Seminar Ultracategories Exercise 3

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Following the exercise, we define $G^+ : \mathsf{Comp}^+ \to \mathsf{Set}^+$ by sending $(X, \mathcal{F})$ to $(X^{\mathrm{disc}}, \mathcal{F}\mid_{\mathrm{disc}})$. Monadicity of G^+ means there is a left adjoint $F^+ \dashv G^+$ and the comparison functor K^T is an equivalence. Since theorem 3.4.4 states that a specific functor is an equivalence, we will show that theorem 3.4.4 can be stated as the monadicity of G^+, by showing the comparison functor corresponds to this equivalence of theorem 3.4.4.

1 Finding the adjunction

First of all, we define the left adjunct $F^+ : \mathsf{Set}^+ \to \mathsf{Comp}^+$, as follows. Let $\beta : \mathsf{Set} \to \mathsf{Comp}$ be the Stone-Čech compactification functor. For an object $(S, \mathcal{F})$ of Set^+, we need to find a compact Hausdorff space and a sheaf on this space. As a first component we can take βS, and let $\mathcal{F}\mid^\beta$ be the unique extension of $\mathcal{F}$ onto βS as follows: A basis for βS is given by sets of the form $U_A = \{\nu \in \beta S \mid A \in \nu\}$ for each $A \subseteq S$. Each sheaf is uniquely specified by their restriction to this basis. Moreover, for all subsets A, B we have that $A \subseteq B$ if and only if $U_A \subseteq U_B$, so we can define $\mathcal{F}\mid^\beta(U_A) = \mathcal{F}(A)$ and $\mathcal{F}\mid^\beta(U_A \subseteq U_B) = \mathcal{F}(A \subseteq B)$. Since $\mathcal{F}$ is a sheaf for the discrete topology on S, we find that $\mathcal{F}\mid^\beta$ is the unique extension of $\mathcal{F}$ onto βS. More explicitly, the singleton sets form a basis of S (with the discrete topology), meaning a sheaf $\mathcal{F}$ on S is uniquely specified by the collection of stalks $\mathcal{F}_s$ for $s \in S$, and we can write each open $U \subseteq \beta S$ as the disjoint union of the smallest basis opens, so we find $\mathcal{F}\mid^\beta(U) = \prod_{s \in S \mid \delta_s \in U} \mathcal{F}_s$.

Given a map $(f : S \to T, \phi : (f^*\mathcal{F}') \to \mathcal{F})$, functoriality of β gives a $\beta f : \beta S \to \beta T$, so it remains to find a map of sheaves $\phi\mid^\beta : (\beta f)^*\mathcal{F}'\mid^\beta \to \mathcal{F}\mid^\beta$. Since βf is the ultraproduct of f, which commutes with small limits, we can set $\phi\mid^\beta$ as the unique extension of ϕ by taking ϕ on every set in the basis of βS.

Next, we show that F^+ is the left adjoint of G^+, i.e. $\mathrm{Hom}((\beta S, \mathcal{F}\mid^\beta), (X, \mathcal{F}')) \simeq \mathrm{Hom}((S, \mathcal{F}), (X^{\mathrm{disc}}, \mathcal{F}'\mid_{\mathrm{disc}}))$. Given $(f : \beta S \to X, \phi : \mathcal{F}\mid^\beta \to f^*\mathcal{F}')$, we can compose with δ to restrict f and ϕ to the elements, respectively stalks, of S for a map from $(S, \mathcal{F})$ to $(X^{\mathrm{disc}}, \mathcal{F}'\mid_{\mathrm{disc}})$. Conversely, given $(g : S \to X^{\mathrm{disc}}, \psi : g^*\mathcal{F}'\mid_{\mathrm{disc}} \to \mathcal{F})$, the unique property of β gives that there is a $\bar{g} : \beta S \to X$ given by sending $\nu \in \beta S$ to $\int_{s \in S} g(s)\,\mathrm{d}\nu$, and similarly for the map of sheaves from $\bar{g}^*\mathcal{F}' \to \mathcal{F}\mid^\beta$, we can extend ψ by taking the ultraproduct, which again commutes with the pullback $\_^*$. These maps are natural since composition with δ and ultraproducts are natural transformations, and they give a bijection since $\bar{g}$ is the unique map such that $\bar{g} \circ \delta = g$, and similarly on the sheaves the integral over δ_s is naturally isomorphic to the identity.

2 Monadicity and theorem 3.4.4

Now that we have the left adjoint F^+ to G^+, monadicity of G^+ is equivalent to having the comparison functor be an equivalence, and we want to show this is equivalent to theorem 3.4.4. Theorem 3.4.4 states that for each compact Hausdorff space X, the construction $G \mapsto \mathcal{F}_G$ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathsf{Set}) \to \mathsf{Shv}(X)$. Taking the dependent sum over $X \in \mathsf{Comp}$, this is the same as an equivalence of categories $\sum_{X \in \mathsf{Comp}} \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathsf{Set})$ to $\sum_{X \in \mathsf{Comp}} \mathsf{Shv}(X)$ that is the identity on the first component. Note that $\sum_{X \in \mathsf{Comp}} \mathsf{Shv}(X)$ is exactly the category Comp^+ of the exercise. So to finish the exercise, it suffices to show that G^+F^+-algebras on Set^+ are equivalent (or even isomorphic) to $\sum_{X\in\mathsf{Comp}} \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathsf{Set})$ in such a way that the comparison functor gets sent to $[\mathsf{id}, \mathcal{F}\_]$.

The objects of the G^+F^+-algebras on Set^+ consists of objects $((S,\mathcal{F}) \in \mathsf{Set}^+, \nu : (\beta S^{\mathrm{disc}}, \mathcal{F} \mid^{\beta} \mid_{\mathrm{disc}}) \to (S,\mathcal{F}))$, such that $\nu\circ\mu = \nu\circ(G^+F^+)(\nu)$ and $\nu\circ\eta = \mathsf{id}$ where $\mu : (G^+F^+G^+F^+)(S,\mathcal{F}) \to (G^+F^+)(S,\mathcal{F})$ is the monad multiplication $G^+ \circ \epsilon_{F^+\_}$, and arrows $f : ((S,\mathcal{F}),\nu) \to ((T,\mathcal{F}'),\xi)$ are given by $f : (S,\mathcal{F}) \to (T,\mathcal{F}')$ such that $f \circ \nu = \xi \circ (G^+F^+)(f)$.

By construction of the adjunction on the first component, the unit $\eta_S : S \to \beta S^{\mathrm{disc}}$ is just given by $s \mapsto \delta_s$, while the counit $\epsilon_X : \beta X^{\mathrm{disc}} \to X$ is given by taking the ultraproduct $\int_{s\in S} s \,\mathrm{d}\mu$. Thus we find that the monad multiplication on the first component comes down to taking the integral of the identity over a given ultrafilter of ultrafilters. Meanwhile, on the sheaves the unit $\eta^+_(S,\mathcal{F}) : S \mid^{\beta} \mid_{\mathrm{disc}} \to \eta_S^*\mathcal{F}$ is given by applying δ to the points at which we take the stalk, and the counit is given by taking the ultraproduct of stals $\int_{s\in S} \mathcal{F}_s \,\mathrm{d}\nu$. By the definition of algebras and the discussion after question 3.4.1, we find that each sheaf $\mathcal{F}$ on a compact Hausdorff space X gives rise to an ultrafunctor $x \mapsto \mathcal{F}_x$.

Thus, if we have an object of the algebra $(S,\mathcal{F},\nu)$, we can send it to the compact Hausdorff space and ultrafunctor $(\beta S, \xi \mapsto \mathcal{F}_{\nu\xi}, \sigma)$, where the unit and multiplication conditions on the first component of ν are equivalent to having σ respect the coherence conditions of ϵ and Δ respectively. Moreover, the conditions on the second component of ν mean that it is equivalent to the identity. Thus, we find an isomorphism between the G^+F^+-algebras on Set^+ and compact Hausdorff spaces X together with ultrafunctors to Set. If we compose this with the comparison functor, we get exactly the pseudo-inverse of $\mathcal{F}\_$ (as in theorem 3.4.4).

Therefore, applying the isomorphism gives that the comparison functor from Comp^+ to G^+F^+-Alg is an equivalence iff $\mathcal{F}\_$ is an equivalence, i.e. G^+ is monadic iff theorem 3.4.4 holds.