sheaf functor preserve coproducts and $\Omega_{\neg\neg}$.

□

Finally, we mention without proof that if $\mathcal{F} \to \mathcal{E}$ is a geometric inclusion and $\mathcal{E}$ has a generic proof, then there is a pullback square of toposes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_{ex} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}_{ex}
\end{tikzcd}$$

Open Subtoposes and Internal Choice. In some cases, $\mathcal{E}$ is an open subtopos of $\mathcal{E}_{ex}$. Take for example $\mathcal{E}$ to be the topos of G-Sets for some group G. Then if we denote the full subcategory of category of transitive G-sets by $\mathcal{G}$, then there is an inclusion of G (seen as a one-object category) into $\mathcal{G}$, which induces the geometric inclusion $\mathbf{Sets}^{G^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{G}^{\mathrm{op}}} \simeq (\mathbf{Sets}^{G^{\mathrm{op}}})_{ex}$. This geometric morphism is essential, and the inverse image functor is logical, because G is a subterminal object in $\mathbf{Sets}^{\mathcal{G}^{\mathrm{op}}}$ (see [44] for details). Therefore, the inclusion is open and $\mathbf{Sets}^{G^{\mathrm{op}}} \simeq \mathbf{Sets}^{\mathcal{G}^{\mathrm{op}}}/G$.

If is is the case that $\mathcal{E}$ is an open subtopos of $\mathcal{E}_{ex}$, then $\mathcal{E} \models IAC$. To see this, we need the following lemma:

Lemma 6.1.6 *In $\mathcal{E}_{ex}$, the projective and internally projective objects coincide.*

Proof. Since the projectives in $\mathcal{E}_{ex}$ are closed under finite limits, in particular the terminal object is projective; using this, it is immediate that internal projectivity implies projectivity. For the converse, let P be projective, and consider an epi $e : X \to Y$, and any $f : T \times P \to Y$. Cover T by a projective T', and since $T' \times P$ is projective, we find the required $T' \times P \to X$.

□

Now if the associated sheaf functor $a : \mathcal{E}_{ex} \to \mathcal{E}$ is a pullback functor, then it preserves internally projective objects, and therefore $\mathcal{E} \models IAC$.

We conclude that $\mathcal{E}$ is not always an open subtopos of $\mathcal{E}_{ex}$, examples being provided by continuous G-Sets, where internal choice may fail.

6.1.3 Regular Completions and Toposes?

For sake of completeness, we remark that the question when a *regular* completion is a topos has a fairly simple answer:

Proposition 6.1.7 *For any regular category $\mathcal{E}$, the regular completion $\mathcal{E}_{reg}$ is a topos if and only if $\mathcal{E}$ is already a topos satisfying the axiom of choice (and thus $\mathcal{E} \simeq \mathcal{E}_{reg}$).*

Proof. The embedding $\mathcal{E} \hookrightarrow \mathcal{E}_{reg}$ preserves epimorphisms. So, if $\mathcal{E}_{reg}$ is balanced, then the embedding would preserve all regular structure, and therefore be an equivalence. But any topos is balanced.

□

6.2 Coproduct Completions and Toposes

In this section we investigate the relationship between the exact completion and the coproduct completion. In particular, we show that if some small category $\mathcal{C}$ has the property that $\mathcal{C}_+$ is a topos, then this topos is *atomic*. The exact completion of this topos is then the presheaf topos $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Before we get into this, we recall some definitions. First of all, a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is called *locally connected* if p^* has an $\mathcal{S}$-indexed left adjoint. For $\mathcal{S} = \mathbf{Set}$, this simply means that p^* has a left adjoint, of which we think as taking connected components. Equivalently, a Grothendieck topos is locally connected if every object can be decomposed as a sum of connected objects.

A stronger property is atomicity; a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is called *atomic* if p^* is a logical functor. For Grothendieck toposes, this amounts to saying that each object of $\mathcal{E}$ has a decomposition as a sum of *atoms*, i.e. objects without non-trivial subobjects. (For more on local connectedness or atomicity, see [44, 7, 8].)

6.2.1 Toposes that are coproduct completions

Menni [57] has characterized all presheaf toposes which have generic proofs:

Theorem 6.2.1 (Menni) *Let $\mathcal{D}$ be a small category. Then the following are equivalent:*

1. $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ *has a generic proof;*
2. $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ *is boolean;*
3. $\mathcal{D}$ *is a groupoid;*
4. $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ *is a coproduct completion of a small category.*

So in the case of presheaves, there is a generic proof precisely when the topos is a coproduct completion. We will not rehearse the proof, but we observe that the implication from (4) to (1) follows from a more general fact, also proved in [57], that *any* category which is of the form $\mathcal{C}_+$ for small $\mathcal{C}$ has a generic proof. In fact, in the proof of this proposition the full strength of the hypothesis (namely that $\mathcal{E}$ is the *free* coproduct completion of $\mathcal{C}$) is not fully used. One has the stronger

Proposition 6.2.2 *Let $\mathcal{E}$ be a category that has all coproducts, and that has a* set *of objects $\mathcal{C}$, such that every object of $\mathcal{E}$ can be written as a sum of objects from $\mathcal{C}$. Then $\mathcal{E}$ has a generic proof.*

We will exploit this in the next section on strong bounds.

The only examples of toposes with a generic proof that we know of either satisfy the axiom of choice or are coproduct completions of a small category. Those satisfying the axiom of choice are not particularly interesting for our purposes, since they coincide with their exact completion. So we concentrate on the coproduct completions and characterize toposes that arise in that way.

Theorem 6.2.3 *Let $\mathcal{C}$ be a small category. Then the following are equivalent:*

1. $\mathcal{C}_+$ *is a topos;*
2. $\mathcal{C}_+$ *is an atomic topos;*
3. $\mathcal{C}_+ \simeq \mathbf{Sh}(\mathcal{C}, \neg\neg)$*;*
4. $\mathcal{C}$ *is an atomic category.*

Proof. The implications (2) ⇒ (1) and (3) ⇒ (1) are trivial. (2) ⇔ (3) and (2) ⇔ (4) are well-known from the theory of atomic toposes and sites. So we only have to prove the implication (1) ⇒ (2).

The first observation is that if $\mathcal{C}_+$ is a topos, then it must be a Grothendieck topos, because it is cocomplete and generated by objects of $\mathcal{C}$, which was assumed to be small. Moreover, because every object of the topos can be written as a sum of connected objects, the topos is locally connected. To show that the topos is atomic, it suffices to show that the connected objects of $\mathcal{C}_+$ are precisely the atoms. Hence we need to show that any monomorphism in $\mathcal{C}$ is in fact an isomorphism. This is done in the following lemma:

Lemma 6.2.4 *Let* $u : A \to B$ *be an monomorphism in* $\mathcal{C}$. *Then* u *is an isomorphism.*

Proof. For any set I, we can define P_I to be the colimit of the diagram (in the topos $\mathcal{C}_+$)

$$\begin{tikzcd}
& & A \arrow[lld, "u"'] \arrow[ld, "u"] \arrow[rd, "u"] & \\
B & B \arrow[rr, dotted, no head] & & B
\end{tikzcd}$$

where we take I copies of the arrow u. Hence for $I = 2$ we get an ordinary pushout of u along itself, and for $I > 2$ we get a so-called wide pushout. Now the rest of the proof consists of two claims; first, that for each set I, the newly formed object P_I is again connected. And next, that if two sets I and J have different cardinality, then P_I and P_J cannot be isomorphic. Then we conclude that there is a proper class of connected objects, which contradicts the smallness of $\mathcal{C}$.

So first we show that P_I is connected. This is equivalent to $Hom(P_I, -)$ preserving sums. Now

$$\begin{aligned}
Hom(P_I, (X_m)_{m\in M}) &\cong \{(f_i : B \to (X_m)_{m\in M})_{i\in I}) \mid f_i u = f_j u \text{ for } i, j \in I\} \\
&\cong \coprod_{m\in M} \{(f_i : B \to X_m)_{i\in I} \mid f_i u = f_j u \text{ for } i, j \in I\} \\
&\cong \coprod_{m\in M} Hom(P_I, X_m),
\end{aligned}$$

the first isomorphism coming from the fact that P_I is a colimit, the second because B is connected and because two maps from B to $(X_m)_{m\in M}$ can only be equal if they land on the same X_m, and the last again by virtue of P_I being a colimit.

Next we show that if I is a proper subset of J, then $P_I \ncong P_J$. It is sufficient to present an object C such that $Hom(P_I, C) \ncong Hom(P_J, C)$. To find such a C, take an equalizer diagram in $\mathcal{C}_+$ into which u fits:

$$\begin{tikzcd}
A \arrow[r, tail, "u"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (C_k)_{k\in K}
\end{tikzcd}$$

The two composites fu and gu can only be equal if there is a $k \in K$ such that

$$\begin{tikzcd}
A \arrow[r, tail, "u"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C_k
\end{tikzcd}$$

commutes (and is in fact an equalizer in $\mathcal{C}$). If u is not an isomorphism, then f and g are different arrows. Put $C := C_k$.

Now we define an equivalence relation on $Hom(B, C)$ by putting $s \sim t$ if and only if $su = tu$. Let V be an indexing for the equivalence classes and write $Hom(B, C) = \coprod_{v \in V} W_v$. Note that every family $(h_i : B \to C)_{i \in I}$ that constitutes a map $h : P_I \to C$ has the property that $h_i \sim h_{i'}$ for all $i, i' \in I$. Therefore, any such map $h : P_I \to C$ determines, and is itself determined by, an element $v \in V$ and a function $I \to W_v$ and thus we find that

$$|Hom(P_I, C)| = |\coprod_{v \in V} W_v^I|.$$

But since f and g are not equal but have $f \sim g$ there must be at least one $v \in V$ such that $|W_v| > 1$. Therefore we conclude that if $|I| \leq |J|$ then $|Hom(P_I, C)| < |Hom(P_J, C)|$, and hence $P_I \ncong P_J$.

□

We have established that any topos that is a coproduct completion of a small category $\mathcal{C}$ must be atomic, in which case the exact completion is the presheaf topos $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

6.3 Strong Bounds

In this section we consider a slightly stronger property than that of having a generic proof. The reason for this is, that generic proofs do not have a particularly strong topos-theoretic flavor, and are therefore hard to exploit. The stronger question that we will ask is: when is the exact completion of a topos a Grothendieck topos?

6.3.1 Strong Bounds and Exact Completions

Recall first that a *bound* for a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is an object B of $\mathcal{E}$ such that each object X of $\mathcal{E}$ fits into a diagram of the form

$$\begin{tikzcd}
Y \arrow[r, tail] \arrow[d, two heads] & p^*I \times B \\
X &
\end{tikzcd}$$

for some object I in $\mathcal{S}$. We say that X is a *subquotient* of $p^*I \times B$. If $p : \mathcal{E} \to \mathcal{S}$, has a bound (which is far from unique) then p is called *bounded*, and in case $\mathcal{S} = \mathbf{Set}$, this is the same as saying that $\mathcal{E}$ is a Grothendieck topos. For more on these matters, see [44].

Definition 6.3.1 Let $p : \mathcal{E} \to \mathcal{S}$ be an $\mathcal{S}$-topos. A *strong bound* for p is an object B of $\mathcal{E}$ such that every object X of $\mathcal{E}$ is a subobject of an object of the form $p^*I \times B$. We call $\mathcal{E}$ a *strongly bounded* $\mathcal{S}$-topos.

If $\mathcal{S} = \mathbf{Sets}$, then the above condition means that $\mathcal{E}$ is Grothendieck (for a strong bound is evidently a bound, since we have replaced "subquotient" by "subobject" in the definition of a bound), and that there exists a subcanonical site $(\mathcal{C}, J)$ for $\mathcal{E}$, such that every object of $\mathcal{E}$ is a sum of representables (i.e. objects of the form $ay(C)$, where y is Yoneda, and a is the associated sheaf functor. To see why this is so, consider first the case that such a site exists. Then the sum of objects of the form $ay(C)$ is a strong bound for $\mathcal{E}$ over **Sets**. Conversely, given a strong bound B, we may build a subcanonical site by taking the full subcategory on all subobjects of B.

By the strengthening of Menni's result on coproduct completions 6.2.2, we may conclude that a topos which is strongly bounded over **Sets** has a generic proof. But in fact, we may draw a stronger conclusion, namely that its exact completion is a Grothendieck topos. For this, we use the following lemma (which can also be found in [21]):

Lemma 6.3.2 *Let $\mathcal{C}$ be a category with finite limits and small, stable, disjoint coproducts. Then $\mathcal{C}_{ex}$ also has small coproducts, and the embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{ex}$ preserves them.*

Proof. This is straightforward but tedious; the coproduct of a family of pseudo-equivalence relations $(\begin{tikzcd} R_i \arrow[r, shift left, "r_{i0}"] \arrow[r, shift right, "r_{i1}"'] & X_i \end{tikzcd})_{i \in I}$ must be constructed as the object $\begin{tikzcd} \coprod_{i \in I} R_i \arrow[r, shift left] \arrow[r, shift right] & \coprod_{i \in I} X_i \end{tikzcd}$. The structure maps for reflexivity, symmetry and transitivity are now the sums of those for the R_i. And the universal property is also uncomplicated.

□

Using this lemma, we see that $\mathcal{E}_{ex}$ inherits cocompleteness from $\mathcal{E}$ and therefore we only need to find a bound for $\mathcal{E}_{ex}$. But every object of $\mathcal{E}_{ex}$ can be covered by an object from $\mathcal{E}$ (since it is an exact completion), and every such cover is a sum of objects from the site $(\mathcal{C}, J)$. Hence every object of $\mathcal{E}_{ex}$ can be covered with a sum of $\mathcal{C}$-objects, and there is only a set of those. So the strong bound of $\mathcal{E}$ becomes a bound of $\mathcal{E}_{ex}$.

We also have a converse: if $\mathcal{E}_{ex}$ has a bound, then $\mathcal{E}$ has a strong bound. For let B be a bound for $\mathcal{E}_{ex}$. For every subobject of $B_i \hookrightarrow B$, we may choose a projective cover P_i. Then every object in the image of $i : \mathcal{E} \to \mathcal{E}_{ex}$ can be covered by a sum of objects of the form P_i, as in the diagram

$$\coprod_{i \in I} P_i \twoheadrightarrow iE$$

But, coming from $\mathcal{E}$, E is projective, so this cover splits, exhibiting $E \cong \coprod_{i \in I} C_i$, with all $C_i \subseteq P_i$. But this says precisely that the sum of all P_i's is a strong bound for $\mathcal{E}$.

We summarize all of the above considerations in the following theorem:

Theorem 6.3.3 *Let $\mathcal{E}$ be a Grothendieck topos. Then the following are equivalent:*

- *$\mathcal{E}$ has a strong bound;*
- *There is a site for $\mathcal{E}$ such that each object of $\mathcal{E}$ can be written as a sum of representables;*
- *$\mathcal{E}_{ex}$ is a Grothendieck topos.*

To close this section, we make a few trivial observations: first, the notion of a strongly bounded $\mathcal{S}$-topos is stable under slicing and taking products/coproducts of toposes. In general it is not stable under change of base (note that we don't have a "relative" notion of a generic proof), but it is stable under change of base along inclusions and open surjections. Also:

Lemma 6.3.4 *Let a commutative triangle of geometric morphisms be given:*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "p"] \arrow[dr, "r"'] & \mathcal{F} \arrow[d, "q"] \\
 & \mathcal{G}.
\end{tikzcd}$$

If p and q are strongly bounded, so is r. If r is strongly bounded, so is p.

Proof. It is easy to see that $B \times p^*C$ is a strong bound for r if B and C are strong bounds for p and q, respectively. And any strong bound for r is also a strong bound for p.

□

We also remark that if $p : \mathcal{E} \to \mathcal{S}$ is strongly bounded, then not every bound is automatically a strong one. Consider a group G and the topos $\mathbf{Sets}^{G^{\mathrm{op}}}$. This topos is strongly bounded, for the sum of all transitive G-sets is a strong bound. The object G itself is a bound, but not a strong one.

Finally, there is a simple reformulation of the axiom of choice in terms of strong bounds:

Proposition 6.3.5 *Let $\mathcal{E}$ be a Grothendieck topos. Then the following are equivalent:*

1. *$\mathcal{E} \models AC$*
2. *1 is a strong bound.*

Proof. First of all, if $\mathcal{E} \models AC$, then any bound is a strong bound. Moreover, Grothendieck toposes satisfying choice are precisely those of the form sheaves over a complete Boolean algebra (see [44]). So, in particular they are localic, i.e. 1 is a bound, and hence also a strong bound.

Conversely, if 1 is a strong bound, then $\mathcal{E}$ is a localic topos in which every object is a subobject of a constant object. But this condition is known (see [10]) to be equivalent to the axiom of choice. (This can be seen by first noticing that all objects, being subobjects of constant (hence decidable) objects, are decidable;

then if every object is decidable, the topos is Boolean. But a Boolean localic topos satisfies AC.)

□

6.3.2 Atomicity

Now we make some connections with better-known topos theoretic properties such as atomicity and local connectedness.

First of all, any atomic topos is strongly bounded, since every object is a sum of atoms, and there is only a set of atoms. Next, we observe that although every atomic topos is locally connected, not every strongly bounded topos is: as a counterexample we may take a non-atomic complete Boolean algebra; the topos of sheaves over such an algebra is not atomic (but satisfies AC). Thirdly, the conjunction of local connectedness and strong boundedness gives atomicity: for, given a strong bound B in a locally connected topos we may consider the set of connected subobjects of B. These include all connected objects, for if C is connected, it can be written as a subobject of a sum of subobjects of B, say as $C \to \coprod_{i \in I} B_i$. But because C is connected this inclusion has to factor through one of the coproduct inclusions $B_i \to \coprod_{i \in I} B_i$. Moreover, since every object may be decomposed as a sum of connected objects, we see that the connected objects form a strong bound. But if, in a cocomplete category, every object is a sum of connected objects and if there is only a set of connected objects, then that category is the free coproduct completion of its full subcategory on the connected objects. And we have already seen that toposes of the form $\mathcal{C}_+$ are atomic.

Theorem 6.3.6 *For a Grothendieck topos $\mathcal{E}$, the following are equivalent:*

1. *$\mathcal{E}$ is atomic;*
2. *$\mathcal{E}$ is locally connected and has a strong bound.*

Corollary 6.3.7 *For a presheaf topos $\mathcal{E}$, the following are equivalent:*

1. *$\mathcal{E}$ is atomic;*
2. *$\mathcal{E}$ has a strong bound.*

Proof. Presheaf toposes are locally connected.

□

Note that both Menni's proof of theorem 6.2.1 and our proof of lemma 6.2.4 make an appeal to a distinction between sets and classes. It seems that in order to extract information from generic proofs or strong bounds, one needs to use cardinality arguments.

We close this section with a list of open questions.

1. First of all, we have seen that "locally connected + strong bound = atomic". On the other hand, we have "locally connected + Boolean = atomic". Thus "locally connected + strong bound = locally connected + Boolean". Can we drop local connectedness and show that every topos with a strong bound is Boolean?
2. A second question concerns the comparison between generic proofs and strong bounds. Does there exist a Grothendieck topos with a generic proof but without a strong bound?
3. I would like to see an example of a Grothendieck topos which is not a coproduct completion, but which is strongly bounded.
4. We still do not know what to do in the elementary case, such as for the Effective Topos.

6.4 Sites and Exact Completions

We investigate when a topos of the form $\mathbf{Sh}(\mathcal{C}, J)$, is an exact completion in the weak sense. As a special case, we retrieve a characterization of those locales for which the associated sheaf topos is an exact completion, a result which is widely known; the key ingredient to the proof is in the book "Categories, Allegories", by Freyd and Scedrov.

Fix a site $(\mathcal{C}, J)$. For convenience, we assume that the site is subcanonical, although this makes no difference for the results. We will denote the associated sheaf functor by a, and the Yoneda embedding by $y : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$.

Definition 6.4.1 We call an element $C \in \mathcal{C}$ *trivial* if C is J-covered by the empty sieve. Also, we call $C \in \mathcal{C}$ *projective* if $y(C)$ is a projective object in the topos $\mathbf{Sh}(\mathcal{C}, J)$.

The first step is to characterize which objects in the site $(\mathcal{C}, J)$ are projective. To this end, we need to make clear what we mean by saying that a sieve R on an object C is a partition. First of all, two sieves M, N on C are disjoint (as subobjects) precisely when $f : X \to C \in M \cap N$ implies that X is trivial. Now if we write $R = \{f_i : D_i \to C | i \in I\}$, then we say that R is a partition if there is a subset $I' \subseteq I$ with the property that $R' = \{f_i : D_i \to C | i \in I'\}$ generates R and for each $i \neq j \in I'$, the prinicpal sieves (f_i) and (f_j) are disjoint. This definition is a straightforward generalization of a partition of a space; the trivial object is the empty set, in that case.

Lemma 6.4.2 *For any $C \in \mathcal{C}$, C is projective if and only if every J-covering sieve R of C can be refined to a partition.*

Proof. Take C to be projective, and take a covering sieve $R = \{f_i : D_i \to C | i \in I\}$. This gives a surjection $\coprod_{i \in I} y(D_i) \twoheadrightarrow y(C)$ in $\mathbf{Sh}(\mathcal{C}, J)$, which splits because $y(C)$ is projective. This splitting induces a decomposition $y(C) \cong$

$\coprod_{i \in I} R_i$, with all R_i a closed sieve on D_i. Consider I', consisting of those $i \in I$ for which R_i is not the initial object. Then the family $(f_i)_{i \in I'}$ is a partition and it refines $(f_i)_{i \in I}$.

Conversely, if any covering sieve of C refines to a partition, we may consider a surjection $e : F \twoheadrightarrow y(C)$ in $\mathbf{Sh}(\mathcal{C}, J)$, where F is any sheaf. Because the representables generate $\mathbf{Sh}(\mathcal{C}, J)$, we can cover F with a coproduct of representables, like $\coprod_{i \in I} y(D_i) \twoheadrightarrow F \twoheadrightarrow y(C)$, so that C is covered by the sieve generated by $\{D_i \to C | i \in I\}$. Refine this as $\{D_i \to C | i \in I'\}$ for suitable $I' \subseteq I$, so that $y(C) \cong \coprod_{i \in I} y(D_i)$. But then there is a map $y(C) \cong \coprod_{i \in I'} y(D_i) \to \coprod_{i \in I} y(D_i)$, and composing this map with $\coprod_{i \in I} y(D_i) \twoheadrightarrow F$ gives a splitting for e.

□

For a locale H, an element $x \in H$ is projective (considered as a subterminal in $\mathbf{Sh}(H)$), if and only if every cover of x can be refined to a partition (in the ordinary sense of partition). It is this characterization which is mentioned in [27], and that we have used for our generalization.

Lemma 6.4.3 *Let F be an object of $\mathbf{Sh}(\mathcal{C}, J)$. Then F is projective if and only if it is a sum of retracts of projective representables.*

Proof. One direction is trivial, since retracts of projectives are again projective, and since projective objects are closed under sums. For the other direction, take F projective and note that F can be covered by a sum of representables. This cover splits, which gives a decomposition of F in retracts of representables. All these are projective, being complemented subobjects of a projective.

□

Lemma 6.4.4 $\mathbf{Sh}(\mathcal{C}, J)$ *has enough projectives if and only if every $C \in \mathcal{C}$ can be covered by a sieve R which is generated by $\{f_i : D_i \to C | i \in I\}$, where the D_i are projective.*

Proof. Suppose $\mathbf{Sh}(\mathcal{C})$ has enough projectives, and take $C \in \mathcal{C}$. Cover $y(C)$ with a projective, which, by the previous lemma, may be taken to be a sum of retracts of projective representables. Thus, $y(C)$ can be covered by a sum of projective representables, say as $\coprod_{i \in I} y(C_i) \twoheadrightarrow y(C)$. But this means precisely that C is covered by the sieve generated by the set $\{C_i \to C | i \in I\}$.

For the converse consider an object F of $\mathbf{Sh}(\mathcal{C}, J)$, and cover it with a sum of representables, as $\coprod_{i \in I} y(C_i) \twoheadrightarrow F$. Every C_i can be covered by a sieve generated by $R_i = \{f_{ij} : D_{ij} \to C_i | j \in J_i\}$ where all D_{ij} are projective, so we get a cover $\coprod_{i \in I, j \in J_i} y(D_{ij}) \twoheadrightarrow \coprod_{i \in I} y(C_i) \twoheadrightarrow F$, with projective domain.

□

In the situation of lemma 6.4.4, we say that the site has *enough projectives.* Now we can formulate the characterization of Grothendieck toposes which are an exact completion as follows.

Proposition 6.4.5 $\mathbf{Sh}(\mathcal{C}, J)$ *is an exact completion in the weak sense if and only if the site $(\mathcal{C}, J)$ has enough projectives*

Proof. This is immediate by the characterization of weak exact completions (theorem 2.2.6) and the previous lemma.

□

If we apply this to locales, we get that $\mathbf{Sh}(H)$ is an exact completion if and only if every element $x \in H$ can be covered by a family $(x_i)_{i\in I}$ where all x_i have the property that any cover can be refined to a partition.

Let us consider two extreme examples: for the first one, take H to be a complete Boolean algebra. Then every object is projective, so the condition of the proposition is trivially valid, and $\mathbf{Sh}(H)$ is an exact completion (namely of itself).

For the second example, take a poset P, and put $H = DP$, the frame of all downsets in P. Then it is known that $\mathbf{Sh}(H)$ is equivalent to $\mathbf{Sets}^{P^{\mathrm{op}}}$, which is an exact completion of the category P_+ (always in the weak limit sense, and also in the strong limit sense if P is actually a meet-semilattice). Note that the projectives are actually join-irreducible elements in this case.

In general, however we don't find $\mathbf{Sh}(H) \simeq (P_+)_{ex}$. In fact, this is equivalent to saying that the induced covering system on the poset (see [43] for an explanation) is trivial. In other words, if there are non-trivial covers, then the full subcategory of $\mathbf{Sh}(H)$ on the projectives is not the *free* coproduct completion of P. Rather, it is the category obtained by adjoining coproducts to P, while respecting those that happened to exist in P.

Chapter 7

Conclusions

In this chapter, I will give a brief overview of the main achievements of this thesis and sketch some possibilities for future research.

7.1 Main results

The central issue that is addressed in this thesis is the presentation of realizability toposes. There are basically two ways of approaching this matter, namely via the theory of exact completions and via tripos theory.

7.1.1 Ordered PCAs

The first contribution consists of a systematic analysis of the construction of a realizability topos out of a partial combinatory algebra (PCA). It turns out that we get a good picture of this, when we consider a relaxation of PCAs, namely *ordered partial combinatory algebras*. Given an ordered PCA $\mathbb{A}$, the standard construction of a tripos and hence of a realizability topos $\mathbf{RT}[\mathbb{A}]$ goes through without significant modifications. We propose a category **OPCA** of ordered PCAs, and show that there is a monad T on this category. This monad takes an ordered PCA $\mathbb{A}$, and forms a new ordered PCA $T\mathbb{A}$ which consists of the non-empty downward closed subsets of $\mathbb{A}$. The central result is now, that the Kleisli category for this monad is dual to the category of realizability toposes and geometric morphisms respecting the inclusion of **Set**. More concretely, if $\mathbb{A}$ and $\mathbb{B}$ are ordered PCAs, then a geometric morphism from $\mathbf{RT}[\mathbb{A}]$ to $\mathbf{RT}[\mathbb{B}]$ is the same as a morphism $\mathbb{B} \to T\mathbb{A}$ of ordered PCAs.

Using the monad T, we also get characterizations of properties of the categories of Partitioned Assemblies and Assemblies: for an ordered PCA $\mathbb{A}$, $\mathbf{PAss}(\mathbb{A})$ is a regular category if and only if $\mathbb{A}$ is a T-algebra, and $\mathbf{PAss}(\mathbb{A})$ is a regular completion if and only if $\mathbb{A}$ is a free T-algebra.

Finally, we have analysed the hierarchies of realizability toposes that arise from iterated application of T to an ordered PCA. Using the fact that these consist of a chain of localic extensions, we have made some first steps towards a possible colimit of such a chain.

7.1.2 Relative Completions

As we have argued, the classical result that the realizability topos is the exact completion of partitioned assemblies is a bit too classical, because it essentially depends on the axiom of choice. Therefore it breaks down when we replace the base category **Set** by an arbitrary base topos. The repairment that we come up with, consists of introducing the notion of a *relative completion.* The relative regular completion is an operation on a category $\mathcal{C}$ equipped with a functor $\mathcal{E} \to \mathcal{C}$, rather than on a category alone. The construction is performed in two steps, by first forming $\mathcal{C}_{reg}$, the free regular completion of $\mathcal{C}$, and then formally inverting all those maps which prevent the composite $\mathcal{E} \to \mathcal{C} \to \mathcal{C}_{reg}$ from being a regular functor. Similarly, we define the relative exact completion.

The key result is, that the category of assemblies is the relative regular completion of the category of partitioned assemblies, and that the realizability topos is the relative exact completion of partitioned assemblies. These results generalize the classical results to realizability over an arbitrary base topos.

In order to establish these results, we have made a connection between Menni's concept of a topology on a category and our relative completion. In many cases, this enables a better comparison between the ordinary completion and the relative version.

7.1.3 Analysis of Realizability Triposes

Our contribution to the theory of realizability triposes consists of the use of completions of indexed preorders for the analysis of triposes. The idea is describe the tripos for a PCA by means of a free construction on a simpler kind of indexed preorder.

The main results are the following: first, that on a certain class of indexed preorders, there is a monad which generalizes the non-empty downset monad on ordered PCAs and on locales. Second, that a free algebra for this monad is a tripos precisely when the original indexed preorder carries an ordered PCA structure, together with a filter of truth-values. This generalizes a well-known theorem from Carboni, Freyd and Scedrov. And third, that we may iterate this construction to obtain new realizability triposes. It must be remarked, that surprisingly many triposes, such as for modified realizability, fit into this framework.

7.1.4 Completions and Toposes

The results in chapter 6 are a bit different in style; all of them are related to the question when the exact completion of a topos is again a topos. First, we show that the toposes which arise as a coproduct completion of a small category are precisely the atomic toposes. Then, we introduce the notion of a strong bound, which strengthens the usual notion of a bound for a geometric morphism. Any Grothendieck topos with a strong bound has a generic proof, and we show that the Grothendieck toposes which have a strong bound are exactly those which have a Grothendieck topos for their exact completion. We also show that if such a topos is locally connected, then it is atomic. Finally, we give a site characterization of those Grothendieck toposes which are exact completions.

7.2 Future work

There are various loose ends in the work presented in this thesis, and many questions are left open. I describe the ones which I think are most interesting and deserve further investigation.

7.2.1 Ordered PCAs

First, there are some questions about (ordered) PCAs. The category in which we have organized them is certainly suitable for realizability purposes, but is still not very well understood. In particular, it would be good to know some closure properties (such as weak limits or -colimits). By our characterization of geometric morphisms of realizability toposes, this would help us understand the 2-categorical properties of the category of realizability toposes. It would be even better to have some kind of representation theorems for ordered PCAs, like we have in group theory.

Next, the hierarchies of realizability toposes and their colimits should be investigated from a more logical point of view. Can we obtain a more direct description of the colimit and describe the logic of its natural number object?

7.2.2 Relative Completions

The central question here is: how do we characterize those categories $\mathcal{C}$, equipped with a functor $\mathcal{E} \to \mathcal{C}$ which are of the form $\mathcal{D}_{\mathcal{E}/reg}$?

We have seen that subcategories with certain covering properties play a key role (these were called *minimal covers*). This gives rise to the following questions. A category may admit many minimal covers; how are these related? Are there sufficient conditions for a category to have a non-trivial minimal cover?

Given a category $\mathcal{D}$, we may consider the poset of all full subcategories of $\mathcal{D}$, ordered by inclusion. On the other hand, we may consider the poset of collections of regular epimorphisms in $\mathcal{D}$, also ordered by inclusion. As explained, there is a Galois connection between these posets; for a subcategory $\mathcal{C}$, consider the collection of regular epimorphisms with respect to which every object of $\mathcal{C}$ is projective. On the other hand, given a class of regular epimorphisms, one can associate the subcategory on those objects which are projective with respect to all of those epimorphisms. There are several questions that one could ask here, but it would already be interesting to know what characterizes the fixed points of this operation.

Finally, for the ordinary completions one obtains results which relate properties of the completion to weak versions of those properties of the original category. It is not obvious how one can tackle such problems in the relative case. As I have indicated, in case the relative completion is a reflective subcategory of the ordinary completion, some results carry over, but in general it is not clear what can be done here.

7.2.3 What is Realizability?

There is a vast family of syntactical realizability notions, and just as vast a family of realizability triposes and toposes. The question of what counts as a realizability notion is therefore a legitimite one. A priori, it is not obvious why there should be a clear-cut answer to it; for all we know, we might very well be in the kind of situation that one often encounters in analytical philosophy, where it is impossible to define a certain concept, and where all purported examples of the concept are only tied together by a form of family resemblance. But, since we are doing mathematics and not philosphy, and since many researchers in the field have expressed a strong intuition that there should be a common core to all realizability notions, it is certainly worth pursueing this quest for the essence of realizability.

Now we have seen that many notions of realizability, such as modified realizability, extensional realizability or the dialectica interpretation, may be comprised by the notion of an ordered PCA together with a filter of truth values. We have also seen that we can give a precise meaning to the intuition that sheaves over a locale is not realizability by observing that in that case, the filter of truth-values is trivial. We may try to formulate an answer to the question: "What is realizability?" using this framework. What I envisage here is a formulation of a class of "basic realizabilities", arising from ordered PCAs and non-trivial filters, and some operations on this class, such as extensionalizing, taking subtriposes, or glueing. Hopefully, something along these lines might bring some structure into the "stamp collection"[1] of realizability notions.

7.2.4 Generic Proofs and Strong Bounds

Carboni's problem of characterizing those toposes of which the exact completion is again a topos is still open. We have shown that the corresponding problem for Grothendieck toposes has an answer in terms of strong bounds, but the relation between strong bounds and other, better-known topos-theoretical properties remains to be investigated. Is any strongly bounded topos Boolean, for example?

Then, we know that a topos with a strong bound has a generic proof. But is the converse also true? And finally, what can one say about strongly bounded geometric morphisms in general? Is there a connection with (some suitably indexed and relativised version of) the exact completion?

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[1] This qualification is P.T. Johnstone's, but I'm afraid it is essentially correct.

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Samenvatting

Realizeerbaarheid. Realizeerbaarheid is een interpretatie van een theorie over natuurlijke getallen, Heyting rekenkunde genaamd. Deze theorie is constructief, hetgeen betekent dat in het afleiden van stellingen geen gebruik gemaakt kan worden van het klassieke principe van de uitgesloten derde. Hierdoor is het klassieke modelbegrip niet geschikt als semantiek voor een dergelijke theorie. Zeer schetsmatig gezien werkt realizeerbaarheid als volgt: men definieert een relatie tussen natuurlijke getallen en zinnen in de taal van de rekenkunde, "n realizeert ϕ". Dit wordt gedaan met inductie naar de structuur van de zin ϕ. Beschouw als voorbeeld de clausule voor implicatie: als ϕ van de vorm $\psi \to \chi$ is, dan geldt "n realizeert $\chi \to \psi$" wanneer n een partieel recursieve functie codeert, die, toegepast op een m waarvoor "m realizeert ψ" geldt, een getal $n \bullet m$ geeft zodat "$n \bullet m$ realizeert χ".

Sinds S.C. Kleene in 1945 deze definitie presenteerde, zijn er vele variaties op ontstaan. De belangrijkste toepassingen liggen in de bewijstheorie, waar realizeerbaarheid bijvoorbeeld gebruikt wordt om consistentieresultaten te bewijzen.

Categorische Logica. Een ander soort semantiek voor constructieve theorieën is de categorische logica. Het cruciale idee hier is, dat men kan redeneren over de objecten en pijlen van een categorie, als waren deze verzamelingen en functies, zolang dat redeneren maar constructief geschiedt, en men geen gebruik maakt van het keuze-axioma.

Zeer kort samengevat werkt de categorische logica als volgt: gegeven een categorie $\mathcal{C}$, beschouwt men voor een object X de verzameling van subobjecten van X, genoteerd $Sub_{\mathcal{C}}(X)$. Deze verzameling is partieel geordend door inclusie. Men denkt aan een subobject A van X als een predicaat met een vrije variabele van type X, en aan de relatie $A \leq B$ als de afleidbaarheid van B uit A. Mits de categorie $\mathcal{C}$ bepaalde structuur heeft, bestaan er op de partiële ordeningen $Sub_{\mathcal{C}}(X)$ operaties analoog aan de verzamelingstheoretische operaties vereniging, doorsnede en complement, die dienst doen als interpretatie van de logische connectieven en quantoren.

De logica die men aldus verkrijgt lijkt, zoals reeds opgemerkt, veel op de logica van verzamelingen en functies, alleen is de logica constructief. (De reden hiervoor is het feit, dat in de ordeningen $Sub_{\mathcal{C}}(X)$ een predicaat A in het algemeen niet gelijk is aan het complement van het complement van A.)

Topossen. Een belangrijke klasse van categorieën wordt gevormd door de *topossen*. Een topos is een categorie met eindige limieten, exponenten en een subobject classifier. Deze structuur garandeert, dat de interpretatie van hogere-orde logica mogelijk is. Hierdoor kan men een topos goed opvatten als een universum, waarin men (constructieve) wiskunde kan bedrijven. Enkele belangrijke voorbeelden van topossen zijn: de categorie van verzamelingen, de categorie van verzamelingen met een groepswerking, of de categorie van schoven op een topologische ruimte.

In de jaren '70 zijn topossen intensief gebruikt om onafhankelijkheidsresultaten te bewijzen. Zo kan men een topos construeren waarin het keuze-axioma geldig is, maar de continuumhypothese faalt. Ook zijn er topossen waarin er maar aftelbaar veel functies van $\mathbb{N}$ naar $\mathbb{N}$ bestaan, of waar elke functie van $\mathbb{R}$ naar $\mathbb{R}$ continu is.

Realizeerbaarheidstopossen. De beide vormen van semantiek, categorische logica en realizeerbaarheid, kunnen gecombineerd worden. Het resultaat wordt een *realizeerbaarheidstopos* genoemd; dit is een topos met de bijzondere eigenschap, dat een zin in de taal van de rekenkunde waar is in de topos, precies wanneer deze realizeerbaar is. Omdat een topos de interpretatie van hogere-orde logica ondersteunt, geeft dit dus een uitbreiding van realizeerbaarheid naar hogere-orde rekenkunde.

Behalve in de studie van realizeerbaarheid, zijn realizeerbaarheidstopossen ook van groot belang voor de theoretische informatica. De zogeheten PER-modellen zijn inmiddels een essentieel ingrediënt voor de semantiek van programmeertalen.

Inhoud van het Proefschrift. Het hoofdthema van dit proefschrift is een systematische studie naar realizeerbaarheidstopossen en hun constructie.

In hoofdstuk 3 staat de constructie van een realizeerbaarheidstopos uit een partieel combinatorische algebra (PCA) centraal. Elke PCA $\mathbb{A}$ geeft aanleiding tot een topos $\mathbf{RT}[\mathbb{A}]$, en we vragen ons af, in hoeverre deze constructie functorieel is. Het blijkt beter, om een met generalizatie van PCA's te werken, die we *geordende PCA's* noemen. De constructie van een topos uit een geordende PCA vindt vrijwel onveranderd doorgang. We presenteren een definitie van een homomorfisme van geordende PCA's, en laten zien dat elk homomorfisme tussen geordende PCA's (op contravariante wijze) aanleiding geeft tot een exacte functor tussen de geassocieerde topossen. Daarnaast identificeren we een klasse homomorfismen van geordende PCA's, welke precies de geometrische morfismen tussen de realizeerbaarheidstopossen karakterizeert.

Omdat realizeerbaarheidstopossen een ingewikkelde structuur hebben, zijn er in de loop der tijd technieken ontwikkeld om goede presentaties van deze topossen te geven. Een van deze technieken maakt gebruik van zogeheten exacte completeringen. Men moet aan een completering denken als een vrije manier om structuur aan een categorie toe te voegen. In het geval van de exacte completering betekent dat, dat men met een categorie $\mathcal{C}$ begint (deze moet eindige limieten hebben), en dat men daaraan vrij quotienten van equivalentierelaties toevoegt. Elke topos is exact, en het blijkt dat een realizeerbaarheidstopos te presenteren is als exacte completering van een simpele, beter te begrijpen en te manipuleren categorie. Een nadeel van deze techniek is het feit dat ze alleen werkt, wanneer de basistopos over welke men werkt voldoet aan het keuzeaxioma. Om deze beperking te omzeilen heb ik een nieuw soort completering ontwikkeld, die in elke situatie toepasbaar is. Dit wordt in hoofdstuk 4 gepresenteerd. Deze completering wordt een *relatieve completering* genoemd, omdat er informatie over de basistopos meegewogen wordt.

Een andere techniek om realizeerbaarheidstopossen te presenteren, gebaseerd op de zogeheten tripostheorie, is tamelijk logisch van aard, en geeft meer inzicht in de logische eigenschappen van de resulterende topos. Men bouwt dan eerst een tripos, hetgeen een speciaal soort geïndexeerde preordening is met een dusdanig rijke structuur dat hogere-orde logica geïnterpreteerd kan worden (maar zonder gelijkheidspredicaten). Vervolgens construeert men een topos door, informeel gesproken, niet-standaard gelijkheidspredicaten toe te voegen. In hoofdstuk 5 presenteer ik een analyse van deze tripossen, door te laten zien dat deze opgevat kunnen worden als een zeker soort completering van een eenvoudiger soort geïndexeerde preordening. Ook karakteriseer ik díe geïndexeerde preordeningen, waarvoor toepassing van deze vrije constructie een tripos oplevert. Deze karakterizering is geformuleerd in termen van de eerder genoemde geordende PCA's, hetgeen een bekende stelling van Carboni, Freyd en Scedrov generalizeert. Tenslotte laat ik zien, dat herhaalde toepassing van deze constructie aanleiding geeft tot torens van tripossen, en dus ook van realizeerbaarheidstopossen. Vele hiervan waren tot dusver onbekend.

In hoofdstuk 6 tenslotte presenteer ik een aantal resultaten die betrekking hebben op de vraag, wanneer de exacte completering van een topos wederom een topos is. Aangetoond wordt, dat de klasse van topossen van welke de exacte completering een *Grothendieck* topos is gekarakteriseerd wordt door een sterke begrenzingsconditie op het geometrisch morfisme naar **Set**. Ook wordt bewezen dat deze begrenzingsconditie voor locaal samenhangende topossen equivalent is met atomiciteit. Daarnaast laat ik zien dat elke topos die een coproduct completering van een kleine categorie is, atomair is. Tenslotte wordt een sitekarakterizering gegeven van díe Grothendieck topossen die een exacte completering zijn.

Curriculum Vitae

Pieter Jan Wouter Hofstra was born 27th March 1975 in Amsterdam. He attended the Stedelijk Gymnasium in 's Hertogenbosch where he graduated in 1993. After studying History of Arts for one year, he switched to Cognitive Artificial Intelligence (CKI) in Utrecht. His Master thesis, written under supervision of Dr. Jaap van Oosten and Prof. Albert Visser, was completed in 1999. Then he started as a Ph.D. student at the Mathematical Institute in Utrecht, where Dr. Jaap van Oosten was his supervisor.