Seminar on Models of Intuitionism - Läuchli realizability

Homework 8

April 13, 2017 (due April 20)

Exercise 1. Construct, for each of the following three $\mathcal{L}$-sentences φ, a simple functional θ such that for all proof assignments p, we have $\theta \in p(\varphi)$. You may find it useful to first construct your favorite (intuitionistic) proof tree for φ. *All parts are worth 2 points.*

a. $\varphi = \forall x\,(A(x) \wedge B(x)) \to \forall x\, A(x) \wedge \forall x\, B(x)$.

b. $\varphi = (A \to B) \vee (A \to C) \to (A \to B \vee C)$.

c. $\varphi = \neg\neg(A \vee \neg A)$.

(Capital Latin letters stand for $\mathcal{L}$-formulae, and all their free variables are displayed.)

Exercise 2. Let $P(x)$ be an atomic $\mathcal{L}$-formula, and let Q be an atomic $\mathcal{L}$-sentence.

a. *3 points.* Construct a (not necessarily simple or invariant) functional θ, such that for all proof assignments p, we have
$$\theta \in p[\forall x\,(P(x) \vee Q) \to \forall x\, P(x) \vee Q].$$

b. *1 point.* Show that there does not exist a functional θ such that for all proof assignments p, we have
$$\theta \in p[Q \vee \neg Q].$$

We thus see that, if we consider the set of sentences such that there exist a functional θ that is in $p(A)$ for all p, we end up with something that is neither intuitionistic nor classical logic. In fact, this logic resides strictly in between the latter two.