From sets and types to topology and analysis—towards practicable foundations for constructive mathematics. Edited by Laura Crosilla and Peter Schuster, Oxford Logic Guides, vol. 48. Clarendon Press, 2005.

This book reminded me of another collection of papers, also co-edited by Peter Schuster: ***Reuniting the antipodes—constructive and nonstandard views on the continuum***, Kluwer, 2001. It brought together various non-classical approaches to Analysis. When I reviewed this book for ***Nieuw Archief voor Wiskunde*** I naturally raised the question: have the antipodes been reunited? Of course the answer was *no*: with few exceptions (most notably Erik Palmgren's contribution), the authors stayed firmly within their specialism.

A similar reunification project has led to the present book: like the ***Reuniting*** volume, it is also the proceedings of a workshop in Venice, this one from 2003. And in this case the 'antipodes' are the students of formal foundational systems (and their semantics) on the one hand, and the 'practicioners', the researchers doing 'constructive mathematics' on the other. Accordingly, the papers have been arranged in two parts: Part I—Foundations and Part II—Practice; an Introduction is meant to provide a link.

Indeed, the relationship between the people who study formal systems (whether from the point of view of proof theory, model theory or category theory) and the 'practical' constructivists has been delicate for decades.

I think the problem has two sides. On the one hand, constructivists of various schools (Brouwer, Bishop, different types of computable analysis) were loath to commit themselves to any well-specified theory of sets to work in, following Brouwer's aversion to Logic in general. In the Introduction, the editors mention this point of view, albeit (in my opinion) with rather more respect than it deserves. Can't we, at last, after 100 years agree that Brouwer's quarrels with formal logic (a subject then in its infancy), much similar to Kronecker's hostility towards Cantorian set theory, did not represent anything revolutionary but were *rearguard actions*, the conservative stance of an otherwise brilliant mathematician? In fact, it is hard to disagree with Kreisel's scathing review of the reprinted Brouwer ***Cambridge Lectures on Intuitionism*** from 1946–51 ("This sad little book … ", ***Canadian Philosophical Review***, vol. 2 (1982), pp. 249–251) in which the great Intuitionist completely ignores the work of Kleene, Heyting and Kolmogorov but finds time to spend an entire paragraph to a silly theorem as 'absurdity of absurdity of absurdity is equivalent to absurdity'.

The hostility towards formalism, until recently quite widespread among construcivists, is the more curious if one observes that the majority of them are logicians.

On the other hand however, people who were concerned with the semantics of intuitionistic systems, often topos-theorists, showed a very similar reluctance to back up vague and pretentious claims about category theory providing new 'foundations' and toposes as 'worlds of sets' with precise analyses. Recently, I had occasion to ask a number of reputed topos-theorists the same question: 'is N projective in the free topos?' (This question is answered in the affirmative in Lambek and Scott's ***Introduction to higher order categorical logic***, but for a proof they refer to an unpublished paper by Makkai and work by Friedman and Scedrov, of which it is not clear that it means the same thing) All of them said yes, this was well-known, and easy; on further prodding though, some proved to be mistaken as to what the logical meaning of it was, and none of them could even outline a proof. For such a fundamental question!

So, again, have the antipodes been reunited? Maybe surprisingly, my answer now would be: to some extent, *yes*. Researchers from previously disjoint schools have found connections between each other's work and became interested in other fields. Precise work on models of intuitionistic set theory in toposes, and predicative versions, pioneered by Joyal, Moerdijk and Palmgren, has led to a better understanding of these theories, as well as to a lot of variations on them. Models of IZF, CZF and variations can be compared, embedded into each other, axiomatized. Workers in locale theory and formal topology collaborate. A prominent researcher in Algebraic set theory as Alex Simpson takes an active interest in computable analysis. Bas Spitters, raised in Brouwer's tradition, is now looking at topos-theoretic models. The relatively new program of Intuitionistic Reverse Mathematics, advocated by Ishihara, forces the study of concrete formal systems.

A review such as this one is hardly suitable for discussing individual contributions. Instead, for such a discussion the reader is referred to the Introduction by the editors, who do an admirable job in 'talking them together'.

This Introduction is slightly less commendable where it tries to sketch the history of constructive mathematics through the 20th century. Two developments from which the subject has benefited enormously have been completely left out. The first is topos theory (for all its shortcomings); the second is the rise of Computer Science, in particular the abstract semantics of programming languages, but also the development of proof assistants. I find the failure to mention Computer Science at all very strange, given that two of the contributed papers (Hancock-Setzer and Berger-Seisenberger) are explicitly about programs.

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