A Partial Analysis of Modified Realizability

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1 Introduction

A formalized version of Kleene realizability for intuitionistic first-order arithmetic **HA** was axiomatically characterized by Troelstra (see [2](3.2)) as follows: for an arbitrary **HA**-sentence ϕ, $\mathbf{HA} \vdash \exists x(x \text{ realizes } \phi)$ if and only if $\mathbf{HA} + \mathrm{ECT}_0 \vdash \phi$.

Many notions of realizability have been characterized in this fashion: see [2] or [3] for details. For some notions, for example extensional realizability, it is necessary to pass to an extension of **HA**: realizability in **HA** is characterized by deducibility from certain axioms in an extension of **HA**.

The present note is concerned with *modified realizability*, seen as interpretation for **HA**. From semantical considerations (see [4]) it follows that this interpretation can be constructed as a combination of three ingredients:

- i) Kleene realizability;
- ii) Kripke forcing over a 2-element linear order P;
- iii) The Friedman translation ([1]).

This will be shown in section 2. The Friedman translation (in the way we use it) introduces a new propositional constant V; hence we move to an extension

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HA(V) of **HA**. We must then define Kleene realizability and forcing for the extended language. Now let $(\phi)_V$ be the result of the Friedman translation applied to ϕ. We obtain, in **HA**, that the sentence saying that ϕ is modified-realizable, is equivalent to the sentence which says that the statement "$(\phi)_V$ is Kleene-realizable" is forced in P (see section 2).

Therefore, the following programme suggests itself: since we know how to characterize Kleene realizability, if we can characterize also the forcing interpretation, we might be able to put these characterizations together in order to obtain our desired result.

However, in this paper we see that this straightforward approach runs into an obstacle, in a way the author thinks is surprising. A more refined analysis yields a partial result: instead of an equivalence, we only obtain one implication. We formulate a conjecture under which the partial result does give a complete characterization.

2 The theory HA(V), forcing and realizability

HA is the theory of first order intuitionistic arithmetic; we assume function symbols present for each definition of a primitive recursive function, as well as the axioms corresponding to the definitions. There are induction axioms for the full language.

Notation. As in [2](1.3), the symbol T denotes the standard Kleene computation predicate, and U the result-extracting function. We write xy for the result (if any) of the x-th partial recursive function applied to y, so $xy \simeq U(\mu z.T(x,y,z))$; $xy\downarrow$ abbreviates $\exists z T(x,y,z)$.

We assume primitive recursive coding of pairs $\langle \cdot, \cdot \rangle$ with decodings p_0, p_1, satisfying the axioms $x = \langle p_0 x, p_1 x \rangle$, $p_0\langle x, y\rangle = x$, $p_1\langle x, y\rangle = y$.

Furthermore we assume that our coding of pairs and partial recursive functions is done in such a way that $0 = \langle 0, 0\rangle$ and for all x, $0x = 0$.

We use the notation $\Lambda x.t$ for a standard index of the partial recursive function sending x to t.

We refer again to [2](1.3), for details concerning the formalization of elementary recursion theory (including the recursion theorem) in **HA**.

We recall the definition of modified realizability for **HA**: for each **HA**-formula φ, new formulas $x \in D_\varphi$ and $x \text{ mr } \varphi$ ("x modified-realizes ϕ") are defined, where x is a variable not occurring in φ.

Type formulas. The basic type o is the type of natural numbers; if σ and τ are types, then so are $\sigma \to \tau$ and $\sigma \times \tau$. For a type σ and a variable x, we have a formula $x \in \sigma$, defined by induction on σ, as follows.

$$\begin{array}{rcl} x \in \emptyset & \equiv & x = x \\ x \in (\sigma \to \tau) & \equiv & \forall y (y \in \sigma \to xy\downarrow \wedge\, xy \in \tau) \\ x \in \sigma \times \tau & \equiv & p_0 x \in \sigma \wedge p_1 x \in \tau \end{array}$$

For any formula φ, D_φ is the *type* of φ, defined as follows:

$$\begin{array}{rcll} D_P & = & o & \text{if } P \text{ is atomic} \\ D_{\varphi \wedge \psi} & = & D_\varphi \times D_\psi & \\ D_{\varphi \to \psi} & = & D_\varphi \to D_\psi & \\ D_{\forall x \varphi} & = & o \to D_\varphi & \\ D_{\exists x \varphi} & = & o \times D_\varphi & \end{array}$$

Note that D_φ depends on the syntactic structure of φ only.

The formulas $x \text{ mr } \varphi$ are defined as follows:

$$\begin{array}{rcl} x \text{ mr } P & \equiv & P \;\text{ for atomic } P \\ x \text{ mr } \varphi \wedge \psi & \equiv & (p_0 x \text{ mr } \varphi) \wedge (p_1 x \text{ mr } \psi) \\ x \text{ mr } \varphi \to \psi & \equiv & x \in D_{\varphi \to \psi} \wedge \forall y (y \text{ mr } \varphi \to xy \text{ mr } \psi) \\ x \text{ mr } \forall y \varphi & \equiv & x \in D_{\forall y \varphi} \wedge \forall y (xy \text{ mr } \varphi(y)) \\ x \text{ mr } \exists y \varphi & \equiv & p_1 x \text{ mr } \varphi(p_0 x) \end{array}$$

Note that $x \text{ mr } \varphi$ implies $x \in D_\varphi$, for all φ.

Apart from modified realizability we shall also use Kleene realizability, denoted $x \text{ r } \varphi$. See [2](3.2), for details.

Extend the language of **HA** by a propositional constant (0-ary relation symbol) V. The theory $\mathbf{HA}(V)$ extends **HA** in the extended language; the only extra axioms are induction axioms for the full extended language.

We consider the following translations between **HA** and $\mathbf{HA}(V)$:

1. The Friedman translation $(\cdot)_V$ from **HA** to $\mathbf{HA}(V)$: for P atomic, $(P)_V \equiv P \vee V$ and $(\cdot)_V$ commutes with all logical structure.

2. The "internal forcing" translations $0 \Vdash \varphi$ and $1 \Vdash \varphi$ from $\mathbf{HA}(V)$ to **HA**:

   $0 \Vdash P \equiv 1 \Vdash P \equiv P$ if P is an atomic **HA**-formula;

   $0 \Vdash V \equiv \bot$; $1 \Vdash V \equiv \top$;

   $1 \Vdash$ commutes with all logical operations;

   $0 \Vdash$ commutes with all logical operations except $\to$;

   $0 \Vdash \varphi \to \psi \equiv (0 \Vdash \varphi \to 0 \Vdash \psi) \wedge (1 \Vdash \varphi \to 1 \Vdash \psi)$

The reader should note that this is a formalized special case of Kripke forcing (on a 2-element poset).

Furthermore, we extend Kleene realizability to $\mathbf{HA}(V)$ by the atomic clause:

$$x \text{ r } V \equiv V$$

First, we remark that Troelstra's characterization of Kleene realizability goes through without problems. The class of *almost negative* $\mathbf{HA}(V)$-formulas is defined by: every quantifier-free $\mathbf{HA}(V)$-formula is almost negative; if φ is a quantifier-free $\mathbf{HA}$-formula, then $\exists x\varphi$ is almost negative; the almost negative formulas are closed under $\wedge, \rightarrow, \forall$.

ECTV is the axiom scheme:

$$\forall x(A(x) \rightarrow \exists y B(x,y)) \rightarrow \exists z \forall x(A(x) \rightarrow zx\downarrow \wedge B(x,(zx)))$$

where $A(x)$ must be almost negative.

By a trivial adaptation of the methods of [2](3.2) we obtain the following result.

Lemma 2.1 *The following statements hold for an arbitrary* $\mathbf{HA}(V)$*-formula* φ*:*

$$\mathbf{HA}(V) \vdash \varphi \;\Rightarrow\; \mathbf{HA}(V) \vdash \exists x(x \text{ r } \varphi)$$

$$\mathbf{HA}(V) + ECTV \vdash \varphi \leftrightarrow \exists x(x \text{ r } \varphi)$$

$$\mathbf{HA}(V) \vdash \exists x(x \text{ r } \varphi) \;\Leftrightarrow\; \mathbf{HA}(V) + ECTV \vdash \varphi$$

Next, let us note some basic properties of the translations $(\cdot)_V$ and $\Vdash$, which all follow by straighforward induction.

Lemma 2.2 *The following properties hold for* $(\cdot)_V$ *and* $\Vdash$*:*

i) $\mathbf{HA}(V) \vdash V \rightarrow (\varphi)_V$

ii) $\mathbf{HA} \vdash \varphi \;\Rightarrow\; \mathbf{HA}(V) \vdash (\varphi)_V$

iii) $\mathbf{HA}(V) \vdash \varphi \;\Rightarrow\; \mathbf{HA} \vdash 0 \Vdash \varphi$

iv) $\mathbf{HA} \vdash (0 \Vdash \varphi) \rightarrow (1 \Vdash \varphi)$

v) $\mathbf{HA} \vdash \varphi \leftrightarrow (0 \Vdash (\varphi)_V)$

vi) $\mathbf{HA} \vdash (\varphi \leftrightarrow (0 \Vdash \varphi)) \wedge (\varphi \leftrightarrow (1 \Vdash \varphi))$

where in i),ii),v) and vi), φ is an **HA***-formula, and in iii) and iv), φ is an* **HA**(V)*-formula.*

Now we relate modified realizability for **HA** to Kleene realizability for **HA**(V) via $(\cdot)_V$ and $\Vdash$, obtaining the equivalence mentioned in the Introduction:

Lemma 2.3 *For every* **HA***-formula φ the following holds:*

$$\mathbf{HA} \vdash \exists x(x \text{ mr } \varphi) \leftrightarrow \exists x(0 \Vdash x \text{ r } (\varphi)_V)$$

Proof. For every **HA**-formula φ there are **HA**-terms s_φ and t_φ, containing at most the free variables of φ, such that the following are provable in **HA**:

$$\begin{array}{l}\forall x(x \text{ mr } \varphi \to t_\varphi x\downarrow \wedge (0 \Vdash t_\varphi x \text{ r } (\varphi)_V))\\ \forall x(x \in D_\varphi \to t_\varphi x\downarrow \wedge (1 \Vdash t_\varphi x \text{ r } (\varphi)_V))\\ \forall x((0 \Vdash x \text{ r } (\varphi)_V) \to s_\varphi x\downarrow \wedge s_\varphi x \text{ mr } \varphi)\\ \forall x((1 \Vdash x \text{ r } (\varphi)_V) \to s_\varphi x\downarrow \wedge s_\varphi x \in D_\varphi)\end{array}$$

The definition of the terms s_φ and t_φ is by recursion on φ, and completely routine. ∎

Our final result in this section is a characterization of the **HA**(V)-formulas φ for which $\mathbf{HA} \vdash 0 \Vdash \varphi$. Consider the following axiom schemes of **HA**(V)-formulas:

G $(V \to \varphi) \to \varphi$

H $(\varphi \to (\psi \vee V)) \to (\varphi \to \psi) \vee V$

K $\forall x(\varphi(x) \vee V) \to (\forall x \varphi(x)) \vee V$

where all formulas φ, ψ and $\varphi(x)$ are supposed to be **HA**-formulas.

Lemma 2.4

i) *For every instance ϕ of either G, H or K we have*

$$\mathbf{HA} \vdash 0 \Vdash \phi$$

ii) *For every* **HA**(V)*-formula ϕ we have*

$$\mathbf{HA}(V) + G, H, K \vdash \phi \ \leftrightarrow \ (0 \Vdash \phi) \vee ((1 \Vdash \phi) \wedge V)$$

iii) For every $\mathbf{HA}(V)$*-formula* ϕ *we have*

$$\mathbf{HA} \vdash 0 \Vdash \phi \;\Leftrightarrow\; \mathbf{HA}(V) + G, H, K \vdash \phi$$

Proof. Statement i) is a straightforward verification.
Statement ii) is proved by induction on ϕ. By way of example we do the implication case $\phi \to \psi$. By induction hypothesis, $\phi \to \psi$ ie equivalent to

$$[(0 \Vdash \phi) \vee ((1 \Vdash \phi) \wedge V)] \to [(0 \Vdash \psi) \vee ((1 \Vdash \psi) \wedge V)]$$

By propositional logic and the valid implication $0 \Vdash \phi \to 1 \Vdash \phi$, this is equivalent to

$$((0 \Vdash \phi) \to (1 \Vdash \psi)) \wedge ((0 \Vdash \phi) \to ((0 \Vdash \psi) \vee V)) \wedge ((1 \Vdash \phi) \to (V \to (1 \Vdash \psi)))$$

Applying H and G to the second and third conjunct, we get

$$((0 \Vdash \phi) \to (1 \Vdash \psi)) \wedge (((0 \Vdash \phi) \to (0 \Vdash \psi)) \vee V) \wedge ((1 \Vdash \phi) \to (1 \Vdash \psi))$$

which is easily seen to be equivalent to

$$(0 \Vdash \phi \to \psi) \vee ((1 \Vdash \phi \to \psi) \wedge V)$$

In a similar way, the scheme K is applied in the induction step for $\forall$.
Finally, statement iii) follows from i) and ii): suppose $\mathbf{HA} \vdash 0 \Vdash \phi$. It is then immediate from ii) that $\mathbf{HA}(V) + G, H, K \vdash \phi$. The converse follows from i). ∎

3 Failure of a direct approach

Lemmas 2.3, 2.4 and 2.1 suggest that one could axiomatize mr-realizability as follows:

Suppose ϕ is mr-realizable. Then by 2.3,

$$\mathbf{HA} \vdash 0 \Vdash \exists x\, (x \text{ r } (\phi)_V)$$

By 2.4,

$$\mathbf{HA}(V) + G, H, K \vdash \exists x\, (x \text{ r } (\phi)_V)$$

so by 2.1,

$$\mathbf{HA}(V) + G, H, K + ECTV \vdash (\phi)_V$$

Suppose F is a conjunction of instances of ECTV such that $\mathbf{HA}(V) + G, H, K + F \vdash (\phi)_V$. Then $\mathbf{HA}(V) + G, H, K \vdash F \to (\phi)_V$. By 2.4,

$\mathbf{HA} \vdash 0 \Vdash (F \to (\phi)_V)$. Since $1 \Vdash (\phi)_V$ is always true and $0 \Vdash (\phi)_V$ is (in **HA**) equivalent to ϕ, $\mathbf{HA} + (0 \Vdash F) \vdash \phi$. So we obtain

$$\mathbf{HA} + (0 \Vdash ECTV) \vdash \phi$$

The converse could be proved, if one knew that the implication

$$\mathbf{HA} \vdash 0 \Vdash \phi \ \Rightarrow \ \mathbf{HA} \vdash 0 \Vdash \exists x(x \text{ r } \phi)$$

was true.

The following result shows that this is *not* the case, and that the reasoning above is, actually, completely uninformative.

Proposition 3.1

i) $\mathbf{HA}(V) + G \vdash \exists x.x \text{ r } G$

ii) $\mathbf{HA}(V) + G + H + ECTV \vdash V$,
hence $\mathbf{HA} + (0 \Vdash ECTV)$ *is inconsistent;*

iii) *The implication:*
$\mathbf{HA} \vdash 0 \Vdash \phi \ \Rightarrow \ \mathbf{HA} \vdash 0 \Vdash \exists x(x \text{ r } \phi)$
does not hold;

iv) *There is an instance* F *of the scheme* H *such that*
$\mathbf{HA}(V) + G, H, K \nvdash \exists x(x \text{ r } F)$

Proof. i): If $y \text{ r } (V \to \phi)$ with ϕ V-free, then $V \to (y0{\downarrow} \wedge y0 \text{ r } \phi)$, since $V \to 0 \text{ r } V$. An application of G yields $y0{\downarrow} \wedge y0 \text{ r } \phi$. Hence $\Lambda x.x0$ realizes every instance of G.
ii): We reason in $\mathbf{HA}(V) + G, H + ECTV$. The following sentence is (equivalent to) an instance of H:

(1) $\forall e[(\exists x(ex = 0) \to V) \to ((\neg \exists x(ex = 0)) \vee V)]$

Now $\exists x(ex = 0) \to V$ is almost negative; hence ECTV yields

(2) $\exists z \forall e[(\exists x(ex = 0) \to V) \to ze{\downarrow} \wedge$
$(ze = 0 \to \neg \exists x(ex = 0)) \wedge$
$(ze \neq 0 \to V)]$

Take a z satisfying (2). By the recursion theorem, let e be such that

(3) $\forall y(ey \simeq ze)$

Now assume V. Then $\exists x(ex = 0) \rightarrow V$, so (2) gives

(4) $ze\downarrow \wedge (ze = 0 \rightarrow \neg\exists x(ex = 0)) \wedge (zx \neq 0 \rightarrow V)$

Clearly, $ze = 0$ gives a contradiction with (3), hence $ze \neq 0$.

We have proved: $V \rightarrow ze \neq 0$. By G, $ze \neq 0$. Now (3) gives $\neg\exists x(ex = 0)$, so e satisfies the hypothesis of (2). By the conclusion of (2),

(5) V

as claimed.

The inconsistency of $\mathbf{HA} + (0 \Vdash ECTV)$ now follows easily.

iii): Let F be the instance of H, chosen in the proof of ii). Clearly $\mathbf{HA} \vdash 0 \Vdash F$. Also, we have

$$\mathbf{HA}(V) + ECTV + G \vdash F \rightarrow V$$

By i), the theory $\mathbf{HA}(V) + ECTV + G$ is sound for r-realizability, hence

$$\mathbf{HA}(V) + G \vdash \exists x(x \text{ r } (F \rightarrow V))$$

Whence $\mathbf{HA} \vdash 0 \Vdash (\exists x(x \text{ r } (F \rightarrow V)))$, so

$$\mathbf{HA} \vdash [0 \Vdash \exists x(x \text{ r } F)] \rightarrow [0 \Vdash \exists x.x \text{ r } V]$$

Therefore, $\mathbf{HA} \vdash \neg(0 \Vdash \exists x.x \text{ r } F)$.

iv) is clear from iii). ∎

Remark. Just to avoid any confusion: the system $\mathbf{HA}(V) + ECTV + G, H, K$ is as consistent as $\mathbf{HA}$ is; the translation $1 \Vdash (-)$ turns G, H, K into tautologies and $ECTV$ into the scheme ECT_0 of [2].

4 A Partial Characterization

In section 3 we considered the scheme $ECTV$ which characterizes *all* realizable $\mathbf{HA}(V)$-formulas; actually, we are more interested in realizable formulas of the form $(\varphi)_V$. We shall work with the theory $\mathbf{HA}(V)+G$ which is sound for realizability. Moreover,

$$\mathbf{HA}(V) + G \vdash \exists x(x \text{ r } (\varphi)_V) \;\Rightarrow\; \mathbf{HA} \vdash \exists x(x \text{ mr } \varphi)$$

Hence, if we can characterize over $\mathbf{HA}(V) + G$ the realizable formulas of form $(\varphi)_V$ we have a partial result on the modified-realizable formulas.

First, we simplify $x \text{ r } (\varphi)_V$ somewhat. Define, for $\mathbf{HA}$-formulas φ, the formula $x \text{ r}' \varphi$ by: $x \text{ r}' P \equiv P \vee V$, for atomic P; and r' has the same clauses as Kleene realizability.

Lemma 4.1 *For all* **HA***-formulas* φ*,*

$$\mathbf{HA}(V) + G \vdash \exists x(x \text{ r } (\varphi)_V) \leftrightarrow \exists x(x \text{ r}' \varphi)$$

Proof. Trivial. ∎

Definition 4.2 The class of V-*Harrop* formulas is defined as follows: $P \vee V$ is V-Harrop, if P is an atomic **HA**-formula; the V-Harrop formulas are closed under $\wedge$ and $\forall$; and if φ is V-Harrop and ψ is an arbitrary $\mathbf{HA}(V)$-formula, then $\psi \to \varphi$ is V-Harrop.

Note at once, that $\mathbf{HA}(V) \vdash V \to \varphi$, for each V-Harrop formula φ.

Lemma 4.3 *For each* **HA***-formula* φ*, there is an almost negative* V*-Harrop formula* $A(x, \varphi)$ *such that the equivalence*

$(*)$ $\forall x(x \text{ r}' \varphi \leftrightarrow x \in D_\varphi \wedge A(x, \varphi))$

is provable in $\mathbf{HA}(V) + G$*.*

Proof. By induction on φ. If φ is an atomic formula P, let $A(x, \varphi)$ be $P \vee V$. $(*)$ is clear.

For $\varphi \wedge \psi$, let $A(x, \varphi \wedge \psi)$ be $A(p_0 x, \varphi) \wedge A(p_1 x, \psi)$. $(*)$ is clear from induction hypothesis.

For $\varphi \to \psi$ let $A(x, \varphi \to \psi)$ be

$$\forall yz(y \in D_\varphi \wedge A(y, \varphi) \wedge Txyz \ \to A(Uz, \psi))$$

To prove $(*)$, suppose $x \text{ r}' (\varphi \to \psi)$. By induction hypothesis

$$\forall y(y \in D_\varphi \wedge A(y, \varphi) \to xy\downarrow \wedge xy \in D_\psi \wedge A(xy, \psi))$$

Since $V \to A(y, \varphi)$ we have

$$\forall y(y \in D_\varphi \to (V \to xy\downarrow \wedge xy \in D_\psi))$$

so applying G we find $x \in D_{\varphi \to \psi}$; $A(x, \varphi \to \psi)$ also follows. The converse is just as straightforward.

For $\exists y \varphi(y)$ let $A(x, \exists y \varphi(y))$ be $A(p_1 x, \varphi(p_0 x))$. For $\forall y \varphi(y)$ let $A(x, \forall y \varphi(y))$ be $\forall yz(Txyz \to A(Uz, \varphi(y)))$. The proof of $(*)$ in these cases is left to the reader. ∎

Definition 4.4 The following principle will be called CST for "Choice for Subtypes":

$$\text{CST}\left\{\begin{array}{c}\forall x(x\in\sigma\wedge A(x)\to\exists y(y\in\tau\wedge B(y)))\\ \to\\ \exists z(z\in(\sigma\to\tau)\wedge\forall x(x\in\sigma\wedge A(x)\to B(zx)))\end{array}\right.$$

where σ and τ are types, and $A(x)$ and $B(y)$ must be almost negative V-Harrop formulas.

Theorem 4.5

i) $\mathbf{HA}(V)+G\vdash\exists x(x\ \mathrm{r}\ F)$
for every instance F of the principle CST.

ii) $\mathbf{HA}(V)+G+CST\vdash(\varphi)_V\leftrightarrow\exists x(x\ \mathrm{r}'\ \varphi)$
*for every **HA**-formula φ.*

iii) $\mathbf{HA}(V)+G\vdash\exists x(x\ \mathrm{r}'\ \varphi)\Leftrightarrow\mathbf{HA}(V)+G+CST\vdash(\varphi)_V$
*for **HA**-formulas φ.*

Proof. i) Consider an instance of CST. Write $\Psi(z)$ for the formula $z\in(\sigma\to\tau)\wedge\forall x(x\in\sigma\wedge A(x)\to B(zx))$.

Since the formulas $x\in\sigma$, $A(x)$ and $\Psi(z)$ are almost negative, there are, by the standard treatment of realizability in [2](3.2), terms t,s,u such that

$$\begin{array}{l}x\in\sigma\to tx\downarrow\wedge(tx\ \mathrm{r}\ x\in\sigma)\\ A(x)\to sx\downarrow\wedge(sx\ \mathrm{r}\ A(x))\\ \Psi(z)\to uz\downarrow\wedge(uz\ \mathrm{r}\ \Psi(z))\end{array}$$

In particular, since $A(x)$ is V-Harrop, we have $V\to sx\downarrow$.

Suppose $\alpha\ \mathrm{r}\ \forall x(x\in\sigma\wedge A(x)\to\exists y(y\in\tau\wedge B(y)))$.

Let $\zeta=\Lambda x.p_0(\alpha x\langle yx,sx\rangle)$.

I claim $\zeta\in(\sigma\to\tau)$. For suppose $(x\in\sigma)\wedge V$. Then we have $tx\ \mathrm{r}\ x\in\sigma$ and $sx\ \mathrm{r}\ A(x)$, so $\alpha x\langle tx,sx\rangle\downarrow$, and

$$p_1(\alpha x\langle tx,sx\rangle)\ \mathrm{r}\ [\zeta x\in\tau\wedge B(\zeta x)]$$

Since this realized formula is almost negative, it holds. In particular we have: $x\in\sigma\to(V\to\zeta x\downarrow\wedge\zeta x\in\tau)$. Using G we conclude that $\zeta\in(\sigma\to\tau)$.

Now suppose $x\in\sigma\wedge A(x)$. Again we see $B(\zeta x)$. Hence $\Psi(\zeta)$ and therefore $u\zeta\ \mathrm{r}\ \Psi(\zeta)$.

We conclude that $\Lambda\alpha.u\zeta$ realizes the considered instance of CST.

ii) This is proved by induction on φ. CST comes in at the implication step (also at the universal quantification step), where we also use Lemma 4.3.

By induction hypothesis, $(\varphi \to \psi)_V$, which is $(\varphi)_V \to (\psi)_V$, is equivalent to $\forall x(x \text{ r}' \varphi \to \exists y(y \text{ r}' \psi))$. By 4.3 this is equivalent to

$$\forall x(x \in D_\varphi \wedge A(x,\varphi) \to \exists y(y \in D_\psi \wedge A(y,\psi)))$$

Applying CST we find z such that

$$z \in D_{\varphi\to\psi} \wedge \forall x(x \in D_\varphi \wedge A(x,\varphi) \to A(zx,\psi))$$

So $z \text{ r}' (\varphi \to \psi)$. Hence $(\varphi \to \psi)_V$ implies $\exists z(z \text{ r}' (\varphi \to \psi))$; the converse is easy.
iii) If $\mathbf{HA}(V) + G \vdash \exists x(x \text{ r}' \varphi)$ then $\mathbf{HA}(V) + G + CST \vdash (\varphi)_V$ by ii).

Conversely, if $\mathbf{HA}(V) + G + CST \vdash (\varphi)_V$ then by i), $\mathbf{HA}(V) + G \vdash \exists x(x \text{ r } (\varphi)_V)$. By Lemma 4.1, $\mathbf{HA}(V) + G \vdash \exists x(x \text{ r}' \varphi)$. ∎

Corollary 4.6 *For all* **HA***-sentences* ϕ *the following implication holds:*

$$\mathbf{HA}(V) + G + CST \vdash (\phi)_V \;\Rightarrow\; \mathbf{HA} \vdash \exists x(x \text{ mr } \phi)$$

Proof. Assume $\mathbf{HA}(V) + G + CST \vdash (\phi)_V$. By 4.5i), $\mathbf{HA}(V) + G \vdash \exists x(x \text{ r } (\phi)_V)$. By 2.4iii), $\mathbf{HA} \vdash 0 \Vdash \exists x(x \text{ r } (\phi)_V$ so by 2.3, $\mathbf{HA} \vdash \exists x(x \text{ mr } \phi)$. ∎

Let $A(x,\varphi)$ again be the formula defined in 4.3.

Conjecture For any number n and any **HA**-sentence ϕ, if $\mathbf{HA} \vdash \overline{n} \in D_\phi$ and $\mathbf{HA} + G, H, K \vdash A(\overline{n},\phi)$, then $\mathbf{HA} + G \vdash A(\overline{n},\phi)$.

Proposition 4.7 *Under the Conjecture, the implication of Corollary 4.6 is an equivalence.*

Proof. Using 2.4iii), it is easy to see that $\mathbf{HA}(V) + G, H, K$ has the numerical existence property.

Assume $\mathbf{HA} \vdash \exists x(x \text{ mr } \phi)$. We get, using 2.3,2.4,4.1 and 4.3, $\mathbf{HA}(V) + G, H, K \vdash \exists x(x \in D_\phi \wedge A(x,\phi))$. Applying the numerical existence property we find a number n such that $\mathbf{HA}(V) + G, H, K \vdash (\overline{n} \in D_\phi \wedge A(\overline{n},\phi))$

Since $\mathbf{HA}(V) + G, H, K$ is conservative over **HA** we have $\mathbf{HA} \vdash \overline{n} \in D_\phi$. So we can apply the Conjecture which gives $\mathbf{HA}(v) + G \vdash A(\overline{n},\phi)$, from which, by 4.5, we get $\mathbf{HA}(v) + G + CST \vdash (\phi)_V$. ∎

Remarks 1. The characterization over $\mathbf{HA}(V)$+G *cannot* be easily replaced by a characterization over HA using $0 \Vdash (-)$: the principle $0 \Vdash CST$ is inconsistent, by the argument of proposition 3.1: the formulas used in that proof are almost negative V-Harrop formulas, so one can use CST in the same way as ECTV, to obtain the contradiction.

2. One might be tempted to formulate the Conjecture in a simpler way, saying that $\mathbf{HA}(V)+G,H,K$ is conservative over $\mathbf{HA}(V)+G$ w.r.t. almost negative V-Harrop sentences. But this is probably false: $\mathbf{HA}(V)+G,H,K$ proves the almost negative V-Harrop sentence $\forall x[(\neg\neg\exists yTxxy\wedge\forall y(Txxy\to V))\to V]$. But I don't see how this sentence can be proved in $\mathbf{HA}(V)+G$.
3. In $\mathbf{HA}(V)+G+CST$ one can prove the following principle $\mathrm{Pr}(V)$ which states that V is a "prime element" in the Lindenbaum algebra of $\mathbf{HA}(V)+G+CST$:

$$\mathrm{Pr}(V)\quad ((A\to V)\to\exists xBx)\to\exists x((A\to V)\to Bx)$$

which extends the V-translation of the "Independence of Premiss" principle of [2](1.11.6).

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References

[1] H.M. Friedman. Classically and Intuitionistically Provably Recursive Functions. In G.H. Müller and D.S. Scott, editors, *Higher Set Theory*, pages 21–27. Springer (Lecture Notes in Mathematics 669), 1978.

[2] A.S. Troelstra, editor. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Springer (Lecture Notes in Mathematics 344), 1973. With contributions by A.S. Troelstra, C.A. Smoryński, J.I. Zucker and W.A. Howard.

[3] A.S. Troelstra. Realizability. In S.R. Buss, editor, *Handbook of Proof Theory*, pages 407–473. North-Holland, 1998.

[4] J. van Oosten. The modified realizability topos. *Journal of Pure and Applied Algebra*, 116:273–289, 1997.