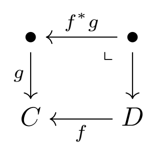
Homework 7 Solutions

March 24, 2024

Exercise 1. (3 points)

Let $\mathcal{C}$ be a small category with pullbacks. Grothendieck topologies are supposed to represent choices of 'covering families'. This suggests the following construction. Assign to each $C \in \mathcal{C}$ the following collection of sieves $J(C) = \{S \subset y(C) \mid S \text{ jointly epi}\}$.

For a family of arrows A (all with the codomain C) we write $\mathrm{Pb}_f(A) = \{f^*g \mid g \in A\}$ for some $f : D \to C$, where f^*g is given by the pullback square



Prove that if for any jointly epi familty A the set $\mathrm{Pb}_f(A)$ is jointly epi then J is a Grothendieck topology on $\mathcal{C}$.

Solution

We need to check the three axioms for Grothendieck topologies.

(i) First of all the maximal sieve $y(C)$ contains id_C, which is epi. This makes $y(C)$ jointly epi and hence $y(C) \in J(C)$.

(iii) Let $R \subset y(C)$ and suppose that there is some $S \in J(C)$ such that for any $(g : D \to C) \in S$ we have $g^*R \in J(D)$. We need to prove that R is jointly epi. Let $f, f' : C \to B$ such that for all $g \in R$ we have $fg = f'g$. Since S is jointly epi it suffices to show that for all $g \in S$ we have $fg = f'g$. Let $g \in S$ and note that for any $h \in g^*R$ we have $gh \in R$ and hence by assumption $fgh = f'gh$. Moreover, by another assumption g^*R is jointly epi and hence $fg = f'g$, which finishes the argument.

(ii) Let $S \in J(C)$ and consider an arrow $f : D \to C$. We note that $Pb_f(S) \subset f^*S$, and hence it suffices to show that $Pb_f(S)$ is jointly epi. This last fact follows from the assumption that jointly epi familis are stable under pullback.

Grading

- Prove (i)
- Prove (ii)
- Prove (iii)

Exercise 2. (7 points)

Prove the uniqueness part of the universal property of coproducts of sheaves. In other words, prove that for sheaves $\mathcal{F}_i$ the morphisms $\sigma_i : \mathcal{F}_i \to \sum_i \mathcal{F}_i$ are jointly epi.

Solution

Let $f, f' : \sum_i \mathcal{F}_i \to \mathcal{G}$ be such that $f\sigma_i = f'\sigma_i$ for all i. We need to show that for all $C \in \mathcal{C}$ and for all $[\alpha] \in \sum_i \mathcal{F}_i(C)$ we have $f_C([\alpha]) = f'_C([\alpha])$. Let $[(\alpha_i : S_i \to \mathcal{F}_i)_i] \in \sum_i \mathcal{F}_i(C)$. Since $\mathcal{G}$ is a sheaf it suffices to show that there is some cover S of C such that for all $g \in S$ we have

$f_C([\alpha])|_g = f'_C([\alpha])|_g$. Choose $S = \bigcup_i S_i$ and let $(g : D \to C) \in S$. There is some i such that $g \in S_i$. Let R_j be the family of sieves on D given by

$$R_j = \begin{cases} \emptyset & \text{if } j \neq i \\ y(D) & \text{if } j = i, \end{cases}$$

and let $\beta_j : R_j \to \mathcal{F}_j$ be given by

$$\beta_j = \begin{cases} ! & \text{if } j \neq i \\ h \mapsto \alpha_j(gh) & \text{if } j = i, \end{cases}$$

where $! : \emptyset \to F_j$ is the unique natural transformation out of the initial presheaf $\emptyset$. Since $y(D) = R_i$ we have $\bigcup_j R_j = y(D) \in J(D)$. Because $g \in S_i$ we have $g^*S_i = y(D)$ and hence $R_j \subseteq g^*S_j \cap R_j$ for all j. Moreover, it is clear that $g^*\alpha_j|_{R_j} = \beta_j|_{R_j}$ and hence $[g^*\alpha] = [\beta]$. The morphism $\beta_i : y(D) \to \mathcal{F}_i$ corresponds through the Yoneda lemma to $\alpha_i(g) \in \mathcal{F}_i$. The map σ_i then sends this to a familty of morphisms which in the i'th component is $(h \mapsto \alpha_i(g)|_h) = \beta_i$ and in any other j'th component is $! = \beta_j$. Therefore we obtain $[g^*\alpha] = [\beta] = \sigma_i(\alpha_i(g))$ and hence

$$f_C([\alpha])|_g = f_C([g^*\alpha]) = f_C(\sigma_i(\alpha_i(g))) = f'_C(\sigma_i(\alpha_i(g))) = f'_C([g^*\alpha]) = f'_C([\alpha])|_g.$$

Grading

- Unfold the definitions
- Recognise the need for a local to global argument
- Find the covering sieve S
- Define β
- Use $g^*S = y(D)$ to conclude $[g^*\alpha] = [\beta]$
- Prove $\sigma_i(\alpha_i(g)) = [g^*\alpha]$
- Complete the argument

Exercise 3. (7 points)

Prove the case for $\vee$ of the Lemma on page 11. Is the induction hypothesis necessary for this case?

Solution

(i) Suppose that $C \Vdash \varphi \vee \psi(\bar{a})$ then there is some cover $S \in J(C)$ such that for all $(h : D \to C) \in S$

$$D \Vdash \varphi(\bar{a}|_h) \vee D \Vdash \psi(\bar{a}|_h)$$

holds. Let $g : B \to C$ be some arrow. We need to show $B \Vdash \varphi \vee \psi(\bar{a}|_g)$. To this end consider the cover g^*S. Let $h \in g^*S$ and note that $gh \in S$. As a result $B \Vdash \varphi(\bar{a}|_{gh}) \vee B \Vdash \psi(\bar{a}|_{gh})$, which is equivalent to $B \Vdash \varphi(\bar{a}|_g|_h) \vee B \Vdash \psi(\bar{a}|_g|_h)$.

(ii) Let $S \in J(C)$ and suppose that for all $(g : D \to C) \in S$ we have $D \Vdash \varphi \vee \psi(\bar{a}|_g)$. Then there are covers $S_g \in J(\text{dom}(g))$ for each $g \in S$, such that for all $h \in S_g$ we have

$$\text{dom}(h) \Vdash \varphi(\bar{a}|_g|_h) \vee \text{dom}(h) \Vdash \psi(\bar{a}|_g|_h).$$

Now let $R = \{gh \mid g \in S, h \in S_g\}$ and note that it is a sieve on C.

We use the transitivity axiom to prove that R is in fact a cover of C. Let $g \in S$ and observe that if $h \in S_g$ then $gh \in R$ and hence $S_g \subseteq g^*R$. Since S_g is a cover this implies (by the transitivity axiom) that g^*R is a cover and hence by transitivity again $R \in J(C)$.

Now to finish of, consider an element $gh \in R$, where $g \in S$ and $h \in S_g$, then we have

$$\mathrm{dom}(h) \Vdash \varphi(\bar{a}|_{gh}) \vee \mathrm{dom}(h) \Vdash \psi(\bar{a}|_{gh}),$$

and hence $C \Vdash \varphi \vee \psi(\bar{a})$.

Grading

For (i)

- Unfold the definitions
- Find the cover g^*S
- Finish the proof

For (ii)

- Unfold the definitions
- Find the sieve R
- Prove that R is a cover
- Complete the proof