Realizability and Independence of Premiss – a note by Jaap van Oosten, June 15, 2004

Independence of premiss is the axiom scheme

$$\forall x[(\neg A(x) \to \exists y B(x,y)) \to \exists y(\neg A(x) \to B(x,y))]$$

The principle is underivable in **HA**, since it is inconsistent with ECT_0. However, **HA** is closed under the derived rule: if $\mathbf{HA} \vdash \neg A \to \exists y B(y)$, then $\mathbf{HA} \vdash \neg A \to B(n)$, for some natural number n.

A variation is the principle IP_0:

$$\forall x(Ax \vee \neg Ax) \wedge (\forall x Ax \to \exists y By) \;\to\; \exists y(\forall x Ax \to By)$$

Proposition 0.1 IP_0 *is provable in* $\mathbf{HA} + \mathrm{MP} + \mathrm{ECT}_0$.

Proof. Reason in **HA**. Suppose $\forall x(Ax \vee \neg Ax)$. Then by ECT_0 there is a total recursive function n such that $\forall x(Ax \leftrightarrow nx = 0)$.

Suppose $(\forall x Ax \to \exists y By)$, that is $(\forall x(nx = 0) \to \exists y By)$, then again by ECT_0 there is m such that $(\forall x(nx = 0) \to m0 \text{ defined } \wedge B(m0))$. Let k be an index of a partial recursive function, such that for a pair $\langle a, b\rangle$:

$$k\langle a, b\rangle = \mu x.[ax \neq 0 \vee T(b, 0, x)]$$

(T the Kleene T-predicate) Since n is total we have $\neg\neg(\exists x(nx \neq 0) \vee \forall x(nx = 0))$, so $\neg\neg(\exists x(nx \neq 0) \vee m0 \text{ defined})$; so $\neg\neg(k\langle n, m\rangle \text{defined})$, therefore $k\langle n, m\rangle$ defined by MP. Now: if $n(k\langle n, m\rangle) \neq 0$ we have $\neg\forall x Ax$; and if $T(m, 0, k\langle n, m\rangle)$ we have $\forall x Ax \to B(U(k\langle n, m\rangle))$ (where U is the result extraction function). In both cases, $\exists y(\forall x Ax \to By)$, as desired. ∎

Another variation is the propositional version of IP, $\mathrm{IP}_\vee$:

$$(\neg A \to B \vee C) \to ((\neg A \to B) \vee (\neg A \to C))$$

$\mathrm{IP}_\vee$ is not derivable in the intuitionistic propositional calculus IPC, but **HA** is closed under the corresponding derived rule (as follows immediately from the rule for IP).

A formula $\Phi(p_1, \ldots, p_n)$ of IPC with propositional variables $p_1, \ldots, p_n$ is called *effectively realizable* if there is a partial recursive function F such that, whenever $A_1, \ldots, A_n$ are sentences of arithmetic and $N_1, \ldots, N_n$ are the Gödel numbers of $A_1, \ldots, A_n$, then $F(N_1, \ldots, N_n)$ is defined and realizes $\Phi(A_1, \ldots, A_n)$. Not much is known about the set of effectively realizable propositional formulas: examples by Rose and Ceitin show that it differs from the set of IPC-provable formulas, even if one asks F to be constant.

Proposition 0.2 $\mathrm{IP}_\vee$ *is not effectively realizable.*

Proof. It is convenient to assume that our coding of pairs and recursive functions is such that $\langle 0, 0\rangle = 0$ and $0{\cdot}x = 0$ for all x ($a{\cdot}b$ denotes the result of applying the a-th partial recursive function to b); then 0 realizes every true negative sentence. Let $A(f)$ be the sentence $\forall x \exists y T(f,x,y)$ and let $B(f)$ and $C(f)$ be negative sentences, expressing "there is an x on which f is undefined, and the least such x is even" (respectively, odd). Suppose there is a total recursive function F such that for every f, $F(f)$ realizes

$$(\neg A(f) \to B(f) \vee C(f)) \;\to\; ((\neg A(f) \to B(f)) \vee (\neg A(f) \to C(f)))$$

Choose, by the recursion theorem, an index f of a partial recursive function of two variables, such that:

$f{\cdot}(g,x) = 0$ if there is no $w \leq x$ witnessing that $F(S^1_1(f,g)){\cdot}g$ is defined, or if x is the least such witness, and *either* $(F(S^1_1(f,g)){\cdot}g)_0 = 0$ and x is even, *or* $(F(S^1_1(f,g)){\cdot}g)_0 \neq 0$ *and* x is odd;
$f{\cdot}(g,x)$ is undefined in all other cases.

Then for every g we have:

1. $F(S^1_1(f,g)){\cdot}g$ is defined. For otherwise, $f{\cdot}(g,x) = 0$ for all x, hence $S^1_1(f,g)$ is total, so g realizes

$$\neg A(S^1_1(f,g)) \to B(S^1_1(f,g)) \vee C(S^1_1(f,g))$$

2. If $(F(S^1_1(f,g)){\cdot}g)_0 = 0$ then the first number on which $S^1_1(f,g)$ is undefined is odd, so $C(S^1_1(f,g))$ holds;

3. If $(F(S^1_1(f,g)){\cdot}g)_0 \neq 0$ then $B(S^1_1(f,g))$ holds.

Now let, again by the recursion theorem, g be chosen such that for all y:

$$g{\cdot}y = \begin{cases} \langle 1,0\rangle & \text{if } (F(S^1_1(f,g)){\cdot}g)_0 = 0 \\ \langle 0,0\rangle & \text{if } (F(S^1_1(f,g)){\cdot}g)_0 \neq 0 \end{cases}$$

Then g is a realizer for $\neg A(S^1_1(f,g)) \to [B(S^1_1(f,g)) \vee C(S^1_1(f,g))]$.

However, it is easy to see that $F(S^1_1(f,g)){\cdot}g$ makes the wrong choice. ∎

Addendum (26-08-04).

Theorem 0.3 *The formula*

$$\Phi(p) \equiv ((\neg\neg p \to p) \to p \vee \neg p) \to \neg\neg p \vee \neg p$$

is not effectively realizable for Σ^0_2-sentences.

Proof. Let $p(e)$ be the formula $\exists x(e{\cdot}x \uparrow)$. We show that there cannot be a total recursive function F such that for all e, F realizes $\Phi(p(e))$. First a few easy preliminary observations:

a) Suppose $p(e)$ is false. Then every number realizes $\neg\neg p(e) \to p(e)$ and every number realizes $\neg p(e)$, so a number k realizes

$$(\neg\neg p(e) \to p(e)) \to p(e) \vee \neg p(e)$$

if and only if k codes a total recursive function such that $(k{\cdot}x)_0 = 1$ for all x.

b) Suppose $p(e)$ is true. Then every number realizes $\neg\neg p(e)$, and a number m realizes $\neg\neg p(e) \to p(e)$ if m codes a total function and for all x, $e{\cdot}(m{\cdot}x)_0$ is undefined and $(m{\cdot}x)_1$ realizes this fact; hence k realizes

$$(\neg\neg p(e) \to p(e)) \to p(e) \vee \neg p(e)$$

if and only if for all such m, $(k{\cdot}m)_0 = 0$ and $(k{\cdot}m)_1 = \langle a, b\rangle$ where $e{\cdot}a$ is undefined and b realizes this fact.

Now assume, for a contradiction, that F is a total recursive function such that for all e, $F(e)$ realizes $\Phi(p(e))$.
We define by the recursion theorem, a code e of a partial recursive function of 2 variables as follows:
We reserve $Y(e,k)$ for the least computation (if any) which witnesses that $F(S^1_1(e,k)){\cdot}k$ is defined and $(F(S^1_1(e,k)){\cdot}k)_0 \neq 0$.

Let $e{\cdot}(k,x) = 0$ if not $(Y(k,e) < x)$. If $Y(e,k) < x$, put 0 if for some $m \leq Y(e,k)$, $x = (m{\cdot}0)_0$; and undefined else.

One checks that this is a valid definition. Now with e as just defined, again apply the recursion theorem to find a code k such that:

$k{\cdot}m = \langle 1, 0\rangle$ if not $(Y(k,e) < m)$. Otherwise, output $\langle 0, m{\cdot}0\rangle$.

First, I claim that $Y(e,k)$ exists. For otherwise, $e{\cdot}(k,x) = 0$ always and $k{\cdot}m = \langle 1, 0\rangle$ always, so $\neg p(S^1_1(e,k))$ holds and k realizes the premiss of $\Phi(p(S^1_1(e,k)))$ by remark a); so we should have that $F(S^1_1(e,k)){\cdot}k$ should realize $\neg\neg p \vee \neg p$; contradiction.

Since $(F(S^1_1(e,k)){\cdot}k)_0 \neq 0$, hence $Y(e,k)$ exists, we see that $e{\cdot}(k,x)$ is only defined for at most finitely many $x \geq Y(e,k)$. So $p(S^1_1(e,k))$ is true, and we will get a contradiction with the assumption on F (since it clearly makes the wrong choice), if we can show that k realizes the premiss of $\Phi(p(S^1_1(e,k)))$.

Suppose m realizes $\neg\neg p(S^1_1(e,k)) \to p(S^1_1(e,k))$. Then by b), certainly $(m{\cdot}0)_0$ is defined and $S^1_1(e,k){\cdot}(m{\cdot}0)_0$ is undefined; it follows from the definition of e that m cannot be a number $\leq Y(e,k)$. But if $m > Y(e,k)$, it follows from the definition of k that $k{\cdot}m$ realizes $p(S^1_1(e,k)) \vee \neg p(S^1_1(e,k))$. We conclude that k does realize the required formula. ∎