Hand-in 10

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises. We write $\mathbb{T}$ for some subcategory of the category of topological spaces with the following properties:

a. If $X \in \mathbb{T}$ and U is an open subset of X, then U is an object of $\mathbb{T}$ and the inclusion $U \hookrightarrow T$ is an arrow of $\mathbb{T}$;

b. $\mathbb{T}$ is closed under products;

c. $\mathbb{T}$ is a full subcategory of the category of topological spaces;

d. $\mathbb{R} \in \mathbb{T}$ ($\mathbb{R}$ is the real line with the usual topology).

When considering $\mathbb{N}$ or $\mathbb{Q}$ as topological spaces we consider them with the discrete topology.

Exercise 1. (5 points) In the seminar we defined a Grothendieck topology J on $\mathbb{T}$ by

$$S \in J(T) \Leftrightarrow T = \{U | U \text{ is an open subset of } T, \text{ and } (U \hookrightarrow T) \in S\}.$$

Verify that this indeed defines a Grothendieck topology.

From now on we consider $\mathbb{T}$ with the Grothendieck topology as above.

Exercise 2. (7 points) In the seminar we defined for each topological space X a presheaf $C(X)$ on $\mathbb{T}$ by $C(X)(T) = \mathrm{Cts}(T, X)$ on objects T and by precomposition on arrows. Show that $C(X)$ is a sheaf.

Exercise 3. (4 + 4 points) In the seminar we saw in the proof of Proposition 2.2 by unwinding forcing definitions that for each space W from $\mathbb{T}$ we have $W \Vdash \neg\exists q \in C((Q))(q \in U \wedge q \in L)$ being equivalent to the assertion that for any $\beta : W' \to W$ from $\mathbb{T}$ and continuous $q : W' \to \mathbb{Q}$ not both $(\beta, q) \in L(W')$ and $(\beta, q) \in U(W')$. Similarly, show the following:

a. Both $W \Vdash \exists r \in C(\mathbb{Q})(r \in U)$ and $W \Vdash \exists q \in C(\mathbb{Q})(q \in L)$ holding is equivalent to the assertion that there is an open cover $\{W_i\}$ op W such that for each i there are continuous $q_i, r_i : W_i \to \mathbb{Q}$ with $(W_i \hookrightarrow W, q_i) \in L(W_i)$ and $(W_i \hookrightarrow W, r_i) \in U(W_i)$.

b. The forcing $W \Vdash \forall q, r \in C(\mathbb{Q})(q < r \wedge r \in L \Rightarrow q \in L)$ is equivalent to the assertion that for any $\beta : W' \to W$ and continuous $q, r : W' \to \mathbb{Q}$ if $q(x) < r(x)$ for all $x \in W'$ and $(\beta, r) \in L(W')$ then $(\beta, q) \in L(W')$. (*Hint: You may use that under the isomorphism from Proposition 2.1 the ordering on $C(\mathbb{Q}_{dis})$ becomes the pointwise ordering.*)