Hand-in 6

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises.

Exercise 1. (3 + 7 points) In this exercise we consider the category Top of topological spaces and continuous function. We make it into a prop-category by taking for X a topological space, $\mathrm{Prop}_{\mathrm{Top}}(X)$ to be the topology on X ordered by inclusion and for $f : Y \to X$ a continuous function, the pullback operation $f^* : \mathrm{Prop}_{\mathrm{Top}}(X) \to \mathrm{Prop}_{\mathrm{Top}}(Y)$ to be defined by taking the preimage through f. This prop-category has binary meets given by taking the intersection and for each topological space X a top element $\top_X = X$ of $\mathrm{Prop}_{\mathrm{Top}}(X)$. Both are preserved by the pullback operations such that in conclusion Top has finite meets. We will now study if Top has universal quantifiers.

a. Show that for I, X topological spaces the pullback operation $\pi_1^* : \mathrm{Prop}_{\mathrm{Top}}(I) \to \mathrm{Prop}_{\mathrm{Top}}(I \times X)$ induced by the projection $\pi_1 : I \times X \to I$ has a right adjoint $\bigwedge_{I,X} : \mathrm{Prop}_{\mathrm{Top}}(I \times X) \to \mathrm{Prop}_{\mathrm{Top}}(I)$ and describe it explicitly.

b. Show that Top does not have universal quantifiers. *Hint. Suppose for a contradiction that* Top *does have universal quantifiers and consider the case where* $I = \mathbb{R}_{\geq 0}$ *with the regular topology and* $X = \mathbb{R}_{>0}$ *with the discrete topology.*

Exercise 2. (7 points) In this exercise we consider the category Grp of groups and group morphisms. We make it into a prop-category by taking for X a group, $\mathrm{Prop}_{\mathrm{Grp}}(X)$ to be the set of subgroups ordered by inclusion and for $f : Y \to X$ a group morphism, the pullback operation $f^* : \mathrm{Prop}_{\mathrm{Grp}}(X) \to \mathrm{Prop}_{\mathrm{Grp}}(Y)$ to be defined by taking the preimage through f. Show that Grp has equality.

Exercise 3. (3 points) Prove the statement preceding Proposition 5.7.1 that the assignment $(-)^*$ is functorial.