Hand-in 2

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises.

Exercise 1. (5 points) Let M be an interpretation of some language $\mathcal{L}(S)$ of signature S. Then for t_1, t_2 terms of type Y with free variables among $\overline{z} : \overline{Z}$ we have that $\{\overline{z} | t_1 = t_2\}^{(M)}$ is represented by the equalizer of

$$\overline{Z}^{(M)} \underset{t_2^{(M)}}{\overset{t_1^{(M)}}{\rightrightarrows}} Y^{(M)} \ .$$

Show that more generally for $\overline{t_1} = (t_{1,1}, t_{1,2}, ..., t_{1,n}), \overline{t_2} = (t_{2,1}, t_{2,2}, ..., t_{2,n})$ finite tupples of terms with free variables among $\overline{z} : \overline{Z}$ such that $t_{i,j}$ is of type Y_j that $\{\overline{z} | \overline{t_1} = \overline{t_2}\}^{(M)}$ is represented by the equalizer of

$$\overline{Z}^{(M)} \underset{\langle t_{2,1}^{(M)}, ..., t_{2,n}^{(M)} \rangle}{\overset{\langle t_{1,1}^{(M)}, ..., t_{1,n}^{(M)} \rangle}{\rightrightarrows}} \overline{Y}^{(M)} \ .$$

Here $\overline{t_1} = \overline{t_2}$ stands for $\bigwedge_i^n (t_{1,i} = t_{2,i})$ and $\overline{Y}^{(M)} = Y_1^{(M)} \times Y_2^{(M)} \times ... \times Y_n^{(M)}$.

Exericse 2. (3 + 7 points) Let T be a theory and M a model of T. Prove the following:

a. Let $p(\overline{z}), q(\overline{z})$ be formulas with free variables among $\overline{z} : \overline{Z}$. Then we have

$$\{\overline{z} | p(\overline{z}) \wedge q(\overline{z})\}^{(M)} \leq \{\overline{z} | p(\overline{z})\}^{(M)}.$$

b. Let now $p(\overline{x}, y)$ be a formula with free variables among $\overline{x} : \overline{X}$ and $y : Y$. Let also $q(y), r(y)$ be formulas with as free variables y or none such that the sequent $q(y) \Rightarrow r(y)$ is in T. Then we have

$$\{\overline{x} | \exists y (p(\overline{x}, y) \wedge q(y))\}^{(M)} \leq \{\overline{x} | \exists y (p(\overline{x}, y) \wedge r(y))\}^{(M)}$$

Exercise 3. (Exercise E.4, 5 points) Prove the following statement which was used in the proof of Lemma 5.1: For an arrow $f : X \to Y$ a monomorphism m representing the subobject graph(f) is an equalizer of the two parallel arrows $f \circ \pi_1, \pi_2 : X \times Y \rightrightarrows Y$.