Filtered colimits in the Effective Topos

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Abstract

It is shown that the "constant sheaves" functor $\nabla : \text{Sets} \to \mathcal{E}\!f\!f$ does not preserve ω_1-filtered colimits, and that as a consequence of this, the full subcategory of $\mathcal{E}\!f\!f$ on the countable projective objects is not dense.

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Introduction

The present note aims to contribute to the study of the *Effective Topos* $\mathcal{E}\!f\!f$. $\mathcal{E}\!f\!f$, introduced in [1], is one of the prime examples of elementary topoi which are not Grothendieck. In fact, $\mathcal{E}\!f\!f$ is not cocomplete, and the global sections functor $\Gamma : \mathcal{E}\!f\!f \to \text{Sets}$ does not have a left adjoint, but a right adjoint $\nabla : \text{Sets} \to \mathcal{E}\!f\!f$.

A fundamental question is: how does $\mathcal{E}\!f\!f$ compare to Grothendieck topoi? Is it possible to embed $\mathcal{E}\!f\!f$ into a Grothendieck topos in a nice way? In [6], a functor from $\mathcal{E}\!f\!f$ into the "recursive topos" of Mulry ([4]) is defined, but this functor does not preserve a lot of structure (it is, for example, not an embedding).

Nice embeddings can be obtained by considering small full dense subcategories of $\mathcal{E}\!f\!f$. Recall that for every category $\mathcal{E}$, a subcategory $\mathcal{C} \subset \mathcal{E}$ is dense if for every object X of $\mathcal{E}$, the natural cocone with vertex X for the diagram $\mathcal{C}{\downarrow}X \stackrel{\text{dom}}{\to} \mathcal{C} \to \mathcal{E}$ is colimiting. If this is the case, and J is the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology on $\mathcal{E}$, then the left Kan extension of the Yoneda embedding on $\mathcal{C}$, the functor from $\mathcal{E}$ to $[\mathcal{C}^{\rm op}, \text{Sets}]$ which sends X to $\mathcal{E}(-,X)$, factors through the sheaf topos $\text{Sh}(\mathcal{C},J)$ and this factorisation is full and faithful, cartesian closed, and preserves all limits and colimits of $\mathcal{E}$; hence also the natural numbers object. This is standard topos theory, for which the most complete reference is now [2].

The category $\mathcal{E}ff$ is an exact completion ([5]) and therefore, if a small full dense subcategory $\mathcal{C}$ of $\mathcal{E}ff$ exists, we may assume that $\mathcal{C}$ consists of projective objects of bounded cardinality. In fact, I started out from the conjecture that the countable projectives might provide such a dense subcategory; to my surprise, this is wrong as this paper shows (it is fairly easy to see that the countable projectives do form a separating set, i.e. that the natural cocone mentioned earlier is always an epimorphic family).

Basically, this note contains two theorems: theorem 1.2 states an equivalent condition for the full subcategory of λ-small (i.e., having underlying set of cardinality less than λ) projectives to be dense in $\mathcal{E}ff$, relating this to the preservation by ∇ of λ-filtered colimits. Then, after a few folklore results included for completeness' sake, theorem 1.5 states that ∇ does not preserve ω_1-filtered colimits.

I have not been able to settle the matter for higher cardinals such as $\mathcal{P}(\omega)^+$. However, the proof of theorem 1.5 carries the suggestion that there is infinitary set-theoretic combinatorics at work here, and that any result might well depend on axioms independent of ZFC.

1 Filtered Colimits and Dense Subcategories in $\mathcal{E}ff$

For definitions and basic facts concerning $\mathcal{E}ff$ the reader is referred to [1]. However there is one further fact, mentioned in [5], which is helpful to understand theorem 1.2 and its proof. Let N denote the natural numbers object of $\mathcal{E}ff$. Then $\Gamma(N) = \mathbb{N}$; let $\eta_N : N \to \nabla(\mathbb{N})$ be the unit of the adjunction, and $\eta_N^* : \mathcal{E}ff/\nabla(\mathbb{N}) \to \mathcal{E}ff/N$ the pullback functor. There is a functor $\nabla_N : \text{Sets}/\mathbb{N} \to \mathcal{E}ff/N$ obtained by composing with η_N^*. Furthermore denote, as usual, the forgetful (domain) functor $\mathcal{E}ff/N \to \mathcal{E}ff$ by Σ_N. Then:

Lemma 1.1 (Robinson, Rosolini) *An object of $\mathcal{E}ff$ is projective if and only if it is isomorphic to one in the image of $\Sigma_N \circ \nabla_N$.*

Theorem 1.2 *Let $\lambda > \omega$ be a regular cardinal. Then the following two assertions are equivalent:*

i) *The full subcategory of $\mathcal{E}ff$ on the λ-small projectives is dense.*

ii) $\nabla : \text{Sets} \to \mathcal{E}ff$ *preserves λ-filtered colimits.*

For $\lambda = \omega$, the implication i)⇒ii) still holds.

Proof. i)⇒ii): First observe that, since ∇ preserves epi-mono factorizations, statement ii) is equivalent to saying that for any set X, $\nabla(X)$ is the vertex of a colimiting cocone for the diagram consisting of all $\nabla(Y)$ for $Y \subseteq X$ λ-small, and (∇-images of) inclusions. Now since for any X, any cocone to $\nabla(X)$ for a diagram of λ-small projectives also yields a cocone for a diagram of ∇'s of λ-small subsets of X (by sheafification), it is clear that i) implies ii).

For ii)⇒i), observe that if $\nabla : \text{Sets} \to \mathcal{E}ff$ preserves λ-small colimits then the same is true for the functor $\nabla/\mathbb{N} : \text{Sets}/\mathbb{N} \to \mathcal{E}ff/\nabla(\mathbb{N})$ because the forgetful functors $\Sigma_{\mathbb{N}} : \text{Sets}/\mathbb{N} \to \text{Sets}$ and $\Sigma_{\nabla(\mathbb{N})} : \mathcal{E}ff/\nabla(\mathbb{N}) \to \mathcal{E}ff$ preserve and create colimits. Since the pullback functor $\eta_N^* : \mathcal{E}ff/\nabla(\mathbb{N}) \to \mathcal{E}ff/N$ has a right adjoint, the composite functor $\nabla_N : \text{Sets}/\mathbb{N} \to \mathcal{E}ff/N$ preserves λ-filtered colimits too.

In order to prove i), it clearly suffices to prove that every projective object X is a colimit of its λ-small sub-projectives. So suppose that for every λ-small sub-projective Y of X we are given a map $\phi_Y : Y \to (Z, =)$ in $\mathcal{E}ff$, such that for $Y' \subset Y$, $\phi_Y{\restriction}Y' = \phi_{Y'}$. Each such projective Y is a set Y together with a map $e : Y \to \mathbb{N}$; equivalently, an $\mathbb{N}$-indexed family of sets $(Y_n)_{n\in\mathbb{N}}$. Any map $(Y_n)_{n\in\mathbb{N}} \to (Z, =)$ is represented by a function $f : Y \to Z$ such that for some partial recursive function p we have that for all n such that $Y_n \neq \emptyset$, $p(n)$ is defined and

$$p(n) \in \bigcap_{y\in Y_n} [f(y) = f(y)]$$

In such a case, one says that p *tracks* f. Two such functions $f, g : Y \to Z$ represent the same morphism iff there is a partial recursive function q such that for all n with $Y_n \neq \emptyset$, $q(n) \in \bigcap_{y\in Y_n}[f(y) = g(y)]$.

Now I claim that for some partial recursive function p, it holds that for $Y \subset X$ λ-small, *every* ϕ_Y has a representative which is tracked by p; for otherwise choose for every p a λ-small $Y_p \subset X$ for which no representative tracked by p exists; since there are only countably many partial recursive functions the union $\bigcup_p Y_p$ is still λ-small (since $\lambda > \omega$); a contradiction is easily obtained.

Fix such a p as in the previous paragraph. Construct an object $(Z', =')$ from $(Z, =)$ by putting

$$Z' = \{(n, z) \mid p(n) \text{ is defined and } p(n) \in [z = z]\}$$

and

$$[(n, z) =' (m, z)] \;=\; \begin{cases} \{n\} \wedge [z = z'] & \text{if } n = m \\ \emptyset & \text{otherwise} \end{cases}$$

Recall that $\{n\} \wedge [z = z']$ is $\{\langle n, a\rangle \mid a \in [z = z']\}$, where $\langle -, - \rangle$ is a recursive bijection $\mathbb{N}^2 \to \mathbb{N}$.

The object Z' comes with maps $Z' \stackrel{\pi_1}{\to} N$ and $Z' \stackrel{\pi_2}{\to} Z$ such that every $\phi_Y : Y \to Z$ factors through some $\phi'_Y : Y \to Z'$ which has the property that if one regards $Y = (Y_n)_{n \in \mathbb{N}}$ as an object of $\mathcal{E}ff/N$, ϕ'_Y is a map over N.

We have therefore a cocone for the λ-filtered diagram of sub-projectives of X, regarded as objects of $\mathcal{E}ff/N$, with vertex the object $Z' \stackrel{\pi_1}{\to} N$. Since the diagram is in the image (under ∇_N) of a λ-filtered diagram in Sets/$\mathbb{N}$ and ∇_N preserves λ-filtered colimits, its colimit is the projective X (as object of $\mathcal{E}ff/N$), and there is a unique mediating map $X \to Z'$ over N. But then the composite $X \to Z$ is the unique mediating map for the original cocone of the ϕ_Y's. ∎

It is worth noting that this result also applies to other realizability toposes based on partial combinatory algebras A, provided (for the implication ii)⇒i)) one replaces ω by $|A|$.

So, we are led to study the preservation of λ-filtered colimits by ∇. The first two results in this direction are easy, and folklore facts. Recall that there is a full subcategory Ass of *assemblies* in $\mathcal{E}ff$ which is reflective and such that ∇ factors through Ass. Ass can be described as follows: objects are pairs (X, E) where X is a set and $E : X \to \mathcal{P}(\mathbb{N})$; morphisms $(X, E) \to (Y, F)$ are functions $f : \{x \in X \mid E(x) \neq \emptyset\} \to Y$ with the property that for some partial recursive function p it holds that whenever $n \in E(x)$ then $p(n)$ is defined and an element of $F(f(x))$ (one says that p *tracks* f, as before). The factorization ∇ : Sets $\to$ Ass sends X to (X, E_∇) where $E_\nabla(x) = \mathbb{N}$ for all $x \in X$.

Clearly, if ∇ : Sets $\to \mathcal{E}ff$ preserves λ-filtered colimits then so does ∇ : Sets $\to$ Ass.

Proposition 1.3 ∇ : Sets $\to$ Ass *does not preserve filtered colimits.*

Proof. Let $e \mapsto [e] : \mathbb{N} \to \mathcal{P}_{\rm fin}(\mathbb{N})$ be a bijective coding of finite subsets of $\mathbb{N}$. Let A be the assembly $(\mathbb{N}, E)$ where $E(n) = \{e \,|\, n \in [e]\}$. Then for any finite subset $[e]$ of $\mathbb{N}$ there is a map of assemblies $\nabla([e]) \to A$, tracked by the function which is constant e; and this system of maps is clearly a cocone for the diagram of ∇'s of finite subsets of $\mathbb{N}$ and inclusions between them. But there is no mediating map: $\nabla(\mathbb{N}) \to A$. ∎

Proposition 1.4 $\nabla : \mathrm{Sets} \to \mathrm{Ass}$ *preserves ω_1-filtered colimits.*

Proof. Easy. ∎

Theorem 1.5 *The functor $\nabla : \mathrm{Sets} \to \mathcal{E}\!f\!f$ does not preserve ω_1-filtered colimits.*

Proof. Let D be the ω_1-filtered diagram of countable subsets of ω_1 and inclusions between them; clearly, in Sets, the cocone $D \to \omega_1$ is colimiting. We shall see that $\nabla(D) \to \nabla(\omega_1)$ is not colimiting in $\mathcal{E}\!f\!f$.

Recall the necessary ingredients of the construction of an ω_1-*Aronszajn tree* (see [3] for the full story). If α is a countable ordinal and $s, t : \alpha \to \omega$, we write $s \sim t$ if the set $\{\xi \in \alpha \,|\, s(\xi) \neq t(\xi)\}$ is finite. If $s \sim t$, let $d(s,t)$ be the cardinality of this set.

It is possible to construct a sequence $\{s_\alpha : \alpha \in \omega_1\}$ such that for each α, s_α is a 1-1 function from α into ω, and such that for $\alpha < \beta$, $s_\alpha \sim (s_\beta {\restriction} \alpha)$.

Let T^* consist of all injective functions $s : \alpha \to \omega$, defined on some countable α, such that $s \sim s_\alpha$. Note that for each $\alpha \in \omega_1$, the set $L_\alpha = \{s \in T^* \,|\, \mathrm{dom}(s) = \alpha\}$ is countable.

Equip T^* with the structure of an object of $\mathcal{E}\!f\!f$, by defining

$$[s = t] \;=\; \begin{cases} \emptyset & \text{if } \mathrm{dom}(s) \neq \mathrm{dom}(t) \\ \{n \,|\, d(s,t) \leq n\} & \text{otherwise} \end{cases}$$

Clearly, if $n \in [s = t]$ and $m \in [t = u]$ then $m + n \in [s = u]$, so this is a well-defined equality relation. $(T^*, =)$ is a uniform object since $0 \in \bigcap_{s \in T^*} [s = s]$.

For each $\alpha \in \omega_1$ let $\phi_\alpha : \alpha \to T^*$ be defined by

$$\phi_\alpha(\beta) = s_\alpha {\restriction} \beta$$

Then for each pair $\alpha < \alpha'$ in ω_1 we have that

$$d(s_\alpha, s_{\alpha'} {\restriction} \alpha) \in \bigcap_{\beta \in \alpha} [\phi_\alpha(\beta) = \phi_{\alpha'}(\beta)]$$

which means that the functions ϕ_α and $\phi_{\alpha'}\lceil\alpha$ define the same morphism from $\nabla(\alpha)$ to $(T^*,=)$ in $\mathcal{E}ff$; we shall denote this morphism also by ϕ_α.

If $A \subset \omega_1$ is a countable set, let $\phi_A : A \to T^*$ be the restriction of ϕ_α to A, where $\alpha = \sup\{\beta+1 \,|\, \beta \in A\}$. Clearly then, the system $\{\phi_A : \nabla(A) \to (T^*,=) \,|\, A \subset \omega_1 \text{ countable}\}$ defines a cocone on $\nabla(D)$ with vertex $(T^*,=)$. I claim that this cocone does not factor through $\nabla(\omega_1)$.

Suppose, to the contrary, that there is a morphism $\Phi : \nabla(\omega_1) \to (T^*,=)$ such that for each $\alpha \in \omega_1$, $\Phi\circ\nabla(\iota_\alpha) = \phi_\alpha$, where ι_α is the inclusion of α in ω_1. Then $\Phi : \omega_1 \to T^*$ has the property that for every α there is an $n \in \omega$ such that

$$n \in \bigcap_{\beta\in\alpha} [\phi_\alpha(\beta) = \Phi(\beta)]$$

Then there must be a number n such that the set

$$A_n \;=\; \{\alpha \in \omega_1 \,|\, n \in \bigcap_{\beta\in\alpha} [\phi_\alpha(\beta) = \Phi(\beta)]\}$$

is *unbounded* in ω_1. Fix such an n for the rest of the proof. If $\alpha < \alpha'$ are elements of A_n, then

$$2n \in \bigcap_{\beta\in\alpha} [\phi_\alpha(\beta) = \phi_{\alpha'}(\beta)]$$

So for each $\beta < \alpha$ there are at most $2n$ ordinals $\xi \in \beta$ such that $s_\alpha(\xi) \neq s_{\alpha'}(\xi)$; it follows that $2n+1 \in [s_\alpha = s_{\alpha'}\lceil\alpha]$.

However, this is a contradiction once we have proved the following

Claim 1. Let A be unbounded in ω_1; then there exist, for each $k \in \omega$, elements $\alpha < \alpha'$ of A such that $d(s_\alpha, s_{\alpha'}\lceil\alpha) \geq k$.
Proof of Claim 1: first observe that if $A \subseteq \omega_1$ is unbounded, then for each $\xi \in \omega_1$ there is at least one n such that the set

$$A_{\xi,n} \;=\; \{\alpha \in A \,|\, \alpha > \xi \text{ and } s_\alpha(\xi) = n\}$$

is unbounded.

Claim 2. Let A be unbounded. Then for each $\eta \in \omega_1$ there is a $\xi > \eta$ such that there are n, m with $n \neq m$ and both $A_{\xi,n}$ and $A_{\xi,m}$ unbounded.
Proof of Claim 2: suppose Claim 2 is false; then by the remark preceding it, there is $\eta \in \omega_1$ such that for each $\xi > \eta$ there is exactly one n such that $A_{\xi,n}$ is unbounded. Then for every $\xi > \eta$ there is a $\beta_\xi \in A$ such that for all $\alpha, \alpha' \in A$ that are $\geq \beta_\xi$, $s_\alpha(\xi) = s_{\alpha'}(\xi)$. But then the function $\xi \mapsto s_{\beta_\xi}(\xi)$ is easily seen to be a 1-1 function from $\{\xi \,|\, \eta < \xi\}$ to ω, which is impossible.

Proof of Claim 1, continued: we construct, for each $k \in \omega$, sequences $(\xi_1, \dots, \xi_k)$ and $((n_1, m_1), \dots, (n_k, m_k))$ with $\xi_1 < \dots < \xi_k < \omega_1$, $n_i, m_i \in \omega$ such that $n_i \neq m_i$ and the sets

$$\begin{aligned} A_{\vec{\xi},\vec{n}} &= \{\alpha \in A \,|\, \alpha > \xi_k \text{ and } \forall i \leq k(s_\alpha(\xi_i) = n_i)\} \\ B_{\vec{\xi},\vec{m}} &= \{\alpha \in A \,|\, \alpha > \xi_k \text{ and } \forall i \leq k(s_\alpha(\xi_i) = m_i)\} \end{aligned}$$

are both unbounded.

For $k = 1$ simply apply Claim 2. Inductively, suppose $(\xi_1, \dots, \xi_k)$ and $((n_1, m_1), \dots, (n_k, m_k))$ have been defined satisfying the conditions. Apply Claim 2 with $A = A_{\vec{\xi},\vec{n}}$ and $\eta = \xi_k$. One finds $\xi_{k+1} > \xi_k$ and $a \neq b$ such that both $A_{\xi_{k+1},a}$ and $A_{\xi_{k+1},b}$ are unbounded.

If $B_{\vec{\xi},\xi_{k+1},\vec{m},a}$ is unbounded, let $n_{k+1} = b, m_{k+1} = a$. If $B_{\vec{\xi},\xi_{k+1},\vec{m},b}$ is unbounded, let $n_{k+1} = a, m_{k+1} = b$. If neither of these two, let $n_{k+1} = a$ and pick m_{k+1} arbitrary, such that $B_{\vec{\xi},\xi_{k+1},\vec{m},m_{k+1}}$ is unbounded. ∎

Remark. Echoing the remark following the proof of theorem 1.2, it is worth noting that theorem 1.5 holds for every realizability topos based on a partial combinatory algebra A, whatever its cardinality; since the object $(T^*, =)$ can be constructed in every such topos.

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