Homework 11 Solutions

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Throughout this document we assume that $\mathbb{T}$ satisfies conditions (a)-(d) on page 24.

You are allowed to use these lemmas and other similar results without proof.

- For a formula ϕ with free variables among $A_1, \ldots, A_n$

$$\mathrm{Sh}(\mathcal{C}, J) \models \forall x_1 \in A_1, \ldots, x_n \in A_n.\ \phi(x_1, \ldots, x_n) \quad \leftrightarrow \quad \forall C \in \mathcal{C}.\ \forall \bar{x} \in \bar{A}(C).\ C \Vdash \phi(\bar{x}). \tag{1}$$

- For the site $\mathbb{T}$ and any $x, y \in C(X)(Z)$ we have

$$Z \Vdash x \neq y \quad \leftrightarrow \quad \forall z \in Z.\ x(z) \neq y(z). \tag{2}$$

- Suppose $\mathbb{T}$ contains a terminal object and let $X \in \mathbb{T}$ and $\bar{a} \in \bar{A}(1)$ for some list of sheaves $\bar{A}$ then

$$1 \Vdash \forall x \in y_X.\ \phi(x, \bar{a}) \quad \leftrightarrow \quad X \Vdash \phi(\mathrm{id}_X, \bar{a}|_{X \to 1}) \tag{3}$$

Exercise 1. (2 points) For the internal Dedekind reals $\mathbb{R}$ we define internal intervals in the usual way $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (and similarly for half open/closed intervals). Prove that $\mathrm{Sh}(\mathbb{T}) \models (-\infty, 0) \cup [0, \infty) \neq \mathbb{R}$.

Solution. Define the relation $\phi \subseteq ((-\infty, 0) \cup [0, \infty)) \times \mathbb{R}$ by

$$\phi(x, y) \quad \leftrightarrow \quad (x < 0 \wedge y = 0) \ \vee \ (x \geq 0 \wedge y = 1).$$

This relation is evidently total and functional hence it defines the function $f : (-\infty, 0) \cup [0, \infty) \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & : \quad x < 0 \\ 1 & : \quad x \geq 0, \end{cases}$$

which is not continuous. If we had $\mathbb{R} = (-\infty, 0) \cup [0, \infty)$ then this would give us a discontinuous function $\mathbb{R} \to \mathbb{R}$ contradicting theorem 3.1.

Grading

- Find the idea. (1pt)
- Write a correct proof. (1pt)

Exercise 2. (3 points) In classical ZF one can prove that there is a bijection $2^{\mathbb{N}} \cong \mathbb{R}$. In HHA one cannot define such a bijection. In fact, something sligthy stronger holds. Prove that if ϕ defines a surjection $\mathbb{R} \to 2^{\mathbb{N}}$ in ZF then HHA $\nvdash$ "ϕ is a function".

Solution. The exponent $2^{\mathbb{N}}$ is the Cantor space in $\mathrm{Sh}(X)$ which is a subspace of the real numbers. Hence the composite $\mathbb{R} \to 2^{\mathbb{N}} \to \mathbb{R}$ would define a continuous function from $\mathbb{R} \to \mathbb{R}$ such that its image is the Cantor space. This is impossible as $\mathbb{R}$ is connected but the Cantor space is not connected.

Grading

- The Cantor space is a subspace of $\mathbb{R}$. (1pt)
- Conclude that the function would be continuous in ZF. (1pt)
- Obtain a contradiction using that the Cantor space is not connected. (1pt)

Exercise 3. (6 points) We define for each sheaf F a sentence $\mathrm{dec}_F^= := (\forall x, y \in F.\ x = y \vee x \neq y)$. We say that a sheaf F has decidable equality if $\mathrm{dec}_F^=$ is valid. What topological property does $\mathrm{dec}_F^=$ characterize? More specifically if $X \in \mathbb{T}$ is a space then what topological property for X is equivalent to y_X having decidable equality?

Solution. A space in $\mathbb{T}$ has decidable equality if and only if it is discrete.

Proof. ($\Rightarrow$) Suppose $\mathrm{Sh}(\mathbb{T}) \models \mathrm{dec}_{y_X}^=$ then for any space $C \in \mathbb{T}$ and any $x, y \in y(X)(C)$ we have $C \Vdash x = y \vee x \neq y$. Let $x \in X$ be arbitrary, let $\Delta_x : X \to X$ be the constant function $\Delta_x(y) = x$ and observe that we have $X \Vdash \mathrm{id}_X = \Delta_x \vee \mathrm{id}_X \neq \Delta_x$. Hence there is an open cover $\{U_i\}_i$ of X such that

$$\mathrm{id}_X|_{U_i} = \Delta_x|_{U_i} \quad \vee \quad U_i \Vdash \mathrm{id}_X|_{U_i} \neq \Delta_x|_{U_i}$$

for all i. In other words

$$\forall y \in U_i.\ y = x \quad \vee \quad \forall y \in U_i.\ y \neq x$$

forall i. There is an i such that $x \in U_i$ and hence we know that $x = y$ for all $y \in U_i$. This implies that $U_i = \{x\}$, and since U_i is open this shows that $\{x\}$ is open.

($\Leftarrow$) We need to show $C \Vdash f = g \vee f \neq g$ for all $C \in \mathbb{T}$ and any $f, g \in y(X)(C)$. Since X is discrete the preimages of single points are open, hence the collection $S = \{f^{-1}(y_1) \cap g^{-1}(y_2) \mid y_1, y_2 \in X\}$ is an open cover of C. For any $y_1, y_2 \in X$ we have $y_1 = y_2 \vee y_1 \neq y_2$ and hence we have either

$$(\forall x \in f^{-1}(y_1) \cap g^{-1}(y_2).\ f(x) = y_1 = y_2 = g(x))$$

or

$$(\forall x \in f^{-1}(y_1) \cap g^{-1}(y_2).\ f(x) = y_1 \neq y_2 = g(x)).$$

□

Grading

- Discrete topology or something clearly equivalent. (2pt)
- Use id_X and Δ_x. (1pt)
- Unfold definitions. (1pt)
- Fibers are open. (1pt)
- Construct S. (1pt)

Exercise 4. (4 points) We assume here that $\mathbb{T}$ has a terminal object for convenience. Find a formula that defines the neighborhoods of any Hausdorff space in $\mathbb{T}$. More concretely give for any sheaf F a formula $\phi_F(U, x)$ with free variables of sort $\mathcal{P}(F)$ and F such that for any Hausdorff space $X \in \mathbb{T}$, any point $x \in X$ and any subset $U \subseteq X$ we have

$$1 \Vdash \phi_{y_X}(U, x) \quad \leftrightarrow \quad U \in \mathcal{N}(x). \tag{4}$$

Solution. Take $\phi_F(U, x) := (\forall y \in F.\ y \in U \vee x \neq y)$. Note that we actually don't quite need our space to be Hausdorff, it suffices to assume that all singletons are closed.

Proof. Let X be such a space, let $x \in X$ and let $U \subseteq X$ then we need to show (4). We have

$$\begin{aligned} 1 \Vdash \forall y \in F.\ y \in U \vee x \neq y \quad &\leftrightarrow \quad X \Vdash \mathrm{id}_X \in U|_! \vee \Delta_x \neq \mathrm{id}_X \\ &\leftrightarrow \quad \exists \text{open cover } \mathcal{U}.\ \forall V \in \mathcal{U}.\ V \Vdash \mathrm{id}_X|_V \in U|_! \vee V \Vdash \Delta_x \neq \mathrm{id}_X|_V \end{aligned}$$

and the condition $V \Vdash \mathrm{id}_X|_V \in U|_! \vee V \Vdash \Delta_x \neq \mathrm{id}_X|_V$ is equivalent to $V \subseteq U \vee \forall p \in V.\ x \neq p$, in other words it is equivalently

$$V \subseteq U \vee V \subseteq X \setminus \{x\}. \tag{5}$$

($\Rightarrow$) If we have such a cover $\mathcal{U}$ of X then there is some V such that $x \in V \in \mathcal{U}$. This implies $V \not\subseteq U \setminus \{x\}$ and hence $V \subseteq U$, which implies $U \in \mathcal{N}(x)$.

($\Leftarrow$) Since $U \in \mathcal{N}(x)$ there is an open V such that $x \in V \subseteq U$ and since singletons are closed there is an open cover of X given by $\mathcal{U} = \{V, X \setminus \{x\}\}$. Each member of this cover obviously satisfies condition (5). □

Grading

- Find a correct formula. (2pt)
- Simplify the validity statement using the provided lemmas. (1pt)
- Complete the argument. (1pt)