Homework 11

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Throughout this document we assume that $\mathbb{T}$ satisfies conditions (a)-(d) on page 24.

You are allowed to use these lemmas and other similar results without proof.

- For a formula ϕ with free variables among $A_1, \dots, A_n$

$$\mathrm{Sh}(\mathcal{C}, J) \models \forall x_1 \in A_1, \dots, x_n \in A_n.\ \phi(x_1, \dots, x_n) \quad \leftrightarrow \quad \forall C \in \mathcal{C}.\ \forall \bar{x} \in \bar{A}(C).\ C \Vdash \phi(\bar{x}). \tag{1}$$

- For the site $\mathbb{T}$ and any $x, y \in C(X)(Z)$ we have

$$Z \Vdash x \neq y \quad \leftrightarrow \quad \forall z \in Z.\ x(z) \neq y(z). \tag{2}$$

- Suppose $\mathbb{T}$ contains a terminal object and let $X \in \mathbb{T}$ and $\bar{a} \in \bar{A}(1)$ for some list of sheaves $\bar{A}$ then

$$1 \Vdash \forall x \in y_X.\ \phi(x, \bar{a}) \quad \leftrightarrow \quad X \Vdash \phi(\mathrm{id}_X, \bar{a}|_{X \to 1}) \tag{3}$$

Exercise 1. (2 points) For the internal Dedekind reals $\mathbb{R}$ we define internal intervals in the usual way $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (and similarly for half open/closed intervals). Prove that $\mathrm{Sh}(\mathbb{T}) \models (-\infty, 0) \cup [0, \infty) \neq \mathbb{R}$.

Exercise 2. (3 points) In classical ZF one can prove that there is a bijection $2^{\mathbb{N}} \cong \mathbb{R}$. In HHA one cannot define such a bijection. In fact, something sligthy stronger holds. Prove that if ϕ defines a surjection $\mathbb{R} \to 2^{\mathbb{N}}$ in ZF then HHA $\not\vdash$ "ϕ is a function".

Exercise 3. (6 points) We define for each sheaf F a sentence $\mathrm{dec}_F^{=} := (\forall x, y \in F.\ x = y \vee x \neq y)$. We say that a sheaf F has decidable equality if $\mathrm{dec}_F^{=}$ is valid. What topological property does $\mathrm{dec}_F^{=}$ characterize? More specifically if $X \in \mathbb{T}$ is a space then what topological property for X is equivalent to y_X having decidable equality?

Exercise 4. (4 points) We assume here that $\mathbb{T}$ has a terminal object for convenience. Find a formula that defines the neighborhoods of any Hausdorff space in $\mathbb{T}$. More concretely give for any sheaf F a formula $\phi_F(U, x)$ with free variables of sort $\mathcal{P}(F)$ and F such that for any Hausdorff space $X \in \mathbb{T}$, any point $x \in X$ and any subset $U \subseteq X$ we have

$$1 \Vdash \phi_{y_X}(U, x) \quad \leftrightarrow \quad U \in \mathcal{N}(x). \tag{4}$$