A Coinductive Definition of Repletion

Jaap van Oosten
Dept. of Mathematics
Utrecht University
The Netherlands
jvoosten@math.uu.nl

We work in an arbitrary topos $\mathcal{E}$, where a subobject Σ of the subobject classifier Ω is given, which contains the truth value *true*. A subobject $A \subseteq B$ is called a Σ-subobject if its classifying map factors through Σ; clearly, the factorization $t : 1 \to \Sigma$ of the generic subobject, classifies Σ-subobjects.

One calls a map $f : X \to Y$ Σ-*equable* if $\Sigma^f : \Sigma^Y \to \Sigma^X$ is an isomorphism. A map $f : X \to Y$ is called Σ-*replete* if it is internally orthogonal to all Σ-equable maps, that is internally, given

$$\begin{array}{ccc} Z & \xrightarrow{g} & W \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

with g Σ-equable, there is a unique fill-in $W \to X$ making both triangles commute.

An object X is called replete if $X \to 1$ is replete. Since any map between replete objects is automatically replete, there is the full subcategory $\mathcal{R}$ of $\mathcal{E}$ of replete objects and replete maps. $\mathcal{R}$ is a reflective subcategory of $\mathcal{E}$, and the reflection functor r has been given by Martin Hyland as: $r(X)$ is the largest subobject A of Σ^{Σ^X} such that $\eta : X \to \Sigma^{\Sigma^X}$ factors through A, and $X \to A$ is Σ-equable.

The following gives a rewrite of this definition in slightly more explicit terms. Note: I confuse between (Σ)-subsets and their classifying functions, writing $x \in f$ for $x \in X$, $f \in \Sigma^X$ e.g. I denote the image of $\eta : X \to \Sigma^{\Sigma^X}$ by $\bar{X}$.

Theorem 0.1 *Define the operator* $\Phi_X : \mathcal{P}(\Sigma^{\Sigma^X}) \to \mathcal{P}(\Sigma^{\Sigma^X})$ *by:*

$$\Phi_X(A) = \begin{array}{l} \{\varphi \in \Sigma^{\Sigma^X} \mid \forall A' \, (\bar{X} \cup A \subseteq A' \subseteq \Sigma^{\Sigma^X} \to \forall \mathcal{U} \in \Sigma^{A'} \\ \qquad\qquad\qquad \varphi \in \mathcal{U} \leftrightarrow \{x \in X \mid \eta(x) \in \mathcal{U}\} \in \varphi)\} \end{array}$$

and let $E(X)$ *be the largest fixed point of* Φ_X*, i.e.* $E(X) = \bigcup\{A \mid A \subseteq \Phi_X(A)\}$*. Then* $E(X)$ *is the repletion of* X*.*

Proof. Φ_X is clearly a monotone operator, so $E(X)$ is well-defined. First, a lemma:

Lemma 0.2 *Suppose $\bar{X} \subseteq A' \subseteq \Sigma^{\Sigma^X}$. Then $X \to A'$ is Σ-equable if and only if for all $\varphi \in A'$ and $\mathcal{U} \in \Sigma^{A'}$:*

$$\varphi \in \mathcal{U} \leftrightarrow \{x \in X \mid \eta(x) \in \mathcal{U}\} \in \varphi$$

Proof. We have the operations $G : \Sigma^X \to \Sigma^{A'}$ and $H : \Sigma^{A'} \to \Sigma^X$ given by $G(U) = \{\varphi \in A' \mid U \in \varphi\}$ and $H(\mathcal{U}) = \{x \in X \mid \eta(x) \in \mathcal{U}\}$. Since $\bar{X} \subseteq A'$, $HG(U) = U$ always holds, so $X \to A'$ is Σ-equable iff always $GH(\mathcal{U}) = \mathcal{U}$, i.e. the equivalence in the statement of the lemma holds. ∎

To conclude the proof the theorem, first note that for $\bar{X} \subseteq A' \subseteq \Sigma^{\Sigma^X}$, $\mathcal{U} \in \Sigma^{A'}$ and $y \in X$, always $\eta(y) \in \mathcal{U}$ iff $y \in \{x \in X \mid \eta(x) \in \mathcal{U}\}$ iff $\{x \in X \mid \eta(x) \in \mathcal{U}\} \in \eta(y)$; hence $\bar{X} \subseteq \Phi_X(\emptyset) \subseteq E(X)$. Thus, using the fixed point property of $E(X)$, for $\varphi \in E(X)$ and $\mathcal{U} \in \Sigma^{E(X)}$, we have that $\varphi \in \mathcal{U}$ iff $\{x \in X \mid \eta(x) \in \mathcal{U}\} \in \varphi$; hence by the lemma, $X \to E(X)$ is Σ-equable.

Now take $A \subseteq \Sigma^{\Sigma^X}$ with $\bar{X} \subseteq A$ and suppose $X \to A$ is Σ-equable. Let $\varphi \in A$, and $\bar{X} \cup A \subseteq A' \subseteq \Sigma^{\Sigma^X}$, $\mathcal{U} \in \Sigma^{A'}$ arbitrary.

Then, since $\bar{X} \subseteq A$ and $\mathcal{U} \cap A \in \Sigma^A$, $\varphi \in \mathcal{U}$ iff $\varphi \in \mathcal{U} \cap A$ iff $\{x \in X \mid \eta(x) \in \mathcal{U} \cap A\} \in \varphi$ iff $\{x \in X \mid \eta(x) \in \mathcal{U}\} \in \varphi$. So $\varphi \in \Phi_X(A)$. We conclude that $A \subseteq \Phi_X(A)$; by definition of $E(X)$ then, $A \subseteq E(X)$.

We conclude that $E(X)$ is the largest subobject of Σ^{Σ^X} with $X \subseteq E(X)$ and $X \to E(X)$ Σ-equable, i.e. $E(X)$ is the repletion of X, as desired. ∎

Note that in fact, for $\bar{X} \subseteq A \subseteq \Sigma^{\Sigma^X}$:

$$A \subseteq \Phi_X(A) \Leftrightarrow X \to A \text{ is } \Sigma\text{-equable} \tag{1}$$

Another remark is, that since $X \subseteq \Phi_X(\emptyset)$ we may put

$$E(X) = \bigcup \{A \mid \bar{X} \subseteq A \subseteq \Phi_X(A)\} \tag{2}$$

Pino Rosolini has given the following characterization of replete objects: an object X is replete if and only if $X \to \Sigma^{\Sigma^X}$ is monic, and

$$\forall B \subseteq \Sigma^{\Sigma^X} (\bar{X} \subseteq B \wedge \forall \mathcal{U}, \mathcal{V} \in \Sigma^B (\mathcal{U} \cap \bar{X} = \mathcal{V} \cap \bar{X} \to \mathcal{U} = \mathcal{V}) \to B \subseteq \bar{X}) \tag{3}$$

This characterization is an easy consequence of theorem 0.1. In fact, suppose X satisfies (3) and let $\bar{X} \subseteq A \subseteq \Phi_X(A)$. Then clearly

$$\forall \varphi \in A \forall \mathcal{U} \in \Sigma^A (\varphi \in \mathcal{U} \leftrightarrow \{x \in X \mid \eta(x) \in \mathcal{U}\} \in \varphi)$$

whence

$$\forall \mathcal{U}, \mathcal{V} \in \Sigma^A (\mathcal{U} \cap \bar{X} = \mathcal{V} \cap \bar{X} \to \mathcal{U} = \mathcal{V})$$

so $A \subseteq \bar{X}$; by (2), $E(X) = \bar{X}$.

For the converse assume $E(X) = \bar{X}$ and let $\bar{X} \subseteq B$ such that

$$\forall \mathcal{U}, \mathcal{V} \in \Sigma^B(\mathcal{U} \cap \bar{X} = \mathcal{V} \cap \bar{X} \to \mathcal{U} = \mathcal{V})$$

Then

$$\forall \mathcal{U} \in \Sigma^B(\mathcal{U} = \{\varphi \in B \,|\, \{x \in X \,|\, \eta(x) \in \mathcal{U}\} \in \varphi\})$$

so by lemma 0.2, $X \to B$ is Σ-equable, hence by (1), $B \subseteq \Phi_X(B)$ so $B \subseteq E(X) = X$. So X satisfies (3).