Seminar on Logic - 2018/2019. Exercise of the 17th of April with Solution.

We said that, since $y^{op}\colon \mathcal{C}^{op} \hookrightarrow \mathrm{Pro}(\mathcal{C})^{op}$ is the free small filtered cocompletion of $\mathcal{C}^{op}$, there is a bijection between the class of continuous presheaves $\mathrm{Pro}(\mathcal{C})^{op} \to \mathrm{S\scriptstyle ET}$ and the class of presheaves $\mathcal{C}^{op} \to \mathrm{S\scriptstyle ET}$. This bijection is the precomposition with y^{op}.

(a) - (3 points) Prove that this bijection is actually an equivalence of categories, that is, it is fully faithful.

Solution. Let F, G be continuous presheaves $\mathrm{Pro}(\mathcal{C})^{op} \to \mathrm{S\scriptstyle ET}$.

Faithfulness: let $\alpha, \beta\colon F \to G$ be arrows of $\mathrm{S\scriptstyle ET}^{\mathrm{Pro}(\mathcal{C})^{op}}$ such that $\alpha y^{op} = \beta y^{op}$. Let P be an object of $\mathrm{Pro}(\mathcal{C})^{op}$. Then $P = \mathrm{colim}_{(a\in\mathcal{A})} y^{op}_{C_a}$, being $\mathcal{A}$ small and filtered and being $C_a \in \mathcal{C}$ for every $a \in \mathcal{A}$. By continuity and by colimit universal property it is the case that $\alpha(P) = \alpha(\mathrm{colim}_{(a\in\mathcal{A})} y^{op}_{C_a}) = \mathrm{colim}_{(a\in A)} \alpha(y^{op}_{C_a}) = \mathrm{colim}_{(a\in A)} \beta(y^{op}_{C_a}) = \beta(\mathrm{colim}_{(a\in\mathcal{A})} y^{op}_{C_a}) = \beta(P)$. Since P is arbitrary, we conclude that $\alpha = \beta$.

Fullness: let $\alpha\colon F y^{op} \to G y^{op}$ be an arrow in $\mathrm{S\scriptstyle ET}^{\mathcal{C}^{op}}$. If $P \in \mathrm{Pro}(\mathcal{C})$, then $P = \mathrm{colim}_{(a\in\mathcal{A})} y^{op}_{C_a}$, being $\mathcal{A}$ small and filtered and being $C_a \in \mathcal{C}$ for every $a \in \mathcal{A}$. We define $\beta(P) := \mathrm{colim}_{a\in A} \alpha(C_a)$. We verify that this definition does not depend on the choice of the small filtered diagram. Then we see that β is natural $F \to G$. Finally, by definition, it is clear that $\beta(y^{op}_C) = \alpha(C)$ for every $C \in \mathcal{C}$, that is, $\beta y^{op} = \alpha$.

A clearer proof of the fact that the precomposition with:

$$\Gamma\colon \mathrm{S\scriptstyle TONE}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$$

induces an equivalence of categories $\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\mathrm{S\scriptstyle TONE}_{\mathcal{C}})$.

Let $\mathcal{C}$ be a small pretopos.

(b) - (4 points) Without using that $\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is a basis for the coherent topology over $\mathrm{Pro}(\mathcal{C})$ (as we did during the seminar), prove that the precomposition with the fully faithful functor:

$$\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$$

is an equivalence $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{wp}(\mathcal{C}))$, exhibiting its pseudo-inverse. *Hint: use Theorem 6.2.12 and look into the proof of Corollary 7.2.4.*

Solution. There are (at least) two ways of exhibiting a pseudo-inverse: the first one is the precomposition with the functor $\lambda\colon \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}^{wp}(\mathcal{C})$ of Theorem 6.2.12, in which case you basically verify that $\lambda \circ (\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C}))$ and $(\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})) \circ \lambda$ are naturally isomorphic to the identities.

The second one is the following. Since it is the case that:

$$\mathcal{F} \cong \mathrm{Eq}(\mathcal{F}\restriction_{\mathrm{Pro}^{wp}(\mathcal{C})} \lambda(-) \rightrightarrows \mathcal{F}\restriction_{\mathrm{Pro}^{wp}(\mathcal{C})} \lambda(\lambda(-) \times_{(-)} \lambda(-)))$$

for every $\mathcal{F} \in \mathrm{Pro}(\mathcal{C})$ (as we can see in the proof of Corollary 7.2.4), we can consider the functor:

$$\mathrm{Pro}^{wp}(\mathcal{C}) \ni \mathcal{F}' \mapsto \mathrm{Eq}(\mathcal{F}'\lambda(-) \rightrightarrows \mathcal{F}'\lambda(\lambda(-) \times_{(-)} \lambda(-))) \in \mathrm{Pro}(\mathcal{C}),$$

that provides a pseudo-inverse of:

$$\mathrm{Pro}(\mathcal{C}) \ni \mathcal{F} \mapsto \mathcal{F} \circ (\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C}))^{op} = \mathcal{F}\restriction_{\mathrm{Pro}^{wp}(\mathcal{C})} \in \mathrm{Pro}^{wp}(\mathcal{C}).$$

In both cases we are using that λ preserves small filtered limits (Theorem 6.2.12).

(c) - (3 points) Prove that this equivalence restrics to an equivalence:

$$\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\mathrm{Pro}^{wp}(\mathcal{C}))$$

and so conclude the usual equivalence:

$$\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\textsc{Stone}_{\mathcal{C}})$$

induced by $\Gamma\colon \textsc{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$.

Solution. Let $\mathcal{F} \in \mathrm{Pro}(\mathcal{C})$. Since $\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ preserves small filtered limits (Remark 6.2.7), whenever $\mathcal{F}$ is continuous, it is the case that $\mathcal{F} \circ (\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C}))^{op}$ is continuous. Viceversa, if $\mathcal{F} \circ (\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C}))^{op} = \mathcal{F} \restriction_{\mathrm{Pro}^{wp}(\mathcal{C})}$ is continuous, then, since λ preserves small filtered limits, since:

$$\mathcal{F} \cong \mathrm{Eq}(\mathcal{F} \restriction_{\mathrm{Pro}^{wp}(\mathcal{C})} \lambda(-) \rightrightarrows \mathcal{F} \restriction_{\mathrm{Pro}^{wp}(\mathcal{C})} \lambda(\lambda(-) \times_{(-)} \lambda(-)))$$

and since in $\textsc{Set}$ small filtered colimits commute with finite limits, it is also the case that $\mathcal{F}$ is continuous. Therefore the equivalence $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{wp}(\mathcal{C}))$ restricts to an equivalence $\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}^{cont}(\mathrm{Pro}^{wp}(\mathcal{C}))$.

Finally, since $\Gamma\colon \textsc{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$ factors as:

$$(\textsc{Stone}_{\mathcal{C}} \xrightarrow{\Gamma} \mathrm{Pro}(\mathcal{C})) = (\textsc{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C}))$$

(Theorem 6.3.14) and since, as we saw, the precompostition with $\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is an equivalence $\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}^{cont}(\mathrm{Pro}^{wp}(\mathcal{C}))$, it is the case that the precompositon with Γ is an equivalence $\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}^{cont}(\textsc{Stone}_{\mathcal{C}})$.