Hand-in 10

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises. We write $\mathbb{T}$ for some subcategory of the category of topological spaces with the following properties:

a. If $X \in \mathbb{T}$ and U is an open subset of X, then U is an object of $\mathbb{T}$ and the inclusion $U \hookrightarrow T$ is an arrow of $\mathbb{T}$;

b. $\mathbb{T}$ is closed under products;

c. $\mathbb{T}$ is a full subcategory of the category of topological spaces;

d. $\mathbb{R} \in \mathbb{T}$ ($\mathbb{R}$ is the real line with the usual topology).

When considering $\mathbb{N}$ or $\mathbb{Q}$ as topological spaces we consider them with the discrete topology.

Exercise 1. (5 points) In the seminar we defined a Grothendieck topology J on $\mathbb{T}$ by

$$S \in J(T) \Leftrightarrow T = \{U | U \text{ is an open subset of } T\text{, and } (U \hookrightarrow T) \in S\}.$$

Verify that this indeed defines a Grothendieck topology.

Solution. We verify the requirements from I.1.5. We start with (i). Let $T \in \mathbb{T}$. Then the identity on T lies in $\mathbb{T}$ and therefor also $1_T \in y(T)$. Noting that 1_T is the inclusion of T in itself we now see

$$T \supseteq \bigcup\{U | U \text{ open in } T, (U \hookrightarrow T) \in y(T)\} \supset \mathrm{dom}(1_T) = T$$

so

$$T = \bigcup\{U | U \text{ open in } T, (U \hookrightarrow T) \in y(T)\}$$

and therefor $y(T)$ is a covering of T. This shows that (i) holds.

We now verify (ii). Let $g : T' \to T$ be a $\mathbb{T}$-arrow and $S \in J(T)$. Then we have

$$T = \bigcup\{U | U \text{ open in } T, (U \hookrightarrow T) \in S\}.$$

Consequently

$$T' = \bigcup\{g^{-1}(U) | U \text{ open in } T, (U \hookrightarrow T) \in S\}$$

Note that for U open in T with $(U \hookrightarrow T) \in S$ we have $g^{-1}(U)$ open by g being continuous and for $i : g^{-1}(U) \hookrightarrow T'$ the inclusion we have $\mathrm{im}(g \circ i) \subseteq U$. By property (a) of $\mathbb{T}$ we have that $g^{-1}(U), i \in T'$. Note now that $g|_{g^{-1}(U)} = g \circ i \in \mathbb{T}$ and consequently $g \circ i = (U \hookrightarrow T) \circ g|_{g^{-1}(U)} \in S$. We see $i \in (y(g))^{-1}(S)$. We now find

$$T' \supseteq \bigcup\{V | V \text{ open in } T', (V \hookrightarrow T') \in (y(g))^{-1}(S)\} \supseteq \bigcup\{g^{-1}(U) | U \text{ open in } T, (U \hookrightarrow T) \in S\} = T'$$

and therefor

$$T' = \bigcup\{V | V \text{ open in } T', (V \hookrightarrow T') \in (y(g))^{-1}(S)\}.$$

We conclude that $(y(g))^{-1}(S)$ is a covering of T' which proves (ii).

We finally verify (iii). Let R be a sieve on some $T \in \mathbb{T}$ and let S be a covering on T such that for each $(g : T' \to T) \in S$ we have $(y(g))^{-1} \in J(T')$. Let $x \in T$. Then by S a covering there is $U \subseteq T$ open such that $(U \hookrightarrow T) \in S$ and $x \in U$. Let i be the inclusion of U in T. Then $(y(i))^{-1}(R) \in J(U)$ and therefor there is $V \subseteq T'$ open in U such that $(V \hookrightarrow U) \in (y(i))^{-1}(R)$. Consequently $(V \hookrightarrow T) = i \circ (V \hookrightarrow U) \in R$. We see that x lies in $\{V | V \text{ open in } T, (V \hookrightarrow T) \in R\}$. As x was arbitrary in T we conclude that

$$T = \{V | V \text{ open in } T, (V \hookrightarrow T) \in R\}$$

or equivalently that R is a covering of T. This proves (iii) and completes the proof. △

From now on we consider $\mathbb{T}$ with the Grothendieck topology as above.

Exercise 2. (7 points) In the seminar we defined for each topological space X a presheaf $C(X)$ on $\mathbb{T}$ by $C(X)(T) = \mathrm{Cts}(T, X)$ on objects T and by precomposition on arrows. Show that $C(X)$ is a sheaf.

Solution. We have already seen during the seminar that $C(X)$ is a presheaf. It remains to show that $C(X)$ has the sheaf property. Suppose that S is a covering of some $T \in \mathbb{T}$ and suppose that we have a morphism $\tau : S \to C(X)$. We need to construct an extension of τ to $y(T)$. Let $f : T' \to T$ be a $\mathbb{T}$-arrow. We need to define a continuous $f^* : T' \to X$. Let $x \in T'$. Then there is open U in T such that $(U \hookrightarrow T) \in S$ and $f(x) \in U$. Then we have the map $\tau((U \hookrightarrow T)) : U \to X$ and consequently $\tau((U \hookrightarrow T))(f(x)) \in X$. We show now that $\tau((U \hookrightarrow T))(f(x))$ does not depend on U. Let V be another open in T such that $(V \hookrightarrow T) \in S$ and $f(x) \in V$. Then $U \cap V$ is open in both U and V so $(U \cap V \hookrightarrow U), (U \cap V \hookrightarrow V)$ are both $\mathbb{T}$-arrows. We now find by τ being a morphism that

$$\begin{aligned}
\tau((U \hookrightarrow T))(f(x)) &= \tau((U \hookrightarrow T))((U \cap V \hookrightarrow U)(f(x))) \\
&= (\tau((U \hookrightarrow T)) \circ (U \cap V \hookrightarrow U))(f(x)) \\
&= \tau((U \hookrightarrow T) \circ (U \cap V \hookrightarrow U))(f(x)) \\
&= \tau((U \cap V \hookrightarrow T))(f(x)) \\
&= \tau((V \hookrightarrow T))(f(x)).
\end{aligned}$$

We see that $\tau((U \hookrightarrow T))(f(x))$ indeed does not depend on U such that more general we may define a function $f^* : T' \to X$ by $f^*(x) = \tau((U \hookrightarrow T))(f(x))$ where U is an open in T such that $(U \hookrightarrow T) \in S$.

We now show that f^* is continuous. Let $x \in T'$ and let U be open in T such that $(U \hookrightarrow T) \in S$ and $f(x) \in U$. Then by f being continuous $f^{-1}(U)$ is open in T' and also $x \in f^{-1}(U)$. As x was arbitrary in T' we see that T' is covered by $\{f^{-1}(U) | U \text{ open in } T, (U \hookrightarrow T) \in S\}$ and that this cover is open. Also on each $f^{-1}(U)$ we have that f^* restricts to $\tau(U \hookrightarrow T) \circ f|_{f^{-1}(U)}$ which is continuous. As f^* is continuous on each element of an open cover we have that f^* is continuous.

We have now a function $\tau' : y(T) \to C(X)$. It remains to show that this is a morphism. Let $(f : T' \to T) \in y(T)$ and $g : T'' \to T'$. Then for $x \in T''$ there is open U of T such that $(U \hookrightarrow T) \in S$ and $f(g(x)) \in U$. We now compute

$$\tau'(f \circ g)(x) = \tau((U \hookrightarrow T))(f(g(x)))\tau((U \hookrightarrow T))(f(g(x))) = \tau'(f)(g(x)) = (\tau'(f) \circ g)(x)$$

so $\tau'(f \circ g) = \tau'(f) \circ g$. We see that τ' is a morphism which completes the proof. △

Exercise 3. (4 + 4 points) In the seminar we saw in the proof of Proposition 2.2 by unwinding forcing definitions that for each space W from $\mathbb{T}$ we have $W \Vdash \neg \exists q \in C((Q))(q \in U \wedge q \in L)$ being equivalent to the assertion that for any $\beta : W' \to W$ from $\mathbb{T}$ and continuous $q : W' \to \mathbb{Q}$ not both $(\beta, q) \in L(W')$ and $(\beta, q) \in U(W')$. Similarly, show the following:

a. Both $W \Vdash \exists r \in C(\mathbb{Q})(r \in U)$ and $W \Vdash \exists q \in C(\mathbb{Q})(q \in L)$ holding is equivalent to the assertion that there is an open cover $\{W_i\}$ op W such that for each i there are continuous $q_i, r_i : W_i \to \mathbb{Q}$ with $(W_i \hookrightarrow W, q_i) \in L(W_i)$ and $(W_i \hookrightarrow W, r_i) \in U(W_i)$.

Solution. Suppose $W \Vdash \exists r \in C(\mathbb{Q})(r \in U)$ and $W \Vdash \exists q \in C(\mathbb{Q})(q \in L)$. Then there is a cover S_L of W such that for each $(g : W' \to W) \in S$ there is $q \in C(\mathbb{Q})(W')$ such that $W' \Vdash q \in (L \cdot g)$ and there is similar cover S_U of W for U. Let S'_L, S'_U be the corresponding open covers. Let now $\{W_i\}$ be the set of elements of the form $O_L \cap O_U$ where $O_L \in S'_L$ and $O_U \in S'_U$. By S'_L, S'_U covering W it now follows that $\{W_i\}$ is also an open cover and therefor it generates a covering of W. Note also that each $O_L \cap O_U \in S'$ is an open in O_L so $(O_L \cap O_U \hookrightarrow O_L) \in \mathbb{T}$ and consequently $(O_L \cap O_U \hookrightarrow W) = (O_L \hookrightarrow W) \circ (O_L \cap O_U \hookrightarrow O_L) \in S_L$ so $O_L \cap O_U \in S'_L$. By choice of S'_L and S we now find that $S \subseteq S_L$. Similarly $S \subseteq S_U$ so $S \subseteq S_L \cap S_U$. Consider now some W_i. Then $(W_i \hookrightarrow W) \in S \subseteq S_L \cap S_U$ and therefor by choice of S_L, S_U we have continuous $q_i, r_i : W_i \to \mathbb{Q}$ such that $W' \Vdash q_i \in (L \cdot (W_i \hookrightarrow W))$ and $W' \Vdash r_i \in (U \cdot (W_i \hookrightarrow W))$. As we're working in the standard interpretation this is equivalent to $(W_i \hookrightarrow W, q_i) \in L(W_i)$ and $(W_i \hookrightarrow W, r_i) \in U(W_i)$.

Conversely if there is an open cover $\{W_i\}$ op W such that for each i there are continuous $q_i, r_i : W_i \to \mathbb{Q}$ with $(W_i \hookrightarrow W, q_i) \in L(W_i)$ and $(W_i \hookrightarrow W, r_i) \in U(W_i)$ we take S to be the covering of T generated by $\{W_i\}$. Let $g : W' \to W$. Then we have by the presheaf property $(g|_{g^{-1}(W_i)}, (q_i \circ g)|_{g^{-1}(W_i)}) \in L(g^{-1}(W_i))$ for each i and as g is continuous we have that $\{g^{-1}(W_i)\}$ is an open cover of W' and therefor it generated a covering of T' such that by L a sheaf we have $(g, q_i \circ g) \in L(W')$ or equivalently $W' \Vdash q_i \circ g \in (L \cdot g)$. We see that $W \Vdash \exists q \in C(\mathbb{Q})(q \in L)$ holds. We may similarly show $W \Vdash \exists q \in C(\mathbb{Q})(q \in U)$. This completes the proof. △

b. The forcing $W \Vdash \forall q, r \in C(\mathbb{Q})(q < r \wedge r \in L \Rightarrow q \in L)$ is equivalent to the assertion that for any $\beta : W' \to W$ and continuous $q, r : W' \to \mathbb{Q}$ if $q(x) < r(x)$ for all $x \in W'$ and $(\beta, r) \in L(W')$ then $(\beta, q) \in L(W')$. (*Hint: You may use that under the isomorphism from Proposition 2.1 the ordering on* $C(\mathbb{Q})$ *becomes the pointwise ordering.*)

Solution. By the forcing definition for universal quantification we have that $W \Vdash q, r \in C(\mathbb{Q})(q < r \wedge r \in L \Rightarrow q \in L)$ is equivalent to that for each $W'' \xrightarrow{\gamma} W' \xrightarrow{\beta} W$, $q \in C(\mathbb{Q})(W')$ and $r \in C(\mathbb{Q})(W'')$ the forcing $W'' \Vdash q \cdot \gamma < r \wedge r \in L \cdot \beta \cdot \gamma \Rightarrow q \cdot \gamma \in L \cdot \beta \cdot \gamma$ holds. By the forcing definition for implication this is equivalent to that for each $W''' \xrightarrow{\delta} W'' \xrightarrow{\gamma} W' \xrightarrow{\beta} W$, $q \in C(\mathbb{Q})(W')$ and $r \in C(\mathbb{Q})(W'')$ the forcing $W''' \Vdash q \cdot \gamma \cdot \delta < r \cdot \delta \wedge r \cdot \delta \in L \cdot \beta \cdot \gamma \cdot \delta$ implies $W''' \Vdash q \cdot \gamma \cdot \delta \in L \cdot \beta \cdot \gamma \cdot \delta$. By the forcing definition for conjunctions this is again equivalent to that for each $W''' \xrightarrow{\delta} W'' \xrightarrow{\gamma} W' \xrightarrow{\beta} W$, $q \in C(\mathbb{Q})(W')$ and $r \in C(\mathbb{Q})(W'')$ the forcings $W''' \Vdash q \cdot \gamma \cdot \delta < r \cdot \delta$ and $W''' \Vdash r \cdot \delta \in L \cdot \beta \cdot \gamma \cdot \delta$ implying $W''' \Vdash q \cdot \gamma \cdot \delta \in L \cdot \beta \cdot \gamma \cdot \delta$. Unpacking the meaning of $<$ and $\in$ this now becomes that for each $W''' \xrightarrow{\delta} W'' \xrightarrow{\gamma} W' \xrightarrow{\beta} W$, $q \in C(\mathbb{Q})(W')$ and $r \in C(\mathbb{Q})(W'')$ if $(q \circ \gamma \circ \delta)(x) < (r \circ \delta)(x)$ for all $x \in W'''$ and $(\beta \circ \gamma \circ \delta, r \circ \delta) \in L(W''')$ then $(\beta \circ \gamma \circ \delta, q \circ \gamma \circ \delta) \in L(W''')$.

Taking in the above $\gamma = \delta = 1_{W'}$ gives us that for each $\beta : W' \to W$ and continuous $q, r \in W' \to \mathbb{Q}$ if $q(x) < r(x)$ for all $x \in W'$ and $(\beta, r) \in L(W')$ then $(\beta, q) \in L(W')$. Conversely if for each $\beta : W' \to W$ this implication holds then for each $W''' \xrightarrow{\delta} W'' \xrightarrow{\gamma} W' \xrightarrow{\beta} W$, $q \in C(\mathbb{Q})(W')$ and $r \in C(\mathbb{Q})(W'')$ we have that $q \circ \gamma \circ \delta, r \circ \delta \in C(\mathbb{Q})(W''')$ such that by the assumption if $(q \circ \gamma \circ \beta)(x) < (r \circ \delta)(x)$ for all $x \in W''''$ and $(\beta \circ \gamma \circ \delta, r \circ \beta) \in L(W''')$ implies $(\beta \circ \gamma \circ \delta, q \circ \gamma \circ \beta) \in L(W''')$. We have seen above that this is equivalent to $W \Vdash q, r \in C(\mathbb{Q})(q < r \wedge r \in L \Rightarrow q \in L)$ so this completes the proof. △