Erratum to "Ordered Partial Combinatory Algebras"

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To our regret, the paper "Ordered Partial Combinatory Algebras" [1], contains a mistake, which we correct here. The flaw concerns the definition of *computational density* (definition 3.5), which appeared in section 3.3, page 451. This definition is too rigid, and, as a consequence, Lemma 3.6 on page 452 is false. We will first give the correct definition of computational density and show how the computationally dense maps give rise to geometric morphisms of triposes. Then we state the correct version of lemma 3.6, and its proof.

Definition 3.5. (***Computational Density***) A morphism $f : \mathbb{B} \to \mathbb{A}$ is said to be *computationally dense* (cd) iff there exists an element $m \in \mathbb{A}$ such that the following condition holds:

$$\forall a \in \mathbb{A} \exists b \in \mathbb{B} \forall b' \in \mathbb{B} : a \bullet f(b')\downarrow \Rightarrow bb'\downarrow \quad \& \; m \bullet f(bb') \leq a \bullet f(b') \qquad \text{(cd)}$$

Consider a morphism $f : \mathbb{B} \to T\mathbb{A}$ in **OPCA**. Then f induces a geometric morphism of triposes:

$$\begin{tikzcd} I(\mathbb{A}) \arrow[r, shift right, "\hat{f}"'] \arrow[r, phantom, "\perp"] & I(\mathbb{B}) \arrow[l, shift right, "\bar{f}"'] \end{tikzcd}.$$

We define the arrows $\bar{f}$ and $\hat{f}$ as

$$\bar{f}(\beta) = \bigcup_{b \in \beta} f(b), \qquad \hat{f}(\alpha) = \{b \in \mathbb{B} \mid m \bullet f(b) \subseteq \alpha\},$$

Now we state the right version of the lemma:

Lemma 3.6. Suppose we have a geometric morphism

$$\begin{tikzcd} I(\mathbb{A}) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & I(\mathbb{B}) \arrow[l, shift right, "f^*"'] \end{tikzcd}.$$

Then there is a map $f : \mathbb{B} \to T\mathbb{A}$ such that $\bar{f} \dashv\vdash f^*$, and f is computationally dense.

Proof. As has already essentially been shown by Pitts, putting $f(b) = f^*(\downarrow(b))$ is the only choice we have, since this gives $f^*(\beta) \dashv\vdash \bigcup_{b\in\beta} f(b) = \bar{f}(\beta)$, because f^*, as a left adjoint, preserves existential quantification. Again from Theorem 3.4, it follows that this is a morphism in **OPCA+**.

Now consider the counit of the adjunction, $\bar{f}f^* \vdash Id$. This means that there is a realizer m such that for all $\alpha \in T(\mathbb{A})$, and for all $x \in \bar{f}f^*(\alpha), mx \in \alpha$. Now take any $a \in \mathbb{A}$, and put $D = \{b' \in B | a \bullet f(b')\downarrow\}$. Then

$$\bar{f}(\downarrow(b')) \vdash_{b'\in D} \downarrow(a \bullet f(b')),$$

and transposing along the adjunction we get

$$\downarrow(b') \vdash_{b'\in D} f^*(\downarrow(a \bullet f(b'))).$$

So there is a $b \in \mathbb{B}$ such that for all $b' \in D : bb'\downarrow$ and $bb' \in f^*(\downarrow(a \bullet f(b')))$. It follows that for all $b' \in D$: $f(bb') \subseteq \bar{f}f^*(\downarrow(a \bullet f(b')))$. Apply m to get $m \bullet f(bb') \subseteq a \bullet f(b')$.

$\square$

It is worth noting that this relaxed version of the lemma does not require the axiom of choice, as did the previous version.

Finally, we remark that the relaxation of the notion of computational density has no consequences for other results in the paper, except for the one mentioned above.

References

[1] P.J.W. Hofstra and J. van Oosten. Ordered partial combinatory algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 134:445–463, 2003.