Relative and Modified Relative Realizability

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1 Introduction

1.1 Background and Motivation

The notion of Relative Realizability goes back to the work of Kleene and Vesley [14] (actually, it is even older than that; see 4.3 of this paper) and it was described by means of tripos theory right from the beginnings of that theory, see, e.g., [17, Section 1.5, item (ii)]. Recently there has been a renewed interest in Relative Realizability, both in Thomas Streicher's "Topos for Computable Analysis" [18] and in [2, 1, 4]. The idea is, that instead of doing realizability with one partial combinatory algebra A one uses an inclusion of partial combinatory algebras $A_\sharp \subseteq A$ (such that there are combinators $k, s \in A_\sharp$ which also serve as combinators for A), the principal point being that "($A_\sharp$-) computable" functions may also act on data (in A) that need not be computable.

In [2] there is an analysis of the relationships between relative realizability over $A_\sharp \subseteq A$ and the ordinary realizabilities over $A_\sharp$ and A. Let $\mathrm{RT}(A_\sharp, A)$ be the relative realizability topos, and $\mathrm{RT}(A_\sharp)$, $\mathrm{RT}(A)$ the ordinary (effective topos-like) realizability toposes; then

- *there is a local geometric morphism from* $\mathrm{RT}(A_\sharp, A)$ *to* $\mathrm{RT}(A_\sharp)$; and
- *there is a logical functor from* $\mathrm{RT}(A_\sharp, A)$ *to* $\mathrm{RT}(A)$.

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The motivation for the present paper was the observation that there is a general pattern underlying these results.

The basic point is the following. We view $\mathcal{A} = (A_\sharp \to A)$ as an *internal partial combinatory algebra in the topos* $\mathrm{Set}^\to$ (sheaves over Sierpinski space) and we write $P_\mathcal{A}$ for the standard $\mathrm{Set}^\to$-indexed realizability tripos over $\mathcal{A}$. This internal partial combinatory algebra is what we call *j-regular* with respect to the open topology $j = \neg\neg$; here this just boils down to the fact that $A_\sharp$ is closed under application in A. Then one has that $P_{\mathcal{A},j}$, the tripos of j-closed subsets of $\mathcal{A}$, is an open subtripos of $P_\mathcal{A}$ which gives the relative relizability topos $\mathrm{RT}(A_\sharp, A)$; and its closed complement $Q_{\mathcal{A},j}$ gives what we call *relative modified realizability.* In the case where $A_\sharp = A = N$, the standard Kleene partial combinatory algebra on the natural numbers, our notion of relative modified realizability topos agrees with the standard modified realizability topos [6, 9, 21].

In this basic setup, there is an embedding from $\mathcal{A}_\sharp = (A_\sharp \to A_\sharp)$ to $\mathcal{A} = (A_\sharp \to A)$ in $\mathrm{Set}^\to$ of partial combinatory algebras, which is what we call *elementary.* Here this just means that whenever $\mathrm{Set}^\to \models \exists x{:}\mathcal{A}$, we also have that $\mathrm{Set}^\to \models \exists x{:}\mathcal{A}_\sharp$, which holds because of the Kripke interpretation of $\exists$ in $\mathrm{Set}^\to$. Because of this elementary embedding it follows from our general results that there is a local geometric morphism $P_\mathcal{A} \to P_{\mathcal{A}_\sharp}$, which restricts to a local geometric morphism $P_{\mathcal{A},j} \to P_{\mathcal{A}_\sharp,j}$ which exactly induces the above mentioned local map $\mathrm{RT}(A_\sharp, A) \to \mathrm{RT}(A_\sharp)$. The local geometric morphism $P_\mathcal{A} \to P_{\mathcal{A}_\sharp}$ also restricts to a local geometric morphism among the modified realizability triposes $Q_{\mathcal{A},j} \to Q_{\mathcal{A}_\sharp,j}$.

There is a third internal partial combinatory algebra in $\mathrm{Set}^\to$, namely $\mathcal{B} = (A \to A)$ and because the subobject $\mathcal{A} \to \mathcal{B}$ is j-dense (still with $j = \neg\neg$) it follows from our general results that there is logical functor $P_{\mathcal{A},j} \to P_{\mathcal{B},j}$, which exactly induces the logical functor $\mathrm{RT}(A_\sharp, A)$ to $\mathrm{RT}(A)$ mentioned above.

1.2 Outline

In Section 2.1 we present the precise definition of an internal partial combinatory algebra and establish some notation to be used in the sequel. Then in Section 2.2 we embark on a general theory of triposes on a topos $\mathcal{E}$. One of the key notions appears to be that of an *elementary map* (Definition 2.2) in $\mathcal{E}$. We show that if there is an elementary map of partial combinatory algebras $A \to B$, then there is a geometric morphism of triposes $P_B \to P_A$, which is local when the map $A \to B$ is monic (in $\mathcal{E}$).

In Section 2.3, we consider realizability triposes relative to internal topologies. Given a topology j on $\mathcal{E}$, we define what it means that A is *j-regular.* When this is the case, we show that there is a geometric inclusion from $P_{A,j}$, the tripos built using only the j-closed subsets of A, to P_A, the standard realizability tripos over A. Moreover, when A and B are both j-regular, and there is an elementary map $A \to B$, then the (local) geometric morphism $P_B \to P_A$ restricts to a (local) geometric morphism $P_{B,j} \to P_{A,j}$.

We then look in particular at the case where j is an open topology; in this situation we find that also the inclusion $P_{A,j} \to P_A$ is open and $P_{B,j}$ is then the pullback of $P_{A,j}$ along $P_B \to P_A$. Since $P_{A,j} \to P_A$ is open it makes sense to look at its closed complement, which we define as the *modified realizability tripos* and denote by $Q_{A,j}$; the topos it represents is denoted $\mathcal{M}_{A,j}$. We show that the local geometric morphism $P_B \to P_A$ restricts to a local geometric morphism $Q_{B,j} \to Q_{A,j}$ and that $Q_{B,j}$ is the pullback of $Q_{A,j}$ along $P_B \to P_A$.

In Section 2.4 we show that if we have a j-dense inclusion $A \to B$ of partial combinatory algebras, then $P_{B,j}$ is a filter-quotient of $P_{A,j}$ and thus there is a logical functor $P_{A,j} \to P_{B,j}$.

In Section 3 we explore the relationship with the topos of sheaves for j. We show that, in general, $\mathrm{Sh}_j(\mathcal{E})$ is the pullback of $\mathcal{E}[P_{A,j}]$ along the inclusion of $\mathcal{E}$ into $\mathcal{E}[P_A]$ and that, in case j is open with closed complement k, $\mathrm{Sh}_k(\mathcal{E})$ is the pullback of $\mathcal{M}_{A,j}$ along the inclusion of $\mathcal{E}$ into $\mathcal{E}[P_A]$.

The final Section 4 contains applications and relations to existing work in the literature.

2 Triposes over Internal Pca's

2.1 Internal Partial Combinatory Algebras

In this section we intend to lay down some basic definitions and to fix notation.

We shall work, throughout this section, in an arbitrary topos $\mathcal{E}$. We shall employ the internal language and logic freely, and assume the reader is familiar with its use.

Let A be an object of $\mathcal{E}$, and $f : A \times A \rightharpoonup A$ a partial map. We shall write D_A for its domain, i.e. the object defined by the pullback diagram

$$\begin{array}{ccc} D_A & \longrightarrow & A \times A \\ \downarrow & & \downarrow{\scriptstyle f} \\ A & \underset{\eta_A}{\longrightarrow} & \tilde{A} \end{array}$$

where $A \stackrel{\eta_A}{\to} \tilde{A}$ is the partial map classifier of A.

We see this as a structure for a language with just a partial binary function symbol, which we write as juxtaposition: $a, b \mapsto ab$. In composite expressions we assume association to the left, i.e. abc is short for $(ab)c$. In manipulating terms in this language we use the symbol "$\downarrow$" ("is defined"). For a term t, composed from variables $x_1, \ldots, x_n$ of type A and juxtaposition, we define its meaning $t[\vec{u}] = t[u_1, \ldots, u_n]$ and the formula $t[\vec{u}]\downarrow$ by a simultaneous induction (here $u_1, \ldots, u_n$ denote generalized elements of type A, i.e. morphisms $U \to A$ for some parameter object U):

$$\begin{array}{ll} x[\vec{u}]\downarrow = \top & x[u] = u \\ ts[\vec{u}]\downarrow = t[\vec{u}]\downarrow \wedge s[\vec{u}]\downarrow \wedge (t[\vec{u}], s[\vec{u}]) \in D_A & ts[\vec{u}] = f \circ \langle t[\vec{u}], s[\vec{u}] \rangle \end{array}$$

Note that $t\downarrow$ implies $t'\downarrow$ for any subterm t' of t. Given two terms t and s, we use the expression $t \sim s$ as an abbreviation for

$$(t\!\downarrow \leftrightarrow s\!\downarrow) \wedge (t\!\downarrow \rightarrow t = s)$$

.

Definition 2.1 a) The structure $(A, D_A \stackrel{f}{\rightarrow} A)$ is called a *partial combinatory algebra* in $\mathcal{E}$, if the statements:

$$\begin{array}{ll} \mathbf{k} & \exists k{:}A\forall xy{:}A.kxy\!\downarrow \wedge\, kxy = x \\ \mathbf{s} & \exists s{:}A\forall xyz{:}A.sxy\!\downarrow \wedge\, sxyz \sim xz(yz) \end{array}$$

are both true in the internal logic of $\mathcal{E}$.

b) Given two partial combinatory algebras $(A, D_A \stackrel{f}{\rightarrow} A)$ and $(B, D_B \stackrel{g}{\rightarrow} B)$ in $\mathcal{E}$, a map $\mu : A \rightarrow B$ is called an *applicative map* if the following conditions hold:

i) the map $D_A \stackrel{\mu\times\mu}{\rightarrow} B \times B$ factors through D_B

ii) the diagram

$$\begin{array}{ccc} D_A & \xrightarrow{\mu\times\mu} & D_B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\mu]{} & B \end{array}$$

is a pullback in $\mathcal{E}$ (in particular, it commutes!)

iii) the formulas

$$\begin{array}{l} \exists k{:}A\forall xy{:}B.\mu(k)xy\!\downarrow \wedge\, \mu(k)xy = x \\ \exists s{:}A\forall xyz{:}B.\mu(s)xy\!\downarrow \wedge\, \mu(s)xyz \sim xz(yz) \end{array}$$

are true in $\mathcal{E}$.

Note that the combinator axioms **k** and **s** do not require k and s to be global elements of A. We find this appropriate because we do not require the maps to preserve the chosen k and s.

The standard facts about partial combinatory algebras (see, e.g.,[3]) that we need, are all constructively valid, and carry over to internal partial combinatory algebras in a topos $\mathcal{E}$. In particular, we shall use

- Schönfinkel's *Combinatory Completeness*: for any term t and any variable x, there is a term $\Lambda x.t$ such that for any term s, $(\Lambda x.t)s \sim t[s/x]$ holds;

- *Pairing*: the sentence

$$\exists p, p_0, p_1{:}A.\forall xy{:}A.pxy\!\downarrow \wedge\, p_0(pxy) = x \wedge p_1(pxy) = y$$

is true in $\mathcal{E}$. In fact, any choice of k and s in A give p, p_0, p_1 definable in k, s.

Given a partial combinatory algebra A we define the two maps

$$\wedge_A, \Rightarrow_A : \Omega^A \times \Omega^A \to \Omega^A$$

internally by:

$$\begin{array}{lcl} X \wedge_A Y & = & \{x \in A \mid p_0 x \in X \text{ and } p_1 x \in Y\} \\ X \Rightarrow_A Y & = & \{a \in A \mid \forall b \in X (ab\downarrow \wedge\, ab \in Y)\} \end{array}$$

The notations $\wedge, \Rightarrow$ will be extended to morphisms: $X \to \Omega^A$ by composition; and the subscript will be used only if confusion is possible.

2.2 Realizability Triposes on $\mathcal{E}$

Let $(A, D_A \stackrel{f}{\to} A)$ be a partial combinatory algebra in $\mathcal{E}$. We shall not define the notion of a tripos (instead, refer the reader to [8]), but just for definiteness we recall the definition of the *standard realizability tripos* on $\mathcal{E}$ with respect to A, which we shall denote by P_A. $P_A(X)$ is the set of arrows: $X \to \Omega^A$ in $\mathcal{E}$. $P_A(X)$ is preordered by: for $\varphi, \psi \in P_A(X)$, $\varphi \leq \psi$ if and only if the sentence

$$\exists a{:}A.\forall x{:}X.a \in [\varphi(x) \Rightarrow \psi(x)]$$

is true in $\mathcal{E}$.

$P_A(X)$ is a Heyting prealgebra, and the (extensions of the) maps $\wedge_A$, $\Rightarrow_A$ serve as *meet* and *Heyting implication*, respectively.

For any arrow $f{:}X \to Y$ we have $P_A(f){:}P_A(Y) \to P_A(X)$ by composition. This map is a morphism of Heyting prealgebras and has both adjoints $\exists_f$ and $\forall_f$:

$$\begin{array}{lcl} \exists_f(\varphi)(y) & = & \{a \in A \mid \exists x{:}X.f(x) = y \wedge a \in \varphi(x)\} \\ \forall_f(\varphi)(y) & = & \{a \in A \mid \forall x{:}X.f(x) = y \to a \in (A \Rightarrow \varphi(x))\} \end{array}$$

Our first proposition concerns *geometric morphisms* between realizability triposes (again, the reader is referred to [8] for a definition). Recall from [12], that a geometric morphism between toposes is called *local* if it is bounded and its direct image part has a full and faithful right adjoint. Since any geometric morphism which arises from a geometric morphism of triposes is automatically bounded (indeed, localic; see [2] for a proof) we shall say that a geometric morphism between triposes is local if its direct image has a full and faithful right adjoint.

Definition 2.2 A morphism $A \stackrel{i}{\to} B$ in $\mathcal{E}$ is said to be *elementary* if every subobject of B with global support intersects the image of i: if $C \subset B$ and $C \to 1$ is an epimorphism, so is $i^{-1}(C) \to 1$.

Note that the map $A \stackrel{i}{\to} B$ is elementary, precisely when the internal logic of $\mathcal{E}$ obeys the following rule:

$$\mathcal{E} \models \exists x{:}B.R(x) \Rightarrow \mathcal{E} \models \exists x{:}A.R(i(x))$$

for any closed formula $\exists x{:}B.R(x)$ of the internal language.

Example 2.3 Let $\mathcal{E}$ be the topos $\mathrm{Set}^{\to}$. Observe that a map

$$(f_1, f_2) : (A_1 \to A_2) \to (B_1 \to B_2)$$

in $\mathcal{E}$ is elementary iff $f_1 : A_1 \to B_1$ is a surjective function. Therefore, if $A_\sharp \subseteq A$ in Set, the inclusion of $(A_\sharp \to A_\sharp)$ in $(A_\sharp \to A)$ in $\mathrm{Set}^{\to}$ is an elementary map.

The following proposition is essentially already in [2].

Proposition 2.4 *Let $i : A \to B$ be an applicative map of partial combinatory algebras in $\mathcal{E}$. If i is an elementary map, there is a geometric morphism of triposes Φ: $P_B \to P_A$.*

If, moreover, i is monic, the geometric morphism Φ is local.

Proof. Define $\Phi^* : P_B \to P_A$ by composition with the map $\Omega^i : \Omega^B \to \Omega^A$ (i.e., inverse image of i). To show that this is order-preserving we use that $A \to B$ is elementary: if $\varphi \leq \psi$ in $P_B(X)$, then

$$\exists a{:}B \forall x{:}X. a \in \varphi(x) \Rightarrow_B \psi(x)$$

hence, by elementariness,

$$\exists a{:}A \forall x{:}X. i(a) \in \varphi(x) \Rightarrow_B \psi(x)$$

and since i is applicative we have

$$\exists a{:}A \forall x{:}X. a \in (i^{-1}(\varphi(x)) \Rightarrow_A i^{-1}(\psi(x))$$

We define $\Phi_! : P_A \to P_B$ by composition with the map $\exists_i : \Omega^A \to \Omega^B$. Clearly, if $\varphi : X \to \Omega^A$ and $\psi : X \to \Omega^B$ then $\Phi_!\Phi^*(\psi) \leq \psi$ and $\varphi \leq \Phi^*\Phi_!(\varphi)$, so $\Phi_! \dashv \Phi^*$ and $\Phi_!$ is order-preserving. Moreover, $\Phi_!$ preserves finite meets: since i is applicative, internally a choice for the pairing combinators exists in A which are also pairing combinators for B. And since A is inhabited, $\Phi_!$ preserves the top element. So $(\Phi^*, \Phi_!)$ is a geometric morphism of triposes: $P_B \to P_A$.

Now assume that i is monic. It is an easy exercise to show, using elementariness of i, that $\Phi_!$ is full and faithful.

We define $\Phi_* : P_A \to P_B$ using the internal logic of the tripos P_B, by letting, for $\psi \in P_A(X)$,

$$\Phi_*(\psi) = \exists \alpha{:}\Omega^B . \alpha \wedge (\Phi_!\Phi^*(\alpha) \to \Phi_!(\psi)).$$

By internal reasoning in P_B it is obvious that Φ_* is order-preserving. For the proof of adjointness $\Phi^* \dashv \Phi_*$, suppose that $\varphi \in P_B(X)$ and that $\psi \in P_A(X)$. Now if $\Phi^*(\varphi) \leq \psi$, that is, if $\forall x{:}X. \Phi^*(\varphi)(x) \to \psi(x)$, then also

$$\forall x{:}X. \varphi(x) \to \exists \alpha{:}\Omega^B . \alpha \wedge (\Phi_!\Phi^*(\alpha) \to \Phi_!(\psi)(x))$$

(just take $\alpha = \varphi(x)$), so also $\varphi \leq \Phi_*(\psi)$. For the converse, suppose that $\varphi \leq \Phi_*(\psi)$, that is,

$$\forall x{:}X. [\varphi(x) \to \exists \alpha{:}\Omega^B . \alpha \wedge (\Phi_!\Phi^*(\alpha) \to \Phi_!(\psi)(x))]$$

Then by functoriality of Φ^* and using that Φ^* preserves $\exists$, $\wedge$, and $\to$ (seen by direct inspection of the definitions) we have that

$$\forall x{:}X.[\Phi^*(\varphi)(x) \to \exists\alpha{:}\Omega^B.\Phi^*(\alpha) \wedge (\Phi^*\Phi_!\Phi^*(\alpha) \to \Phi^*\Phi_!(\psi)(x))]$$

and thus, since $\Phi_!$ is full and faithful, that

$$\forall x{:}X.\Phi^*(\varphi)(x) \to \exists\alpha{:}\Omega^B.\Phi^*(\alpha) \wedge (\Phi^*(\alpha) \to \psi(x))$$

from which the required inequality $\Phi^*(\varphi) \leq \psi$ obviously follows. Note that by elementary category theory, full and faithfulness of Φ_* follows from full and faithfulness of $\Phi_!$.

∎

2.3 Realizability Triposes and Internal Topologies

Let A be a partial combinatory algebra in $\mathcal{E}$. Now suppose that $j{:}\Omega \to \Omega$ is an internal topology in $\mathcal{E}$, i.e. the following axioms are true in $\mathcal{E}$:

$$\begin{array}{l} \forall p{:}\Omega.p \to j(p) \\ \forall pq{:}\Omega.(p \to q) \to (j(p) \to j(q)) \\ \forall p{:}\Omega.j(j(p)) \to j(p) \end{array}$$

Definition 2.5 We call the partial combinatory algebra A *j-regular* if the following statement is true in $\mathcal{E}$:

$$\exists c{:}A\forall ab{:}A.j(ab\downarrow) \to c(pab)\downarrow \wedge\, j(c(pab) = ab)$$

Note, that A is j-regular if the inclusion $D_A \subset A \times A$ is j-closed (but the converse does not seem to be true in general); also note that every total combinatory algebra is j-regular for every j.

Example 2.6 We continue Example 2.3 and now suppose that $A_\sharp \to A$ is an applicative map of partial combinatory algebras in Set. Now regard $(A_\sharp \to A)$ as an internal partial combinatory algebra in the topos $\mathrm{Set}^\to$. This topos has a point $0 : \mathrm{Set} \to \mathrm{Set}^\to$, corresponding to the open point of Sierpinski space: $0_*(X) = (X \stackrel{\mathrm{id}}{\to} X)$, $0^*(X \to Y) = Y$. Moreover, 0_* embeds Set as $\neg\neg$-sheaves into $\mathrm{Set}^\to$. The partial combinatory algebra $(A_\sharp \to A)$ is $\neg\neg$-regular in $\mathrm{Set}^\to$, because $A_\sharp \to A$ is applicative.

Henceforth we shall deal with a topology j for which our partial combinatory algebras are assumed j-regular.

As usual, Ω_j denotes the image of j; Ω_j^A is the object of j-closed subsets of A and $j^A : \Omega^A \to \Omega_j^A$ is the internal closure map. In the logic, $j^A(\alpha) = \{x \mid j(x \in \alpha)\}$. Note that if A is a j-regular partial combinatory algebra, we have

$$\Lambda ab.c(pab) \in \bigcap_{\alpha,\beta\in\Omega^A} (\alpha \Rightarrow j^A(\beta)) \Rightarrow (j^A(\alpha) \Rightarrow j^A(\beta))$$

(where $c \in A$ is an element satisfying Definition 2.5)

Note also, that

$$\forall\alpha\beta{:}\Omega^A . j^A(\alpha \wedge_A \beta) = j^A(\alpha) \wedge_A j^A(\beta)$$

holds in $\mathcal{E}$.

We define the realizability tripos $P_{A,j}$ by: $P_{A,j}(X)$ is the set of arrows $X \to \Omega_j^A$ in $\mathcal{E}$. We regard this as a subset of $P_A(X)$, and give $P_{A,j}(X)$ the sub-preorder. Using the above remarks, the verification that this is a tripos is straightforward. The following easy proposition was essentially in [19].

Proposition 2.7 *A is j-regular if and only if taking pointwise j-closure gives a left adjoint to the indexed inclusion $P_{A,j} \to P_A$ induced by the inclusion $\Omega_j^A \to \Omega^A$. In this case, we have a geometric inclusion of triposes.*

Proof. We shall only show that j-regularity is necessary, leaving the other details (which are straightforward) to the reader. Actually, j-regularity is needed to show that the map

$$\varphi \mapsto \lambda x . j^A(\varphi(x))$$

is order-preserving.

Let $X = \{(a,b) \in A \times A \mid j(ab{\downarrow})\}$. In $P_A(X)$ we have the objects $\psi(a,b) = \{pab \mid ab{\downarrow}\}$ and $\varphi(a,b) = \{ab \mid ab{\downarrow}\}$. Then clearly $\psi \vdash \varphi$. By definition of X, $j^A(\psi(a,b)) = \{pab\}$, so the requirement that $j^A(\psi) \vdash j^A(\varphi)$ gives us a $c \in A$ satisfying Definition 2.5. ∎

Proposition 2.8 *If $A \xrightarrow{i} B$ is an elementary applicative map, the geometric morphism: $P_B \to P_A$ of 2.4 restricts to a geometric morphism $P_{B,j} \to P_{A,j}$. That is, there is a commutative diagram*

$$\begin{array}{ccc} P_{B,j} & \longrightarrow & P_{A,j} \\ \downarrow & & \downarrow \\ P_B & \longrightarrow & P_A \end{array}$$

of geometric morphisms of triposes.

Moreover, if i is monic, the geometric morphism $P_{B,j} \to P_{A,j}$ is also local.

Proof. Adapt the proof of 2.4 by inserting j's at the appropriate points, to obtain j-closed predicates. For example $\Phi_! : P_{A,j} \to P_{B,j}$ sends $\varphi : X \to \Omega_j^A$ to the map $x \mapsto j^B(i[\varphi(x)])$. The adjointness follows readily from elementariness and j-regularity; moreover it is easy that $\Phi_!$ is full and faithful if i is monic.

Define $\Phi_* : P_{A,j} \to P_{B,j}$ by

$$\Phi_*(\varphi)(x) = \{a{:}B \mid j(\exists\alpha{:}\Omega_j^B . a \in \alpha \wedge ((i[i^{-1}(\alpha)] \Rightarrow_B j^B(\varphi(x)))\}$$

Since the proof of the adjunction $\Phi^* \dashv \Phi_*$ in 2.4 is in the tripos logic and uses only that Φ^* preserves $\wedge$, $\to$ and $\exists$ and that $\Phi_!$ is full and faithful, it can be used here verbatim.

Finally, the diagram in the statement of the proposition commutes because j-closed subobjects are preserved by pulling back (intersection). ∎

Remark 2.9 We wish to point out that, in contrast with the special case considered later in this paper, the diagram of toposes resulting from 2.8 is not in general a pullback (of toposes). Our 'running example' ($\mathcal{E} = \mathrm{Set}^{\rightarrow}$) with elementary applicative map $(A_\sharp \to A_\sharp) \to (A_\sharp \to A)$ provides a counterexample, if we let $A_\sharp \to A$ be an applicative map of *total* combinatory algebras in Set, for example the inclusion $P(\omega)_{\mathrm{r.e.}} \to P(\omega)$, and k the unique nontrivial closed topology in $\mathrm{Set}^{\rightarrow}$. Note, that k-closed subobjects of $A_\sharp \to A$ are of form $U \to A$, with $U \subset A_\sharp$. And note that by totality, both algebras are k-regular.

Letting $\mathcal{A} = (A_\sharp \to A_\sharp)$, $\mathcal{B} = (A_\sharp \to A)$, we see that both $P_{\mathcal{A},k}$ and $P_{\mathcal{B},k}$ give the standard realizability topos $\mathrm{RT}(A_\sharp)$; the inclusion of $P_{\mathcal{A},k}$ in $P_{\mathcal{A}}$ is open, but $P_{\mathcal{B},k} \to P_{\mathcal{B}}$ isn't. Since open maps are stable under pullback, the square cannot be a pullback in this case.

Recall that a topology j is *open* if there is a global element u of Ω such that $j(x) = u \to x$ for all $x \in \Omega$. By analogy we say that a geometric inclusion $\Phi^* \vdash \Phi_*$ of triposes: $P \to Q$ is open, if there is an element α of $Q(1)$ such that for every $\varphi \in Q(X)$, $\Phi_*\Phi^*(\varphi)$ is isomorphic to $Q(!)(\alpha) \Rightarrow \varphi$ where ! denotes $X \to 1$, and $\Rightarrow$ is the Heyting implication of $Q(X)$.

It is an easy exercise to show that open inclusions of triposes yield open inclusions between the corresponding toposes, and that the open topology in $\mathcal{E}[Q]$ corresponds to the subobject of 1 determined by α.

Proposition 2.10 *If j is an open topology, then the inclusion $P_{A,j} \to P_A$ is open and, moreover, the square in Proposition 2.8 is a pullback diagram.*

Proof. Let $j(p) = u \to p$ for some $u \in \Omega$; let U be the subobject of 1 classified by u. In $P_A(1)$ we have the image A' of the projection $A \times U \to A$, so $A' = \{a{:}A \mid u\}$.

We calculate, for $\varphi \in P_A(X)$, the element $A' \Rightarrow \varphi$:

$$\begin{aligned} A' \Rightarrow \varphi(x) &= \{a \mid \forall b{:}A.u \to (ab{\downarrow} \wedge ab \in \varphi(x))\} \\ &= \{a \mid \forall b{:}A.ab{\downarrow} \wedge (u \to ab \in \varphi(x)))\} \\ &= A \Rightarrow j^A(\varphi(x)) \end{aligned}$$

Now clearly, $\lambda x{:}X.A \Rightarrow \varphi(x)$ is isomorphic to φ in $P_A(X)$; so $\lambda x{:}X.A' \Rightarrow \varphi(x)$ is isomorphic to $\lambda x{:}X.j^A(\varphi(x))$. Hence, the inclusion $P_{A,j} \to P_A$ is open.

The square in Proposition 2.8 is a pullback diagram because whenever one has an open inclusion $\Phi^* \dashv \Phi_*$ of triposes $P \to Q$ given by an element $\alpha \in Q(1)$, then the pullback along a geometric morphism $f^* \dashv f_* : R \to Q$ is again an open inclusion, determined by the inverse image of α (i.e., the element $f^*(\alpha) \in R(1)$), and here in the case at hand, we clearly have that the inverse image of $A' = \{a{:}A \mid u\}$ along $P_B \to P_A$ is equal to $B' = \{b{:}B \mid u\}$ (and B' of course determines the inclusion $P_{B,j} \to P_B$ by the argument given above). ∎

Definition 2.11 Let $\mathcal{E}$ be a topos, j an open topology in $\mathcal{E}$, and A a j-regular internal partial combinatory algebra in $\mathcal{E}$. The *Modified Realizability Topos* $\mathcal{M}_{A,j}$ with respect to A and j, is defined as the closed complement of $\mathcal{E}[P_{A,j}]$ in $\mathcal{E}[P_A]$ and the *Modified Realizability Tripos* $Q_{A,j}$ with respect to A and j is defined as the tripos representing $\mathcal{M}_{A,j}$.

We shall see in Section 4.2 that this definition agrees with traditional usage of the term “modified realizability”. Note that we do *not* claim that if k is the closed complement of j, $\mathcal{M}_{A,j}$ is $\mathcal{E}[P_{A,k}]$! In fact this is false for our basic example, see Section 4.2.

We now describe the modified realizability tripos $Q_{A,j}$ explicitly. Suppose j is the open topology $x \mapsto u \to x$, then we saw in 2.10 that the inverse image of the inclusion $P_{A,j} \to P_A$ is given by

$$\varphi \mapsto \lambda x{:}X.A' \Rightarrow \varphi(x),$$

where $A' = \{a{:}A \mid u\}$. Therefore the tripos $Q_{A,j}$ representing $\mathcal{M}_{A,j}$ can be defined by

$$Q_{A,j}(X) = \{\varphi : X \to \Omega^A \mid (\lambda x{:}X.A') \leq \varphi\},$$

where $\leq$ refers to the order in $P_A(X)$. The reflection $P_A(X) \to Q_{A,j}(X)$ is given by $\varphi \mapsto (\lambda x{:}X.A') \vee \varphi$, where $\vee$ is the join in the Heyting algebra $P_A(X)$.

At this point we insert a folklore fact from topos theory which we have not found in text books:

Lemma 2.12 *Suppose $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is a geometric morphism of toposes, and j and k the open and closed topologies in $\mathcal{E}$ corresponding to the subobject $U \subset 1$ in $\mathcal{E}$. Then the pullbacks along f of the sheaf subtoposes $\mathrm{Sh}_j(\mathcal{E})$ and $\mathrm{Sh}_k(\mathcal{E})$ are, respectively, the open and closed subtoposes of $\mathcal{F}$ corresponding to the subobject $f^*(U) \subset 1$.*

Proof. [Sketch] The pullback along f of $\mathrm{Sh}_j(\mathcal{E})$ is the subtopos of $\mathcal{F}$ given by the least topology which makes $f^*(\top) : 1 \to f^*(J)$ dense, where $1 \to J$ is the generic j-dense subobject in $\mathcal{E}$. However, this is equivalent to making $f^*(U) \to 1$ dense, and clearly the open topology corresponding to $f^*(U)$ is the least such.

For the closed case one observes that $0 \to U$ is k-dense; hence an arbitrary geometric morphism $g : \mathcal{G} \to \mathcal{E}$ factors through $\mathrm{Sh}_k(\mathcal{E})$ if and only if $g^*(U)$ is isomorphic to 0 in $\mathcal{G}$. So if now $g : \mathcal{G} \to \mathcal{F}$, then fg factors through $\mathrm{Sh}_k(\mathcal{E})$ if and only if $g^*(f^*(U)) \cong 0$, that is: g factors through the closed subtopos of $\mathcal{F}$ determined by $f^*(U)$. ■

Proposition 2.13 *If $A \xrightarrow{i} B$ is an elementary applicative map, the geometric morphism $P_B \to P_A$ restricts to a geometric morphism $Q_{B,j} \to Q_{A,j}$. Moreover, $Q_{B,j}$ is the pullback of $Q_{A,j}$ along $P_B \to P_A$, that is, there is a pullback diagram*

$$\begin{array}{ccc} Q_{B,j} & \longrightarrow & Q_{A,j} \\ \downarrow & & \downarrow \\ P_B & \longrightarrow & P_A \end{array}$$

of geometric morphisms of triposes.

Moreover, if i is monic, the geometric morphism $Q_{B,j} \to Q_{A,j}$ is local.

Proof. Most of this is immediate from 2.12 and 2.10. We shall only show that if i is monic, the indexed functor $\Phi_* : P_A(X) \to P_B(X)$ restricts to $Q_{A,j}(X) \to Q_{B,j}(X)$.

To this end, suppose that $\varphi \in Q_{A,j}(X)$, that is, that

$$\forall x{:}X.A' \to \varphi(x)$$

holds in $P_A(X)$. Since $A' = \Phi^*(B')$ and since $\Phi_!$ is a functor, it follows that

$$\forall x{:}X.\Phi_!\Phi^*(B') \to \Phi_!(\varphi)(x)$$

holds in $P_B(X)$. Thus also (with $\beta = B'$)

$$\forall x{:}X.B' \to \exists\beta{:}\Omega^B.\beta \wedge \Phi_!\Phi^*(B') \to \Phi_!(\varphi)(x),$$

which is to say that $\Phi_*(\varphi) \in Q_{B,j}$, as required. ∎

2.4 Dense Embeddings and Logical Functors

We now turn to the situation of a monic applicative map $A \to B$ of partial combinatory algebras in $\mathcal{E}$ where A is a j-dense subobject of B, but the embedding is not assumed to be elementary. Generally, we don't have geometric morphisms any more. However, there is an interesting $\mathcal{E}$-indexed functor: $P_{A,j} \to P_{B,j}$.

In order to explain the situation, we recall from Pitts' thesis ([17]) that for any tripos P on $\mathcal{E}$ and any filter Φ on the Heyting pre-algebra $P(1)$, one can consider the *filter quotient* tripos P_Φ: $P_\Phi(X)$ is the same *set* as $P(X)$, but the order is defined by:

$$\varphi \leq_\Phi \psi \text{ iff } \forall_!(\varphi \Rightarrow \psi) \in \Phi$$

where $! : X \to 1$ and $\Rightarrow$ is the Heyting implication in $P(X)$.

Every filter Φ on $P(1)$ gives a filter $\hat{\Phi}$ of subobjects of 1 in the topos $\mathcal{E}[P]$, and the topos $\mathcal{E}[P_\Phi]$ is the filter quotient $\mathcal{E}[P]_{\hat{\Phi}}$ ([17]). The filter quotient construction (which, by the way, is called "filter power" in [10]) is well explained in [15]. For us it is important, that for any filter quotient there is a *logical functor* from the topos to the quotient.

We make the following definition.

Definition 2.14 An $\mathcal{E}$-indexed functor $F : P \to Q$ between $\mathcal{E}$-triposes is called *logical* if the following conditions hold:

i) For any object X of $\mathcal{E}$ and $\varphi, \psi \in P(X)$,

$$F_X(\varphi \Rightarrow \psi) \cong F_X(\varphi) \Rightarrow F_X(\psi)$$

ii) For any map $f : X \to Y$ in $\mathcal{E}$ and any $\varphi \in P(X)$,

$$F_Y(\forall_f(\varphi)) \cong \forall_f(F_X(\varphi))$$

iii) If $\sigma \in P(\Sigma)$ is a generic element for P, then $F_\Sigma(\sigma) \in Q(\Sigma)$ is a generic element for Q.

Since, in a tripos, the whole structure is definable from implication, universal quantification and the generic element, any logical functor between triposes gives rise to a logical functor between the corresponding toposes. Moreover, the filter quotient functor: $P \to P_\Phi$ is a logical functor of triposes.

Proposition 2.15 *Suppose $A \to B$ is a monic applicative map of partial combinatory algebras in $\mathcal{E}$, such that the inclusion $A \to B$ of objects is j-dense. Then there is a filter Φ on $P_{A,j}$ such that the triposes $P_{B,j}$ and $(P_{A,j})_\Phi$ are isomorphic; hence, there is a logical functor of triposes: $P_{A,j} \to P_{B,j}$.*

Proof. Let $\Phi \subseteq P_{A,j}(1)$ be the set of those j-closed subobjects α of A such that

$$\mathcal{E} \models \exists b{:}B.j(b \in \alpha)$$

It is easy to check that this is a filter; we define functors $F : (P_{A,j})_\Phi \to P_{B,j}$ and $G : P_{B,j} \to (P_{A,j})_\Phi$ which are each other's inverse.

$F_X : (P_{A,j})_\Phi(X) \to P_{B,j}(X)$ sends $\varphi : X \to \Omega_j^A$ to

$$\lambda x{:}X.j^B(\varphi(x)) : X \to \Omega_j^B$$

F is order preserving: in $(P_{A,j})_\Phi$, $\varphi \leq \psi$ if and only if

$$\mathcal{E} \models \exists b{:}B.j(\forall x{:}X \forall a \in \varphi(x).ba{\downarrow} \wedge ba \in \psi(x))$$

Clearly, this implies

$$\mathcal{E} \models \exists b{:}B \forall x{:}X \forall a \in j^B(\varphi(x)).ba{\downarrow} \wedge ba \in j^B(\psi(x))$$

which is the definition of $F_X(\varphi) \leq F_X(\psi)$.

$G : P_{B,j} \to (P_{A,j})_\Phi$ is defined by $G_X(\varphi) = \lambda x{:}X.\varphi(x) \cap A$. To show that G is order-preserving, reason internally. $\varphi \leq \psi$ in $P_{B,j}(X)$ means

$$\mathcal{E} \models \exists b{:}B \forall x{:}X \forall a \in \varphi(x).ba{\downarrow} \wedge ba \in \psi(x)$$

so let $b{:}B$ satisfy this formula. Clearly, $b \in A$ implies

$$\forall x{:}X \forall a \in \varphi(x) \cap A.ba{\downarrow} \wedge ba \in \psi(x) \cap A$$

Since A is dense in B, we have therefore

$$\mathcal{E} \models \exists b{:}B.j(\forall x{:}X \forall a \in \varphi(x) \cap A.ba{\downarrow} \wedge ba \in \psi(x) \cap A)$$

so $G_X(\varphi) \leq G_X(\psi)$ in $(P_{A,j})_\Phi(X)$.

Finally, since for $\alpha \in \Omega_j^A$ and $\beta \in \Omega_j^B$ we have the identities $j^B(\alpha) \cap A = j^A(\alpha) = \alpha$, and $j^B(\beta \cap A) = \beta$ (the last one because $A \to B$ is dense), we see that F and G are each other's inverse. ∎

3 Relations with the base topos

In this section we exhibit connections between the toposes $\mathcal{E}$, $\mathcal{E}[P_A]$, $\mathcal{E}[P_{A,j}]$, and $\mathrm{Sh}_j(\mathcal{E})$ (the topos of j-sheaves in $\mathcal{E}$). Recall from the theory of triposes [17] that there is a geometric inclusion $\mathcal{E} \to \mathcal{E}[P_A]$, whose direct image functor is the "constant-objects functor."

Theorem 3.1 *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Sh}_j(\mathcal{E}) & \longrightarrow & \mathcal{E}[P_{A,j}] \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}[P_A] \end{array}$$

which is a pullback in the category of toposes and geometric morphisms.

Proof. Let j_0, j_1, j_2 be the topologies in $\mathcal{E}[P_A]$ whose categories of sheaves are $\mathcal{E}$, $\mathcal{E}[P_{A,j}]$ and $\mathrm{Sh}_j(\mathcal{E})$, respectively. Then we must show that j_2 is the join of j_0 and j_1 in the lattice of internal topologies in $\mathcal{E}[P_A]$. The maps j_0, j_1, j_2 are determined by topologies on the tripos P_A, that is by morphisms $k_0, k_1, k_2 : \Omega^A \to \Omega^A$ in $\mathcal{E}$. Indeed, by [17], we know that j_0 is determined by

$$k_0(\alpha) = \{a{:}A \mid \exists a'{:}A. a' \in \alpha\}$$

and by Proposition 2.7, we know that j_1 is determined by

$$k_1(\alpha) = j^A(\alpha).$$

Finally, j_2 is $j \circ j_0$ (since that indeed is a topology), so is determined by

$$k_2(\alpha) = j^A\{a{:}A \mid \exists a'{:}A. a' \in \alpha\}.$$

Since one easily has that $k_2 = k_1 \circ k_0$ it follows that j_2 indeed is the join of j_0 and j_1, as required. ∎

Remark 3.2 The topos $\mathcal{E}[P_{A,j}]$ can in fact be presented by a tripos R on $\mathrm{Sh}_j(\mathcal{E})$ in such a way that the inclusion $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}[P_{A,j}]$ is the associated constant-objects functor Δ_R. To see this, let us first write $i^* \dashv i_*$ for the geometric inclusion: $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ and note that $\mathrm{Sh}_j(\mathcal{E})$ is of form $\mathcal{E}[Q]$, where Q is the tripos corresponding to the internal locale Ω_j in $\mathcal{E}$, and that $i^* : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$ is the constant objects functor Δ_Q. This functor is a left adjoint, hence preserves epimorphisms, so Pitts' *iteration theorem* ([17], 6.2) applies: for any tripos R on $\mathrm{Sh}_j(\mathcal{E})$, we have that $P = R \circ (i^*)^{\mathrm{op}}$ is a tripos on $\mathcal{E}$, and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Delta_P} & \mathcal{E}[P] \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle K} \\ \mathrm{Sh}_j(\mathcal{E}) & \xrightarrow[\Delta_R]{} & \mathrm{Sh}_j(\mathcal{E})[R] \end{array}$$

where K is an equivalence of categories.

Now it is easy to see that if we compose $P_{A,j}$ with the embedding i_*, we get a tripos on $\mathrm{Sh}_j(\mathcal{E})$, because $P_{A,j}$ has a generic element living in the fibre over Ω_j^A, which is a j-sheaf. We see that if R is the $\mathrm{Sh}_j(\mathcal{E})$-tripos $P_{A,j}\circ(i_*)^{\mathrm{op}}$, the topos $\mathrm{Sh}_j(\mathcal{E})[R]$ is equivalent to

$$\mathcal{E}[P_{A,j}\circ(i_*)^{\mathrm{op}}\circ(i^*)^{\mathrm{op}}] \cong \mathcal{E}[P_{A,j}]$$

Hence, $\mathcal{E}[P_{A,j}]$ is also represented by the tripos R on $\mathrm{Sh}_j(\mathcal{E})$. In particular we have the constant objects functor $\Delta_R : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}[P_{A,j}]$.

Remark 3.3 Having noted that $\mathcal{E}[P_{A,j}]$ can be represented by the tripos R on $\mathrm{Sh}_j(\mathcal{E})$, it is natural to ask if it is also represented by the $\mathrm{Sh}_j(\mathcal{E})$-tripos on the partial combinatory algebra $i^*(A)$, i.e., whether $\mathcal{E}[P_{A,j}]$ is equivalent to $\mathrm{Sh}_j(\mathcal{E})[P_{i^*(A)}]$.

For this question to make sense, one needs to observe that sheafification, like the inverse image of any geometric morphism, preserves partial combinatory algebras. This is true because inverse image functors preserve validity of sentences of the form $\exists u{:}U\forall x{:}X(\varphi \to \psi)$ with φ and ψ geometric, and the combinator axioms for partial combinatory algebras can be brought into this form.

The answer to the question is, in general, *no*; see Section 4.1 for a concrete counter-example.

From theorem 3.1 we draw two inferences: firstly, the implication in Proposition 2.10 is actually an equivalence, because it is well known (e.g.,[11]) that open inclusions are stable under pullback along bounded morphisms.

The second inference is more important for our purposes. Suppose now that j is an open topology, $j(x) = u \to x$, and k its closed complement $k(x) = u \vee x$. We have the following obvious proposition (in view of 2.12):

Proposition 3.4 *Let j be an open topology in $\mathcal{E}$, A j-regular. Let k be j's closed complement. Then*

$$\begin{array}{ccc} \mathrm{Sh}_k(\mathcal{E}) & \longrightarrow & \mathcal{M}_{A,j} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}[P_A] \end{array}$$

is a pullback diagram of toposes.

4 Applications

4.1 Relative Realizability

Given an embedding $A_\sharp \subseteq A$ in Set, [2] defines a tripos P on Set: $P(X) = \mathcal{P}(A)^X$ but $\varphi \leq \psi$ iff there is $a \in A_\sharp$ such that for all $x \in X, b \in \varphi(x)$, ab is defined and an element of $\psi(x)$.

Regard $A_\sharp \to A$ as an internal $\neg\neg$-regular pca $\mathcal{A}$ in the topos $\mathrm{Set}^\to$, as in Example 2.6. In $\mathrm{Set}^\to$, the power object $\Omega^{\mathcal{A}}$ is $(R \stackrel{\pi_2}{\to} \mathcal{P}(A))$ where

$$R = \{(U, V) \mid U \in \mathcal{P}(A_\sharp), V \in \mathcal{P}(A), U \subseteq V\}$$

and π_2 is the second projection.

$(\Omega_{\neg\neg})^{\mathcal{A}}$ is $(R' \stackrel{\pi_2}{\to} \mathcal{P}(A))$ where

$$R' = \{(U, V) \mid V \in \mathcal{P}(A), U = V \cap A_\sharp\}$$

We see that there is a natural 1-1 correspondence between maps $X \stackrel{\varphi}{\to} \mathcal{P}(A)$ in Set, and morphisms $0_*(X) \stackrel{\tilde{\varphi}}{\to} (\Omega_{\neg\neg})^{\mathcal{A}}$ in $\mathrm{Set}^\to$, and we have $\varphi \leq \psi$ in $P(X)$ iff

$$\mathrm{Set}^\to \models \exists a{:}\mathcal{A}\, \forall x{:}0_*(X)\, \forall b \in \tilde{\varphi}(x)\, (ab{\downarrow} \wedge ab \in \tilde{\psi}(x))$$

So in fact, P is $P_{\mathcal{A},\neg\neg} \circ (0_*)^{\mathrm{op}}$, and hence, by Remark 3.2, $\mathrm{Set}[P] \simeq \mathrm{Set}^\to[P_{\mathcal{A},\neg\neg}]$.

The sheafification of $A_\sharp \to A$ is just A and thus the topos induced by the standard realizability tripos on this partial combinatory algebra is just the standard realizability topos on A, which, in general, is *different* from the topos represented by the relative realizability tripos P, thus answering the question put forward in Remark 3.3.

Quite similarly, the standard realizability tripos over a pca A in Set is equivalent to $P_{\mathcal{A},\neg\neg} \circ (0_*)^{\mathrm{op}}$ where now $\mathcal{A} = (A \stackrel{\mathrm{id}}{\to} A)$.

Note, that the requirement of $A_\sharp \to A$ to be a monic applicative map in Set, makes the inclusion of $(A_\sharp \stackrel{\mathrm{id}}{\to} A_\sharp)$ into $(A_\sharp \to A)$ a monic elementary applicative map in $\mathrm{Set}^\to$.

Moreover, there is a $\neg\neg$-dense inclusion of $(A_\sharp \to A)$ into $(A \to A)$. So our propositions 2.8 and 2.15 generalize the theorems in [2] on the existence of a local map of toposes, and a logical functor between toposes.

4.2 Modified and Relative Modified Realizability

Let us look at the special case of the pca $A = (\mathbb{N} \to \mathbb{N})$ in $\mathrm{Set}^\to$ and the open $\neg\neg$-topology there. The open object U is $(0 \to 1)$, and the object A' (see the proof of Proposition 2.10) is $(0 \to \mathbb{N})$. As seen in Section 4.1, $\mathrm{Set}^\to[P_{A,j}]$ is the effective topos. Applying the considerations in Remark 3.2, we see that also $\mathrm{Set}^\to[Q_{A,j}]$ is represented by a tripos over Set. As explained in detail in [21], one can take the tripos R, where $R(X)$ is the set of inclusions $(U \subseteq V)$ of subsets of $\mathbb{N}$, where $0 \in V$ (assuming a Gödelnumbering satisfying $\langle 0, 0\rangle = 0$ and $0x = 0$, for all x). The topos given by this presentation was found around 1980 by Hyland and, independently, Grayson (see [6]) to correspond to *modified realizability*. We see therefore that our usage of "modified realizability" in Definition 2.11 generalizes this.

Let k be the closed complement of $\neg\neg$ in the lattice of topologies in $\mathrm{Set}^\to$. Since

$$\begin{aligned} \mathrm{Set}^\to((X \to Y), \Omega_k^A) &\cong \mathrm{Set}^\to((X \to Y), \Omega_k^A) \\ &\cong \mathrm{Set}(X, P(A)) \end{aligned}$$

one finds that $\mathcal{E}[P_{A,k}]$ is the effective topos. Thus, in general, if k is the closed complement of j, the toposes $\mathcal{M}_{A,j}$ and $\mathcal{E}[P_{A,k}]$ are different.

An example of Relative Modified Realizability occurs in [16]. Here one has $\mathcal{M}_{\mathcal{A},\neg\neg}$ where $\mathcal{A} = (A_\sharp \to A)$ is again the inclusion of total recursive functions into the pca for function realizability.

4.3 Kleene's 1957-realizability

To the best of our knowledge, the first notion of relative realizability was discovered by Kleene in 1951 and published in 1957 in [13]. This was formulated in terms of partial recursive application in function oracles. A rather off-hand remark in [14] observes that this is "equivalent" to the notion of relative realizability given in *loc. cit.*. This means that the two notions coincide on the truth definition for intuitionistic analysis; however, it does not seem straightforward to turn the oracle definition into a tripos.

4.4 An almost-example

N. Goodman ([5]) has the following situation: let T be a set of partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$, ordered by inclusion. A is the internal pca in Set^T where at each partial function r, A_r is the ordinary pca of indices for partial functions recursive in r.

The realizability is defined as follows (we adapt notation to ours): for $\varphi, \psi : X \to \Omega^A$,

> $\varphi \leq \psi$ is forced at r iff for some $a \in A_r$: for all $s \geq r$ and all $x \in X_s, b \in \varphi(x)_s$, there is $t \geq s$ such that ab is defined in A_t and an element of $\psi(x)_t$.

In our tripos-theoretic context this means the following. Let j be the double-negation topology, A the given internal pca. $P(X)$ is the set of arrows: $X \to \Omega^A$ in set^T, and $\varphi \leq \psi$ holds iff

$$\exists a{:}A\, \forall x{:}X\, \forall b \in \varphi(x)\, j(ab{\downarrow} \wedge ab \in \psi(x))$$

is true in Set^T.

It is straightforward to prove that this gives a tripos on Set^T, and also that φ is isomorphic in $P(X)$ to $\lambda x{:}X\, j^A(\varphi(x))$. So P looks very much like our $P_{A,j}$. However, Goodman's pca is *not* $\neg\neg$-regular, and there is no inclusion in the tripos P_A. This is obviously a variation, and the exact connection with our set-up remains to be clarified. It is true that $\mathrm{Sh}_{\neg\neg}(\mathrm{Set}^T)$ is a subtopos of $\mathrm{Set}^T[P]$ ([19]), but we do not know whether it is equivalent to any of the toposes we consider.

A very similar example, where the topology is different from $\neg\neg$ and the pca *is* j-regular, is used in [20].

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