JAAP VAN OOSTEN. ***Realizability: an introduction to its categorical side.*** Studies in Logic and the Foundations of Mathematics, vol. 152. Elsevier Science, Amsterdam, 2008, 328 pp.

Bravely, the author of this book begins his Preface by quoting Martin Hyland's dictum that the time is not yet ripe for a book on realizability toposes. He then goes on to explain in cogent terms why he disagrees with this dictum: but this reviewer, at least, finds himself uncomfortably agreeing with both Hyland and van Oosten. The time may indeed be ripe (indeed, in some respects it is over-ripe) for a book on realizability toposes: but perhaps it isn't quite ripe for this particular book.

The problem (to quote a favourite saying of the reviewer's) is that the study of realizability toposes is still at the 'stamp-collecting' stage: we have lots of exotic and colourful examples, which we can proudly display in our albums, but what we lack is a general theory which would indicate just where the boundaries of the subject lie.

There is an instructive comparison with the study of Grothendieck toposes: almost the first major result in that subject was Jean Giraud's celebrated theorem showing that Grothendieck's intensional notion of a topos, as a category representable as the category of sheaves on a site, was exactly matched by a purely extensional definition. In contrast, thirty years after Hyland discovered the first example of a realizability topos (and, with a little help from Andrew Pitts and the reviewer, parlayed this into the first intensional definition of a family of realizability toposes), we still have nothing approaching an extensional definition of what a realizability topos is.

With these comments in mind, let us examine what the book contains. Of the 288 pages of text, almost half (139 pages) are devoted to chapter 3 which deals with a single example, Hyland's effective topos. This is preceded by 47 pages on partial combinatory algebras and their categories of assemblies (chapter 1), plus 66 pages on triposes and the toposes one constructs from them (chapter 2), and followed by 36 pages on various extensions and generalizations of the realizability notion.

The reviewer has no quarrel with the strong focus on the effective topos: it is by far the prettiest stamp in our collection, and its remarkable and beautiful features deserve to be better known. Indeed, in the present state of the subject, any general account of it has to focus on this particular example, if only to explain to the reader why the search for a general theory is worth pursuing. Moreover, the author's treatment of the effective topos is very well-judged: everything the general reader needs to know about it is here, with the right degree of emphasis.

Where the reviewer does part company with the author is with regard to the relative prominence of the first two chapters. Whilst the tripos-to-topos construction (for which the reviewer would wish to claim partial credit, though the author awards the whole of it to the third author of the Hyland–Johnstone–Pitts joint paper) was the original access route to realizability toposes, and remains an important one, it has over the years become clear that a much gentler approach is provided by regarding them as the exact completions of their subcategories of assemblies.

The fact that any realizability topos is the exact completion of a category of assemblies does appear in the book (Corollary 2.4.5), but not nearly enough use is made of it. And the reader will search in vain for important properties of categories of assemblies, such as the facts that they are quasitoposes and contain exemplary subobjects (that is, monomorphisms satisfying the 'existence' but not the 'uniqueness' part of the definition of a subobject classifier). Even the fact that every category of assemblies has a natural number object is missing: in fact, only the effective topos is shown to possess this amenity (Proposition 3.1.1).

There are one or two other instances where the author's focus on the effective topos has caused him to miss the fact that he could have proved the same results for general realizability toposes at no extra cost. For example, in section 2.4 he introduces uniform objects, and proves (2.4.7) that the power-object of a uniform object is uniform; and in 3.2.6 he proves that all power-objects in the effective topos are uniform. But in fact it is easy to prove that all injective objects (in particular, all power-objects) in any realizability topos are uniform.

This is, then, a partially flawed account of realizability toposes: to that extent, it is not the account of the subject for which the world is ready. (Another symptom of unreadiness is the rather high density of trivial misprints, though most of them will not detain the reader for long.) But if (as the reviewer does) one accepts the author's contention that it is better to have this material available in book form than not to have it, it is still a book to be warmly welcomed.

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