THE JOURNAL OF SYMBOLIC LOGIC
Volume 57, Number 1, March 1992

LAMBEK'S CATEGORICAL PROOF THEORY AND LÄUCHLI'S ABSTRACT REALIZABILITY

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Introduction. In this paper we give an introduction to categorical proof theory, and reinterpret, with improvements, Läuchli's work on abstract realizability restricted to propositional logic (but see [M1] for predicate logic). Partly to make some points of a foundational nature, we have included a substantial amount of background material. As a result, the paper is (we hope) readable with a knowledge of just the rudiments of category theory, the notions of category, functor, natural transformation, and the like. We start with an extended introduction giving the background, and stating what we do with a minimum of technicalities.

0.1. In three publications [L1, 2, 3] published in the years 1968, 1969 and 1972, J. Lambek gave a categorical formulation of the notion of formal proof in deductive systems in certain propositional calculi. The theory is also described in the recent book [LS]. See also [Sz].

The basic motivation behind Lambek's theory was to place proof theory in the framework of modern abstract mathematics. The spirit of the latter, at least for the purposes of the present discussion, is to organize mathematical objects into *mathematical structures*. The specific kind of structure we will be concerned with is *category*.

In Lambek's theory, one starts with an arbitrary theory in any one of several propositional calculi. One has the (formal) proofs (deductions) in the given theory of entailments $A \Rightarrow B$, with A and B arbitrary formulas. One introduces an equivalence relation on proofs under which, in particular, equivalent proofs are proofs of the same entailment; equivalence of proofs is intended to capture the idea of the proofs being only inessentially different. One forms a category whose objects are the formulas of the underlying language of the theory, and whose arrows from A to B, with the latter arbitrary formulas, are the equivalence classes of formal proofs of $A \Rightarrow B$. Let us refer to the category briefly described thus as the *category of proofs*; it depends, among others, on the particular theory under consideration.

Received November 14, 1990; revised March 20, 1991.

This research was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

0022-4812/92/5701-0015/$04.10

Let us note that Lambek's method has been extended by F. W. Lawvere, and others following Lawvere, to cover predicate logic. In this case the categorical structures involved are more complex; they are special kinds of fibrations. See also [M1] and [M2].

The most important consequence of the laying down of a specific kind of structure is the immediate availability of a corresponding concept of *isomorphism* of two structures of the given kind. The notion of isomorphism serves to eliminate inessential features of the original mathematical objects under consideration. Any two isomorphic structures are considered essentially identical; more precisely, only those properties of structures are considered that are invariant under isomorphism. Thus, the *generalization* involved in passing from concrete mathematical objects to a class of abstract ones entails a *narrowing* of the focus of the theory, by restricting attention to properties invariant under isomorphism. A first test of the success of the concept formation is whether the properties of the original objects in which we were interested in the first place are indeed invariant under isomorphism.

In Lambek's theory, the category structure of the category of proofs contains all the necessary abstract information. To be sure, the category of proofs is far from being an arbitrary category; however, any category isomorphic to it is just as good as the one constructed initially out of formulas and proofs in any of the theorems we want to prove. In fact, the first main step of categorical proof theory is to *characterize* the categories obtained as categories of proofs in terms of invariant (that is, category theoretic) properties. The importance of Lambek's theory is largely due to the fact that this characterization can be made in terms of concepts that are familiar in category theory, thus showing that proof theory does not require a "different kind of category theory". In particular, the category of proofs is a cartesian closed category (with additional properties), a central concept in category theory.

The aim of this paper is to show the fruitfulness of Lambek's categorical proof theory by showing that H. Läuchli's main result in [Lä] on abstract realizability has a simple formulation, via Lambek's theory, within category theory as a representation theorem of a familiar kind. The reformulation is a theorem in pure category theory, and it is proved in a correspondingly abstract fashion, without any explicit reference to concepts of logic. Nevertheless, the steps of the proof are related in important ways to logic: in particular, to Kripke's model theory for intuitionistic logic. Rather than trying to "eliminate" logic, we are aiming at establishing its connections to abstract ("structuralist") mathematics.

The category theoretic reformulation suggests ways of strengthening Läuchli's theorem. We give two such strengthenings; their proofs, in our context, are as natural as that of the original Läuchli theorem. Further interesting questions arise that we do not answer.

0.2. The notion of intuitionistic proof of a first order statement has a well-known informal description due to A. Heyting [He]. It proceeds by induction on the complexity of the statement. From the point of view of classical mathematics, all parts of this description except one can be stated in precise terms. The exceptional part concerns the proofs of atomic statements; it says that a proof of such a statement is just a verification of its truth. What such a verification is depends on the particular context; for example, in the framework of arithmetic, a verification is a pair of computations showing that the two terms of an equality have the same value. It is not clear how to formulate this part of Heyting's description in general but precise terms. Nevertheless, the attempts to circumvent this difficulty and come up with a mathematical definition were particularly useful. Several authors equated provability of atomic statements with their truth, and formalized, in one of several possible ways, the rest of Heyting's description, thus obtaining a definition of "realizability". Such a definition tells us when a certain object "realizes" a given formula; the realizing objects are natural numbers in Kleene's version [IM] and functionals of finite type in Kreisel's (see [Tr]).

Läuchli took an interesting departure. (In explaining Läuchli's idea, we restrict ourselves to propositional logic, although [Lä] deals with predicate logic as well; for an analysis of Läuchli's work on predicate logic in a spirit similar to this paper, see [M1] and [M2].) He assigned to every atomic formula A an arbitrary abstract set $p[A]$ of "proofs" of A. Then, using the clauses of Heyting's description, he extended the function p to associate with every formula A the set $p[A]$ of its "proofs". The "proofs" of composite formulas are, in general, maps. For example, the proofs of the implication $A \to B$, i.e. the elements of $p[A \to B]$, are the functions from $p[A]$ to $p[B]$ (intuitively, to prove $A \to B$ is to have a way of transforming a proof of A into a proof of B); in other words, $p[A \to B] = p[B]^{p[A]}$. We are not repeating the full definition of $p[-]$ here, as it will become explicit in the course of the paper anyway.

(In comparison with, e.g., Kleene realizability, an obvious difference is the lack of an *interpretation* giving a priori meaning to the (atomic) formulas. In fact, we should imagine the meaning of A also provided by the elements of the set $p[A]$. Thus, an element of $p[A]$ is a meaning together with a proof that A is true under that meaning.)

With Läuchli, we ask if there is a completeness theorem for intuitionistic (propositional) logic using the notion of abstract proof introduced. For brevity, let us call an abstract proof-assignment p as described above a *model* (of *pure* intuitionistic propositional logic with a given stock of atomic sentences; for generalizations, see below). The completeness theorem should say that for an arbitrary formula A in intuitionistic propositional logic, A has a proof in the usual sense if and only if, *for all models* $p[-]$, A has an abstract proof with respect to p; that is, $p[A]$ is nonempty. The first remark is that the soundness ("only if") part of this assertion is easy (however, we will have a better view of it soon). The essential "if" part is false as things are at the moment, since for $A = (\neg\neg P) \to P$ we always have $p[A] \neq 0$. Läuchli's decisive idea was to replace *sets* as the values of the assignment p by more complex objects, namely, *sets with a distinguished permutation*. The latter we call $\mathbb{Z}$-*sets* (since they are the same as sets with an action of the additive group $\mathbb{Z}$ of integers). The point is that the operations on sets that underlie the inductive clauses of the definition of p (e.g., the exponentiation of sets used in the clause for implication) have natural counterparts for $\mathbb{Z}$-sets. Let us then talk about a $\mathbb{Z}$-*set-valued model* with Läuchli's replacement in mind. The condition of nonemptiness of $p[A]$ is strengthened to the existence of an *invariant* element of $p[A]$; that is, of an element of $p[A]$ fixed by the distinguished permutation of $p[A]$.

0.2.1. Läuchli's Completeness Theorem (for intuitionistic propositional logic). *A formula is provable in intuitionistic propositional logic iff it has an invariant abstract proof in every $\mathbb{Z}$-set-valued model.*

Essentially as a by-product of our analysis, we improve and generalize Läuchli's theorem. First of all, instead of pure logic, we consider an arbitrary underlying theory consisting of a set of axioms T. Now, the notion of a model of T is just that of a model of the pure calculus in which $p[A]$ has an invariant element for any axiom $A \in T$.

0.2.2. Extended Läuchli's Completeness Theorem. *A formula is provable in a countable theory in intuitionistic propositional logic iff it has an invariant abstract proof in every $\mathbb{Z}$-set-valued model of the theory.*

A theory T has the *disjunction property* (a well-known concept; see, for example, [IM] or [LS]) if for all A and B, $T \vdash A \vee B$ only if either $T \vdash A$ or $T \vdash B$. Our second improvement is the

0.2.3. Uniform Extended Läuchli Completeness Theorem. *Suppose that the countable theory T in intuitionistic propositional logic has the disjunction property. Then there is a $\mathbb{Z}$-set-valued model p of T such that, for any formula A, A is provable in T iff A has an invariant abstract proof in p.*

Since pure logic (the empty theory) has the disjunction property, 0.2.3 indeed strengthens 0.1.1.

Although we have not spelled out the definition of "($\mathbb{Z}$-set-valued) model", we have to mention one peculiarity of it, which is the handling of the (identically) false atom **f**. As usual, our primitives are **f**, **t** (= true), $\wedge$, $\vee$ and $\rightarrow$, negation $\neg A$ being understood as $A \rightarrow \mathbf{f}$. Contrary to a natural expectation, $p[\mathbf{f}]$ is not defined to be the empty ($\mathbb{Z}$-) set; as Läuchli points out, his theorem would become false if we were to do so. Instead, **f** is considered to be just another atomic formula, but all axioms $\mathbf{f} \rightarrow A$ are to have (invariant) proofs. Thus, implicitly, a nonempty theory, with axioms the $\mathbf{f} \rightarrow A$, is brought in even in the case of pure logic. (Läuchli deals with this issue in a more direct manner, which results in an, in our minds at least, inessential strengthening of the notion of "model". From our point of view, with arbitrary axioms allowed, one does not have to say anything at all about **f**.)

0.3. Now let us bring the Lambek theory and the Läuchli theory together. In the Lambek theory, the category of proofs is a *bicartesian closed* (b.c.) category. This and related notions will be explained in §1. To give an idea of the notion, it suffices to say here that a b.c. category has, among others, an exponentiation operation on objects, defined within the category structure, through a so-called universal property. In the category of (formulas and) proofs, the formula $A \rightarrow B$ is the exponential B^A. **Set**, the category of sets and functions, is b.c.; in **Set**, exponentiation is the usual set-exponentiation. Also, $\mathbf{Set}^{\mathbb{Z}}$, the category of $\mathbb{Z}$-sets and equivariant maps between them (respecting the action of $\mathbb{Z}$), is b.c. (and more ...). We have the natural notion of a b.c. functor between b.c. categories: a functor preserving the operations defining "b.c.". For example, a b.c. functor F takes B^A to $(FA)^{FB}$.

Because of Läuchli's difficulties with 'false', we give up the initial object in the definition of "bicartesian closed"; we get what we have chosen to call a *connectionally closed* (c.c.) category. So, certainly, the category of proofs can be construed as a c.c. category [although one should note that, as a category, the c.c. category of proofs is not the same as Lambek's bicartesian closed category of proofs; the two have the same objects, and the arrows come from the same proofs, but in the c.c. category the identification of arrows is less stringent]. We also have c.c. functors.

The main point of contact between the two theories is that *a set-valued or $\mathbb{Z}$-set-valued model is the same as the object function of a connectionally closed functor from the category of proofs to* **Set** (*respectively, to* $\mathbf{Set}^{\mathbb{Z}}$).

Note that, in the definition of "proof assignment" (model), Läuchli does not talk about the effect of his $p[-]$ on anything like proofs (although, after the definition, in [Lä] there is talk about lambda-terms giving rise to (simple) functionals, which is related). As the displayed statement shows, the notion of "proof assignment" has a conceptually very clear and simple definition, by the device of considering the proof assignment as acting not just on formulas, but also on *real* proofs.

As it turns out, we can (essentially) characterize the categories that come up as c.c. categories of proofs as *free* c.c. categories; here "free" is an algebraic notion like that in "free group". Let us call a functor F *weakly full* if, for any objects A and B in its domain, if there are no arrows from A to B, then there are no arrows from FA to FB (in the codomain category) either. Clearly, what is usually called a full functor is also weakly full. If I is any (small) set and $\mathbf{A}$ any category, then $\mathbf{A}^I$ is the usual Cartesian power of $\mathbf{A}$. A category with binary coproducts $A + B$ and a terminal object 1 is said to have the *disjunction property* if whenever there is some arrow $1 \to A + B$, then there is either one of the form $1 \to A$, or one of the form $1 \to B$. The category of proofs of a theory T has the disjunction property just in case T has it in the usual sense.

Finally, our main result can be expressed as the following abstract representation theorem.

0.3.1. THEOREM. *Let* $\mathbf{A}$ *be a countable free c.c. category.*

(a) *There is a weakly full c.c. functor of the form* $\mathbf{A} \to (\mathbf{Set}^{\mathbb{Z}})^I$, *with* I *a countable set.*

(b) *If, in addition,* $\mathbf{A}$ *has the disjunction property, then there is a weakly full c.c. functor of the form* $\mathbf{A} \to \mathbf{Set}^{\mathbb{Z}}$.

To see the connection to the Läuchli theorem and its improvements stated above, note that an invariant element of the $\mathbb{Z}$-set X is the same as an arrow $1 \to X$ in $\mathbf{Set}^{\mathbb{Z}}$, with 1 the terminal (one-element) $\mathbb{Z}$-set.

In §4 we give an application of these results to the definability theory of type-theory that makes no reference to $\mathbb{Z}$-sets or the like.

§1. Basic concepts. We explain the operations that occur in the categories used to represent deductive systems. As mentioned in the Introduction, it is remarkable that these operations are all very familiar and described in the literature (e.g. in [CWM] and also in [LS]). Operations in categories can be introduced in two ways: either via universal properties or as specified operations. Contrary to a prevailing view, the choice between these two modes of concept formation is not just a matter of taste. Both have their definite roles in the theory, and their relations should be stated clearly. The second author acknowledges his debt to G. M. Kelly for his enlightenment on this point; see also [BKP].

In the sequel, **C** will be a fixed category, and all objects and arrows coming into consideration will be in **C**. For objects A and B, $\mathbf{C}(A,B)$ will be the set of arrows from A to B; sets of the form $\mathbf{C}(A,B)$ are also called *hom-sets*, and sometimes denoted as $\mathrm{Hom}_{\mathbf{C}}(A,B)$.

We first discuss at some length the categorical operation of product of two objects (also called *binary product* of objects), and start by presenting it via a universal property. A *product diagram based on* (A,B) is given by an additional object D and two arrows as in $A \xleftarrow[p]{} D \xrightarrow[q]{} B$, which are required to satisfy the well-known universal property: for any $A \xleftarrow[f]{} C \xrightarrow[g]{} B$, there is a unique arrow $C \xrightarrow[h]{} D$, such that the diagram

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\
A & & B \\
& D \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

commutes. This property of (p,q) may be stated by saying that we have an operation

(1) $$\langle\ ,\ \rangle^C = \langle\ ,\ \rangle \colon \mathbf{C}(C,A) \times \mathbf{C}(C,B) \to \mathbf{C}(C,D)$$

(giving h above as $h = \langle f,g \rangle$), one for each object C, such that

(1′) $$\begin{aligned} p \circ \langle f,g \rangle &= f, \\ q \circ \langle f,g \rangle &= g, \qquad [f\colon C \to A,\ g\colon C \to B,\ h\colon C \to D], \\ \langle ph, qh \rangle &= h, \end{aligned}$$

for any f, g and h as indicated. Note, in particular, that the last equality is a consequence of the uniqueness part in the universal property.

Conversely, assuming the fixed data $A \xleftarrow[p]{} D \xrightarrow[q]{} B$, and operations $\langle\ ,\ \rangle^C$ as in (1), one for each object C satisfying the three identities (1′) we have that (p,q) is a product diagram; the point is that the third identity ensures the uniqueness of h in the above formulation of the universal property.

C is said to *have binary products* if it has a product diagram based on (A,B) for any objects A and B. As with any categorical operation defined via a universal property, we have a corresponding notion of functor that preserves the given operation. In the case under discussion, a functor F between two categories preserves binary products if it maps any product diagram (based on (A,B)) to a product diagram (based on (FA,FB)).

The most familiar example of a category with products is **Set**. It should be emphasized that the objects A and B do not determine a unique product diagram $A \leftarrow D \to B$ (indeed, in **Set** it is customary to take $D = A \times B = \{(a,b)\colon a \in A, b \in B\}$ but even this does not determine D uniquely as there are different ways of coding ordered pairs as sets!). It is true, however, that the product diagram based on a fixed pair of objects is determined uniquely up to isomorphism. Thus, the usual terminology that speaks about a product operation is justified not entirely, but only in essence; when we speak about "taking the product of A and B", what we really mean is that we choose one of the many product diagrams based on (A, B).

We have an alternative way of formulating a categorical operation, in particular that of product, as a *specified operation.* This makes product an operation in a literal sense. The category $\mathbf{C}$ is said to have specified binary products if there is specified a function that associates with each pair (A, B) of objects a particular product diagram; let us denote the latter as usual:

$$(2) \qquad A \xleftarrow[\pi_{A,B}]{} A \times B \xrightarrow[\pi'_{A,B}]{} B.$$

Instead of saying that (2) is a product diagram we can formulate an *equational presentation* (of a category with specified products) by saying that with each pair of objects (A, B) we have a diagram as in (2) and an operation $\langle\ ,\ \rangle^{\mathbf{C}}_{A,B}$ as in (1) satisfying (1′); that is,

E3a. $\pi\langle f, g\rangle = f,$

E3b. $\pi'\langle f, g\rangle = g, \qquad [f\colon C \to A, g\colon C \to B, h\colon C \to A \times B],$

E3c. $\langle \pi h, \pi' h\rangle = h,$

where the indices on π, π' and $\langle\ ,\ \rangle$ are omitted (they are recoverable from the "sortings" of the variables given) and composition is indicated by juxtaposition. (The numbering follows [LS].)

Obviously, once we have a category with specified binary products, then we have, by forgetting the additional structure, a category with binary products. Conversely, having a category with binary products, we may pass to a category with specified binary products, by making a simultaneous choice of product diagrams for all parameters involved; of course, this may involve a use of the axiom of choice. This latter step of specifying products (and other operations) is often done tacitly in category theoretical discussions.

From a fundamental point of view, the concept defined via a universal property is preferable, simply because it involves less data. The point is that the category structure alone is enough to carry all the information needed in a category with binary products, or, for that matter, in many other structured categories. This circumstance is a basic point contributing to the conceptual economy of the categorical approach.

The category **Set**, as well as other standard examples such as the category $\mathbf{Set}^G$ of G-sets (with a group G), have binary products. In fact, they have naturally specified products as well (without the axiom of choice), although their specification involves such things as defining what ordered pairs should be (a thing a true category theorist finds distasteful).

With any specified categorical operation comes a notion of morphism between categories having that specified operation: this is a functor that preserves the specified operation (this is more than just preserving the given operation in the sense given above!). Thus, a morphism of categories $\mathbf{C}$ and $\mathbf{D}$ with specified binary products is a functor $F\colon \mathbf{C} \to \mathbf{D}$ that takes the specified operation in the domain category into the specified operation in the codomain category:

$$F(A \times B) = (FA) \times (FB), \qquad F(\pi_{A,B}) = \pi_{FA,FB}, \qquad F(\pi'_{A,B}) = \pi'_{FA,FB}.$$

Notice that in this case we automatically have

$$F\langle r, s\rangle^{C}_{A,B} = \langle Fr, Fs\rangle^{FC}_{FA,FB},$$

in the above notation.

If F is a morphism with respect to a certain specified operation, then it is also a functor preserving that operation for the same categories with the specifications forgotten. The reason for this, in the case of the product operation, is that any (binary) product diagram is isomorphic to the corresponding specified product diagram, and any functor preserves isomorphism; hence, the image of any product diagram is isomorphic to a product diagram (the image of the specified product diagram), and thus it is a product diagram itself.

The converse situation is more complicated. It is possible that we have, for categories $\mathbf{C}$ and $\mathbf{D}$ with specified binary products, a functor $F: \mathbf{C} \to \mathbf{D}$ preserving binary products (unspecified), but no morphism $\mathbf{C} \to \mathbf{D}$ in the specified sense. Nevertheless, we can find, for any categories $\mathbf{C}$ and $\mathbf{D}$ with binary products, equivalent categories $\mathbf{C}'$ and $\mathbf{D}'$ with specified binary products so that there is an essentially (up to equivalence of functor-categories) bijective correspondence between the functors $\mathbf{C} \to \mathbf{D}$ preserving binary products, and the morphisms $\mathbf{C}' \to \mathbf{D}'$. The discussion of the "essential equivalence" of the notions with and without specification would necessarily involve higher-dimensional category theory, and we will not say more about it; let us mention that [BKP] takes up this issue in a systematic manner.

In this paper we will use mostly categories with specified operations, although we will make occasional references to the unspecified version.

In the rest of this section, we enumerate the further categorical operations needed. Since these are quite standard (the reader may find them in [CWM]), we will be brief. In giving the equational presentations, we will use a numbering of identities to match that in [LS].

A *terminal object* T is one for which there is exactly one arrow $A \to T$ for any object A. We use $\mathbf{t}$ for the specified terminal object, and $!_A$ for the unique $A \to \mathbf{t}$. In the equational presentation, we have the single identity

E2. $$\mathbf{f} = !_A \qquad [\mathbf{f}: A \to \mathbf{t}].$$

Coproducts are dual to products. $A \xrightarrow[i]{} D \xleftarrow[j]{} B$ is a *coproduct diagram* if for any $A \xrightarrow[f]{} C \xleftarrow[g]{} B$, there is a unique $h: D \to C$ such that $f = hi$ and $g = hj$. In the (equational) specification of binary coproducts, we write $A + B$ for D, $\kappa_{A,B}$ for i, $\kappa'_{A,B}$ for j, and $[f, g]^{C}_{A,B}$ for h. The equations are

E6a. $\qquad [f, g]\kappa = f,$

E6b. $\qquad [f, g]\kappa' = g, \qquad [f: A \to C, g: B \to C, h: A + B \to C],$

E6c. $\qquad [h\kappa, h\kappa'] = h.$

The *initial object* is dual to the terminal one. The equational specification of the initial object $\mathbf{f}$ is given by the single equation

E5. $$g = \Box_A \qquad [g: \mathbf{f} \to A].$$

Given objects A *and* B, an *exponential diagram based on* (A, B) is one of the form

$$\begin{tikzcd} & D \arrow[dl, "p"'] \arrow[rr, "q"] \arrow[dr, "e"] & & E \\ A & & B & \end{tikzcd}$$

such that (p, q) is a product diagram, and for any

$$\begin{tikzcd} & F \arrow[dl, "r"'] \arrow[rr, "s"] \arrow[dr, "h"] & & C \\ A & & B & \end{tikzcd}$$

with (r, s) a product diagram, there is a unique pair (k, l) of arrows $k: C \to E$ and $l: F \to D$ such that the diagram

$$\begin{tikzcd} & F \arrow[dl, "r"'] \arrow[rr, "s"] \arrow[dr, "h"] \arrow[dd, "l"] & & C \arrow[dd, "k"] \\ A & & B & \\ & D \arrow[ul, "p"] \arrow[ur, "e"'] \arrow[rr, "q"'] & & E \end{tikzcd}$$

commutes. Using specified products (that is, $r = \pi_{A,C}, s = \pi'_{A,C}$, etc.), we must have that $l = \langle \pi_{A,C}, k\pi'_{A,C} \rangle_{A,B}$, and the commutativity condition just stated can be written as

$$e\langle \pi_{A,C}, k\pi'_{A,C} \rangle_{A,B} = h.$$

In the equational presentation, we write B^A for E, the already specified product $(A \times E, \pi_{A,E}, \pi'_{A,E})$ for (D, p, q), $\varepsilon_{A,B}$ for e, and $h^{\sim}$ for k given by h. We obtain two equations, the second ensuring the uniqueness of k (given h).

E4a. $\qquad \varepsilon\langle \pi, h^{\sim}\pi' \rangle = h \qquad [h: A \times C \to B],$

E4b. $\qquad (\varepsilon\langle \pi, k\pi' \rangle)^{\sim} = k \qquad [k: C \to B^A].$

A category (assumed to have binary products) is said to *have exponentials* if it has an exponential diagram based on any pair (A, B) of objects. A category has *specified exponentials* if, in addition to having a specified binary product structure, it has, for any pair (A, B) of objects, a specific exponential B^A, and a specified *evaluation arrow* $\varepsilon_{A,B}: A \times B^A \to B$, with the properties given above. A functor *preserves exponentials* if it takes every exponential diagram into an exponential diagram; the meaning of a morphism preserving specified exponentials should be clear.

This completes the listing of the operations. A category is called *cartesian* if it has binary products and a terminal object, *cartesian closed* if it is cartesian and has exponentials, *bicartesian* if it is cartesian and has binary coproducts and an initial object, *bicartesian closed* if it is bicartesian and cartesian closed, and *connectionally closed* (c.c.) if it is cartesian closed and has binary coproducts.

To each of these kinds of structured category, there corresponds a kind of functor preserving the operations in question. We also have a corresponding notion of morphism for each kind of category with specified operations. We will talk about a *connectionally closed* (c.c.) *functor* in case of a functor preserving the connectionally closed operations without specification, and a *c.c. morphism* for the case of specified operations.

A partially ordered set (poset) $(P, \leq)$ is a category in which the objects are the elements of P; for any $x, y \in P$, there is at most one morphism $x \to y$, and there is one precisely when $x \leq y$. The transitivity law ensures that composition is well-defined; reflexivity ensures the existence of the identity arrows. The antisymmetry law $(x \leq y \;\&\; y \leq x \Rightarrow x = y)$ entails that the only isomorphisms are identities. Note that in a poset, product is the same as meet (greatest lower bound), coproduct as join (least upper bound), terminal object as maximum element, and initial object as minimum element. Exponentiation becomes what in lattice theory is called relative complement. As a consequence, when the above-mentioned types of categories are specialized to posets, familiar notions are obtained. For example, a bicartesian poset is the same as a lattice (with 0 and 1), and a bicartesian closed poset is the same as a Heyting algebra. Let us also note that, in the case of a poset, because only trivial isomorphisms exist, the difference between the concepts with and without specified operations disappears. Also, if (e.g.) $\mathbf{C}$ is a c.c. category with specified operations and $\mathbf{H}$ is a c.c. poset, a c.c. functor from $\mathbf{C}$ to $\mathbf{H}$ is automatically a c.c. morphism.

Many naturally occurring categories fall into one or more of the classes defined above. The most important one is **Set**, the category of all small sets and functions; **Set** is bicartesian closed. A reader who has not seen this material before should contemplate the meaning of the operations in **Set**. Let us point out another example, one that plays a leading role in the main result of the paper. With $\mathbb{Z}$ standing for the additive group of the integers (which is, like any group, a (one-object) category), we consider the category $\mathbf{Set}^{\mathbb{Z}}$ of all functors $\mathbb{Z} \to \mathbf{Set}$, or, in a more familiar wording, the category of all $\mathbb{Z}$-sets, with equivariant maps as arrows. Put still another way, the objects of $\mathbf{Set}^{\mathbb{Z}}$ are sets with a distinguished permutation, and the arrows are mappings between the sets respecting the specified permutations. $\mathbf{Set}^{\mathbb{Z}}$ is a bicartesian closed category; in fact it is a topos (see [J]). Moreover, the forgetful functor $\mathbf{Set}^{\mathbb{Z}} \to \mathbf{Set}$, sending each $\mathbb{Z}$-set to its underlying set, is a bicartesian closed functor (in fact, it is logical, in the sense of topos theory); we leave the verification of this fact to the reader. This fact, of course, shows that the bicartesian closed structure of $\mathbf{Set}^{\mathbb{Z}}$ is closely related to that of **Set**.

§2. The category of proofs. Let $\mathscr{L}$ be a language suitable for propositional logic; that is, a set of atomic propositions (briefly, *atoms*). We will adopt the framework of *negationless* propositional logic; negation will be brought in on the extralogical level. A *formula* is built in the usual way out of atoms, the symbol $\mathbf{t}$ ("true", used as an atom) and the binary connectives $\wedge$, $\vee$, $\to$. Formulas are denoted by letters $A, B, C, \ldots$.

The basic ingredient of the proof theory as considered here, and in [LS], is the formal concept of *entailment*; this replaces the notion of Gentzen sequent. An entailment is a pair of formulas, written in the form $A \Rightarrow B$, in reference to the usual notation of Gentzen sequents; A is the *premise* and B is the *consequent* of the entailment $A \Rightarrow B$. In other words, we are using a formalism of Gentzen sequents with exactly one formula on both sides of the symbol $\Rightarrow$. If need be, we refer to entailments by lower case Greek letters.

A *theory* T (in the language $\mathscr{L}$) is a (n arbitrary) set of entailments (in $\mathscr{L}$). We have rules of inference with zero, one, or two hypotheses and one conclusion; the rules with no hypothesis are also called axiom schemes. The rules are grouped into logical and extralogical ones. The list of the rules follows; the symbols to the right of the rules will be used later.

I. Logical Rules.

(TAUT) $$\frac{}{A \Rightarrow A} \qquad 1_A$$

(CUT) $$\frac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C} \qquad \circ_{A,B,C}$$

(TRUE) $$\frac{}{A \Rightarrow \mathbf{t}} \qquad !_A$$

($\wedge$ LEFT1) $$\frac{}{A \wedge B \Rightarrow A} \qquad \pi_{A,B}$$

($\wedge$ LEFT2) $$\frac{}{A \wedge B \Rightarrow B} \qquad \pi'_{A,B}$$

($\wedge$ RIGHT) $$\frac{C \Rightarrow A \qquad C \Rightarrow B}{C \Rightarrow A \wedge B} \qquad \langle\ ,\ \rangle^C_{A,B}$$

($\vee$ RIGHT1) $$\frac{}{A \Rightarrow A \vee B} \qquad \kappa_{A,B}$$

($\vee$ RIGHT2) $$\frac{}{B \Rightarrow A \vee B} \qquad \kappa'_{A,B}$$

($\vee$ LEFT) $$\frac{A \Rightarrow C \qquad B \Rightarrow C}{A \vee B \Rightarrow C} \qquad [\ ,\ ]^C_{A,B}$$

($\rightarrow$LEFT) $$\frac{}{A \wedge (A \rightarrow B) \Rightarrow B} \qquad \varepsilon_{A,B}$$

($\rightarrow$RIGHT) $$\frac{A \wedge C \Rightarrow B}{C \Rightarrow A \rightarrow B} \qquad (\ )^{\sim}_{A,B,C}$$

II. Extralogical Rules.

(T) $$\frac{}{\tau} \quad \text{provided } \tau \in T. \qquad \xi_\tau$$

The calculus axiomatized by the *logical* rules alone is called the *minimal propositional calculus.*

The given theory T enters through the extralogical axiom scheme (T). To recover full intuitionistic propositional logic with negation, we use an additional atom **f** ("false"), interpret $\neg A$ as $A \to \mathbf{f}$ as usual, and add all entailments $\mathbf{f} \Rightarrow A$ to the given theory T.

We have the traditional notion of deduction. A *deduction* is a finite tree with additional data; the nodes are occurrences of entailments; the leaves (nodes without successors) are instances of the axiom schemes, the axiom scheme applying attached as a justification label (the same formula could be an instance of two distinct axiom schemes); every other node has one or two successors, and, if it has two, the order of the two successors is supplied as additional data; every node ρ is the conclusion of an instance of a rule of inference, given as a justification label on ρ, in which the hypothesis is (hypotheses are) the successor(s) of ρ (in the given order as first and second hypothesis, in the case of two hypotheses). The deduction is a deduction *of* the entailment at its root.

Let us write $f\colon A \Rightarrow B$ to indicate that f is a deduction of $A \Rightarrow B$ (in T; T is suppressed in this notation).

It is easy to see that the proof-system just introduced, with **f** and negation treated as explained, is equivalent with respect to provability to the familiar axiomatizations of intuitionistic propositional logic. Nevertheless, the system, which we would like to call Lambek's axiomatization of intuitionistic logic, has certain differences in comparison to sequent calculus, or to natural deduction. In [M2], a comparison of these systems and their extensions to predicate logic will be attempted.

To form the desired category of proofs, certain deductions that are considered only inessentially different will be identified with each other; however, even before that identification, we can develop an algebraic notational system for deductions, eventually used for identifying a categorical structure.

The idea of the notation is to consider the set of all deductions as a many-sorted algebra. The sorts of the underlying similarity type, denoted by $\mathscr{D}_{\mathscr{L}}$, are the entailments themselves. The sorted operation symbols of $\mathscr{D}_{\mathscr{L}}$ are the symbols listed to the right of the logical rules above. The symbols mentioned with the extralogical axiom scheme are individual constants, additional to the language $\mathscr{D}_{\mathscr{L}}$, with appropriate sortings, used as generators of the algebra of proofs of the similarity type $\mathscr{D}_{\mathscr{L}}$. Let us denote the language $\mathscr{D}_{\mathscr{L}}$ with all the individual constants ξ_τ ($\tau \in T$) (the generators) adjoined by $\mathscr{D}_{\mathscr{L}}(T)$.

For example, 1_A is an (individual) constant (nullary operation) of (value-)sort $A \Rightarrow A$, and it denotes the deduction consisting of the single node $A \Rightarrow A$ justified as an application of (TAUT) (it could possibly be justified as an application of (T) if $A \Rightarrow A$ happened to belong to T; with that justification, we would have a different deduction).

ξ_τ is a constant of sort τ ($\tau \in T$).

$\circ_{A,B,C}$, so denoted to suggest composition, is a binary operation, with first and second argument sorts $A \Rightarrow B$ and $B \Rightarrow C$, respectively, and with value-sort $A \Rightarrow C$:

$$\circ_{A,B,C}\colon (A \Rightarrow B) \times (B \Rightarrow C) \to (A \Rightarrow C).$$

Moreover, if $f\colon A \Rightarrow B$, $g\colon B \Rightarrow C$ and $h\colon A \Rightarrow C$ is the deduction pictured as

$$\dfrac{\begin{matrix}\vdots & \quad & \vdots\\ A \overset{f}{\Rightarrow} B & \quad & B \overset{g}{\Rightarrow} C\end{matrix}}{A \Rightarrow C}\ (\mathrm{CUT}),$$

we write $\circ_{A,B,C}(f,g)$, or, more simply, $g \circ f$, or even gf, for h.

The reader will now easily supply the sorting of the rest of the operation symbols, and their use in denoting deductions.

Note that every deduction will have precisely one notation as a closed (variable free) term in the language $\mathscr{D}_{\mathscr{L}}(T)$, and every such term denotes a well-formed deduction. In short, *the deductions are exactly the closed terms of* $\mathscr{D}_{\mathscr{L}}(T)$.

We may also say that the deductions in T form the absolutely free algebra of similarity type $\mathscr{D}_{\mathscr{L}}$ with generators ξ_τ ($\tau \in T$); "absolutely free" because no identities are required to hold.

Next, we consider certain identities over the language $\mathscr{D}_{\mathscr{L}}$. The first three are the following:

E1a. $\quad f1 = f,$

E1b. $\quad 1g = g, \qquad [f\colon A \Rightarrow B, g\colon B \Rightarrow C, h\colon C \Rightarrow D],$

E1c. $\quad (hg)f = h(gf).$

In E1a, we abbreviated the constant 1_A of sort $A \Rightarrow A$ by 1; similarly in E1b; also, all uses of the "composition" symbols $\circ$, with various subscripts, are suppressed in favor of juxtaposition. The rest of the identities are E2a to E6c of §1, with E5 taken out, and with the following modifications. Whenever a variable was meant to be an arrow $E \to F$ before, it is now meant as a variable of sort $E \Rightarrow F$; the symbols $\times$ and $+$ should now be replaced by $\wedge$ and $\vee$; and each B^A should be replaced by $A \to B$ (implication). The symbols A, B, C and D, previously ranging over the objects of a category, now range over the formulas of $\mathscr{L}$. Let us call the identities in the groups E1–E4 and E6 the *c.c. identities.*

Let us impose the c.c. identities on the deductions; in other words, let us consider the free algebra satisfying the c.c. identities of type $\mathscr{D}_{\mathscr{L}}$, freely generated by the generators ξ_τ ($\tau \in T$), and let us denote it by $\mathscr{F}_{\mathscr{L}}(T)$. The elements of the algebra $\mathscr{F}_{\mathscr{L}}(T)$ are our final concept of deduction. $\mathscr{F}_{\mathscr{L}}(T)$ is a many-sorted algebra, with sorts the entailments of $\mathscr{L}$; its elements are equivalence classes $[t]$ of closed terms t of the language $\mathscr{D}_{\mathscr{L}}(T)$, under the congruence relation generated by all instances of the c.c. identities with closed terms of $\mathscr{D}_{\mathscr{L}}(T)$ filled in for the variables.

For example, the associativity identity E1c means that we do not distinguish between the following two deductions:

$$(\mathrm{CUT})\ \dfrac{(\mathrm{CUT})\ \dfrac{\begin{matrix}\vdots & \quad & \vdots\\ A \overset{f}{\Rightarrow} B & \quad & B \overset{g}{\Rightarrow} C\end{matrix}}{A \Rightarrow C} \quad \begin{matrix}\vdots\\ C \overset{h}{\Rightarrow} D\end{matrix}}{A \Rightarrow D}$$

and

$$\dfrac{\overset{\vdots}{A \overset{f}{\Rightarrow} B} \qquad \dfrac{\overset{\vdots}{B \overset{g}{\Rightarrow} C} \qquad \overset{\vdots}{C \overset{h}{\Rightarrow} D}}{B \Rightarrow D}\,(\mathrm{CUT})}{A \Rightarrow D}\,(\mathrm{CUT})$$

E4a identifies any deduction of the form

$$\overset{\vdots}{A \wedge C \overset{h}{\Rightarrow} B}$$

with the following roundabout deduction of the same entailment:

$$\dfrac{(\wedge\mathrm{RIGHT})\,\dfrac{(\wedge\mathrm{LEFT1})\,\dfrac{}{A \wedge C \Rightarrow A} \qquad \dfrac{(\wedge\mathrm{LEFT2})\,\dfrac{}{A \wedge C \Rightarrow C} \qquad \dfrac{\overset{\vdots}{A \wedge C \overset{h}{\Rightarrow} B}}{C \Rightarrow A \to B}\,(\to\mathrm{RIGHT})}{A \wedge C \Rightarrow A \to B}\,(\mathrm{CUT})}{A \wedge C \Rightarrow A \wedge (A \to B)} \qquad \dfrac{}{A \wedge (A \to B) \Rightarrow B}\,(\to\mathrm{LEFT})}{A \wedge C \Rightarrow B}\,(\mathrm{CUT})$$

The identifications of proofs effected by the c.c. identities are closely related to ones considered in the literature in the context of the Gentzen sequent calculus, and natural deduction. However, the precise connections are not easy to state since the Lambek axiomatization of intuitionistic logic differs slightly from either the Gentzen sequent calculus, or natural deduction. In [M2], a comparison will be attempted.

The idea of formulas as sorts (types) and proofs as terms appears in proof theory, independently of categories. In 1969, in a privately circulated manuscript (which later became the paper [Ho]), W. A. Howard regards formulas as types, and denotes natural deduction proofs by terms that at the same time denote "constructions" or functionals of appropriate types. Howard's work was independent of Lambek's. In [Ho], the similarity of Howard's and Läuchli's [Lä] frameworks is noted. In [Lä], using the notation of the lambda calculus, Läuchli implicitly associates terms denoting functionals with proofs in a Hilbert-type calculus for intuitionistic logic.

Let us now turn to the rather straightforward step of construing algebras over the language $\mathscr{D}_{\mathscr{L}}$ as c.c. categories.

Let $\mathbf{C}$ be any algebra of type $\mathscr{D}_{\mathscr{L}}$ satisfying the c.c. identities. We consider the category, also denoted by $\mathbf{C}$, whose objects are the formulas of $\mathscr{L}$ and whose arrows $A \to B$ are the elements of the algebra of sort $A \Rightarrow B$. Furthermore, the identity arrows and composition are given by the operations 1_A and $\circ_{A,B,C}$; the identities E1 say precisely that we do have a category in this way. Moreover, the presence of the rest of the operations of $\mathscr{D}_{\mathscr{L}}$ and the rest of the identities means, according to §1, precisely that **C** is a c.c. category with specified operations: the formulas $A \wedge B$, $A \vee B$ and $A \to B$ are the specified product $A \times B$, coproduct $A + B$, and exponential B^A, respectively.

Applying the last paragraph to the free algebra $\mathscr{F}_{\mathscr{L}}(T)$, we now have that *the (suitably identified) deductions in T form a c.c. category with specified operations.*

Since the algebra $\mathscr{F}_{\mathscr{L}}(T)$ of deductions is a free algebra, it is to be expected that the corresponding c.c. category with specified operations, also named $\mathscr{F}_{\mathscr{L}}(T)$, is free in a suitable sense.

2.1. DEFINITION. Let $\mathscr{L}$ be an arbitrary set. The *c.c. category (with specified operations) freely generated by $\mathscr{L}$ as a set of objects* is the c.c. category $\mathscr{F}_{\mathscr{L}}$ defined, up to isomorphism, by the following universal mapping property:

(α) Each $P \in \mathscr{L}$ gives rise to an object, also denoted P, of $\mathscr{F}_{\mathscr{L}}$; and whenever **D** is a c.c. category with specified operations, with an object $\hat{P}$ assigned to each $P \in \mathscr{L}$, then there is a unique c.c. morphism $F\colon \mathscr{F}_{\mathscr{L}} \to \mathbf{D}$ for which $F(P) = \hat{P}$ for all $P \in \mathscr{L}$.

This definition parallels that of "free group", and those of a number of other free objects in algebra. The fact that any two c.c. categories answering the description of the definition are indeed isomorphic to each other is seen immediately, just as in the case of groups. As for the existence of $\mathscr{F}_{\mathscr{L}}$, the answer is already provided: with $\mathscr{L}$ understood as a set of propositional atoms and T the empty set of (extralogical) axioms, the c.c. category $\mathscr{F}_{\mathscr{L}}(\varnothing)$ constructed above serves as $\mathscr{F}_{\mathscr{L}}$. Indeed, the required c.c. morphism F, in the situation of the definition, is constructed by an obvious recursion on the complexity of the formulas (as objects) and $\mathscr{D}_{\mathscr{L}}$-terms (as "pre-morphisms"); it is clear that F is uniquely determined.

2.2. DEFINITION. Suppose **B** is a c.c. category with specified operations, and X is a set with two objects $d(\xi)$ ("domain of ξ") and $c(\xi)$ ("codomain of ξ") of **B** assigned to each $\xi \in X$. We say that the c.c. category **C** with specified operations is *obtained from* **B** *by adjoining the indeterminate arrows in X* if, with some c.c. morphism F: $\mathbf{B} \to \mathbf{C}$, and arrows (also denoted by) $\xi\colon F(d(\xi)) \to F(c(\xi))$ in **C**, one for each $\xi \in X$, we have the following universal mapping property:

(β) Whenever **D** is a c.c. category with specified operations, $G\colon \mathbf{B} \to \mathbf{D}$ a c.c. morphism, and $\hat{\xi}\colon G(d(\xi)) \to G(c(\xi))$ for each $\xi \in X$, then there is a unique c.c. morphism $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ F$ and $\hat{\xi} = H(\xi)$ for all $\xi \in X$.

Again, the c.c. category obtained by adjoining indeterminate arrows in X to a given c.c. category **B** is determined up to isomorphism, as is easily seen; let us denote it by $\mathbf{B}(X)$. $\mathbf{B}(X)$ always exists; instead of proving this, let us point out that $\mathscr{F}_{\mathscr{L}}(T)$ constructed before serves as $\mathscr{F}_{\mathscr{L}}(X)$, with $X = \{\xi_\tau\colon \tau \in T\}$. Indeed, we have the canonical c.c. morphism $F\colon \mathscr{F}_{\mathscr{L}} \to \mathscr{F}_{\mathscr{L}}(T)$ which is the identity on objects, and for which $F([t]) = [t]'$; here $[t]$ is the arrow defined by t in $\mathscr{F}_{\mathscr{L}}$, and $[t]'$ is the one defined by t in $\mathscr{F}_{\mathscr{L}}(T)$. Moreover, it is easily seen that F satisfies the universal property of the definition. The construction of $\mathbf{B}(X)$ in general is quite similar to that of $\mathscr{F}_{\mathscr{L}}(T)$; in particular, the objects of $\mathbf{B}(X)$ can be taken to be the same as those of **B**, and the canonical morphism $F\colon \mathbf{B} \to \mathbf{B}(X)$ can be taken to be the identity function on objects.

We have described two "free" constructions, each via an appropriate universal property; one with indeterminate objects, the other with indeterminate arrows. These constructions are analogous to the ones given in §§I.4 and I.5 of [LS], given there for cartesian closed categories instead of c.c. ones.

Let us call a c.c. category with specified operations *free* if it is obtained by the two free constructions performed one after the other. That is to say, $\mathbf{C}$ is a free c.c. category if it is of the form $\mathscr{F}_{\mathscr{L}}(X)$, with $\mathscr{L}$ an arbitrary set (of objects) and X a set of indeterminate arrows. We have seen that *the c.c. category of proofs, based on any theory T in minimal propositional logic, is a free c.c. category.*

REMARK (concerning the effect of taking the *intuitionistic*, rather than the *minimal*, as the underlying propositional logic). As mentioned in the Introduction, Lambek considered the category of proofs of intuitionistic, rather than minimal, propositional logic. The construction, which differs from the one described above in its treatment of the false atom $\mathbf{f}$, is described in [LS, Part I, §§1–4, 8]. The difference is that the entailments $\mathbf{f} \Rightarrow A$ are added to the logical rules, and the identities

$$\text{E5.} \qquad g = \square_A \qquad [g\colon \mathbf{f} \Rightarrow A]$$

are added to the c.c. identities E1–4 and E6. Let $\mathscr{F}^*_{\mathscr{L}}$ denote the resulting category of proofs (for *pure* intuitionistic logic). The effect of E5 is that $\mathbf{f}$ is an initial object of $\mathscr{F}^*_{\mathscr{L}}$. Thus, $\mathscr{F}^*_{\mathscr{L}}$, the category of proofs of intuitionistic logic, is a bicartesian closed category. The addition of E5 to the identities has the effect that we identify more arrows than before. This means that when $\mathscr{F}^*_{\mathscr{L}}$ is compared with the c.c. category $\mathscr{F}_{\mathscr{L}}(T)$, where T is the theory consisting of the entailments $\mathbf{f} \Rightarrow A$ as extralogical axioms, the two have the same objects but the former has *fewer* arrows (the former is a surjective image of the latter). The difference is well exemplified by Proposition 8.3 in Part I of [LS], which states that in a bicartesian closed category there is at most one arrow $A \to 0$ for any given object A (0 is the initial object). As a consequence, for any formula A, if the negation $\neg A$ of A (i.e. the formula $A \to \mathbf{f}$) is provable in intuitionistic logic, then all its proofs should be regarded, according to $\mathscr{F}^*_{\mathscr{L}}$, as essentially the same. In particular, no matter how many different proofs a formula A might have, they all lead to the same proof of its double negation $\neg\neg A$. This is counter-intuitive, and provides a reason to prefer our choice of minimal logic as the basic calculus. Another, no less important, reason is that Läuchli's theorem is not true in the bicartesian closed context (cf. §3, below). We conclude our digressive remark, and return to free c.c. categories.

The concept of *projectivity* is a well-known one in algebra, and it is meaningful in a general category. We apply the general concept in the category of c.c. categories with specified operations and c.c. morphisms. Let us call, in a natural way, a functor $G\colon \mathbf{D} \to \mathbf{E}$ *surjective* if it is surjective on objects as well as full (the latter is surjectivity on each separate hom-set).

2.3. DEFINITION. A c.c. category $\mathbf{C}$ with specified operations is *projective* if for any surjective c.c. morphism $G\colon \mathbf{D} \to \mathbf{E}$, and any c.c. morphism $H\colon \mathbf{C} \to \mathbf{E}$, there is at least one c.c. morphism $J\colon \mathbf{C} \to \mathbf{D}$ such that $H = G \circ J$: the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[d, "J"'] \arrow[dr, "H"] & \\ \mathbf{D} \arrow[r, "G"'] & \mathbf{E} \end{tikzcd}$$

commutes.

The following lemma is analogous to the well-known fact that a free module is projective.

2.4. LEMMA. *Any free c.c. category is projective.*

PROOF. In this proof, all categories and functors are c.c.; thus, the c.c. character will not be mentioned.

Let our free c.c. category be $\mathbf{C}$; $\mathbf{C} = \mathscr{F}_{\mathscr{L}}(X)$; let $\mathbf{B} = \mathscr{F}_{\mathscr{L}}$, and let F be the canonical functor $F\colon \mathbf{B} \to \mathbf{C}$.

Let $G\colon \mathbf{D} \to \mathbf{E}$ be surjective, and let $H\colon \mathbf{C} \to \mathbf{E}$. We will construct the diagram

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "F"] \arrow[dr, "I"'] & \mathbf{C} \arrow[d, "J"'] \arrow[dr, "H"] & \\
& \mathbf{D} \arrow[r, "G"'] & \mathbf{E}
\end{tikzcd}$$

Consider $HF(P) \in \mathbf{E}$ for each $P \in \mathscr{L}$. By the surjectivity of G on objects, there is $\hat{P} \in \mathbf{D}$ such that $G(\hat{P}) = HF(P)$. By the universal property of $\mathbf{B}$ (see 2.1), there is $I\colon \mathbf{B} \to \mathbf{D}$ such that $I(P) = \hat{P}$ for all $P \in \mathscr{L}$. Since the composite functors GI and HF agree on the generators $P \in \mathscr{L}$, by the uniqueness part of the universal property of $\mathbf{B}$ we have that $GI = HF$.

Next, consider the arrow $H(\xi)\colon HF(d(\xi)) \to HF(c(\xi))$; that is,

$$H(\xi)\colon GI(d(\xi)) \to GI(c(\xi))$$

for each $\xi \in X$. Since G is full, there is $\hat{\xi}\colon I(d(\xi)) \to I(c(\xi))$ such that $G(\hat{\xi}) = H(\xi)$ for all $\xi \in X$. By the universal property of $\mathbf{C}$ (see 2.2), there is $J\colon \mathbf{C} \to \mathbf{D}$ such that $I = JF$ and $J(\xi) = \hat{\xi}$ for all $\xi \in X$. We have that $HF = GI = GJF$, and also that $GJ(\xi) = H(\xi)$ for all $\xi \in X$. Hence, by the uniqueness part of the universal property of $\mathbf{C}$, $H = GJ$ as required. □2.4

§3. Läuchli's completeness theorem. For any category $\mathbf{C}$, and objects A and B in it, $\mathbf{C}(A, B)$ denotes the set of all arrows $A \to B$. Part (a) of 0.3.1 ("extended Läuchli completeness") can be equivalently stated as

3.1. THEOREM. *Let* $\mathbf{C}$ *be a countable free connectionally closed category, and let* $A, B \in \mathrm{Ob}(\mathbf{C})$ *be such that* $\mathbf{C}(A, B) = \varnothing$. *Then there is a connectionally closed functor* $F\colon \mathbf{C} \to \mathbf{Set}^{\mathbf{Z}}$ *such that* $\mathbf{Set}^{\mathbf{Z}}(FA, FB) = \varnothing$.

The proof will be done in several steps, each involving further definitions and auxiliary statements, some of which have independent interest. Until the end of the proof, $\mathbf{C}$ and the objects A and B are taken to be as in the assumption of 3.1; in particular, $\mathbf{C}(A, B) = \varnothing$.

First step: the poset reflection. When trying to build a functor as desired by the theorem, we will naturally be concerned with the following structural aspect of the given category $\mathbf{C}$: which ordered pairs of objects are connected with arrows and which are not. We can associate with *any* category $\mathbf{C}$ a poset $\mathrm{Po}(\mathbf{C})$, called the *poset reflection* of $\mathbf{C}$, that is a simplified version of $\mathbf{C}$, containing the information on the said structural aspect of $\mathbf{C}$. The reader is reminded that each poset can be regarded as a category; see §1.

The poset reflection is obtained by first considering the *preorder reflection*, which is the preorder with elements the objects of $\mathbf{C}$, and for which $X \leq Y$ just in case there is an arrow $X \to Y$ in $\mathbf{C}$. The poset reflection is then the poset obtained from the preorder reflection by identifying any X and Y for which both $X \leq Y$ and $Y \leq X$. We have an obvious surjective functor $\gamma\colon \mathbf{C} \to \mathrm{Po}(\mathbf{C})$ for which $\mathbf{C}(X, Y) = \varnothing$ iff $\mathrm{Po}(\mathbf{C})(\gamma X, \gamma Y) = \varnothing$; we refer to γ as the poset reflection functor (of $\mathbf{C}$).

It is an obvious and important observation that if $\mathbf{C}$ is c.c., then so is $\mathrm{Po}(\mathbf{C})$, and γ is a c.c. functor. Notice that this assertion, although it is seen immediately by inspecting the definitions involved, nevertheless sensitively depends on the particulars of the definition; e.g., pullbacks in $\mathbf{C}$ are not necessarily preserved by γ. At the same time, the same assertion does hold for variants such as cartesian closed or bicartesian closed categories.

Let us call a c.c. poset an *almost Heyting* (aH) *algebra*; an aH algebra is a Heyting algebra "possibly without 0". We may summarize the first step by saying that we have constructed a c.c. functor $\gamma\colon \mathbf{C} \to \mathrm{Po}(\mathbf{C})$ from the given category $\mathbf{C}$ into an aH algebra, preserving the emptiness of the hom-set $\mathbf{C}(A, B)$; $\mathrm{Po}(\mathbf{C})(\gamma A, \gamma B) = \varnothing$, that is, not $\gamma A \leq \gamma B$.

Our overall plan is to find a c.c. functor, preserving the emptiness of $\mathbf{C}(A, B)$, from $\mathbf{C}$ to the poset reflection of $\mathbf{Set}^{\mathbb{Z}}$, and then, using the projectivity of $\mathbf{C}$, lift it to a functor into $\mathbf{Set}^{\mathbb{Z}}$ itself.

We could have considered, instead of c.c. categories, bicartesian closed categories in this subsection. When we do so, we get a Heyting algebra as the poset reflection of a bicartesian closed category. If the bicartesian closed category taken is the category of proofs of a theory, for the poset reflection we get the *Lindenbaum-Tarski algebra* of the theory; in intuitionistic logic, this is a Heyting algebra instead of a Boolean algebra as in classical logic. Of course, the Lindenbaum-Tarski algebra may be described directly as a poset whose underlying set consists of equivalence classes of formulas under the equivalence relation for which A and B are equivalent iff $A \leftrightarrow B$ is provable.

Second step: building the canonical Kripke model. The Kripke completeness theorem for propositional logic has an elegant algebraic form, which we now explain for the case of aH-algebras (we could do this just as well for Heyting algebras). In this form, the theorem states the existence of an embedding of an arbitrary aH algebra H into another, canonically constructed aH algebra of a special type. The construction is due, in fact in a much more general form concerning predicate logic, to A. Joyal; see Theorem 6.3.5 in [MR]. We will comment on the relations of Joyal's form to the usual form of the Kripke theorem.

Let 2 denote (also) the two-element poset $\{0, 1\}$ (with $0 < 1$); 2 is an aH algebra (of course, 2 is even a Boolean algebra).

Let H be an aH algebra. Let $I = [H, 2]$ be the set of all *almost lattice* (aL) *homomorphisms* $i\colon H \to 2$, i.e. the order-preserving maps $i\colon H \to 2$ that also preserve binary meets, the maximal element, and binary joins, but not necessarily implication. I is a poset with $i \leq j$ iff $i(x) \leq j(x)$ for all $x \in H$.

By identifying $i \in I$ with the set $F = \{x \in H\colon i(x) = 1\}$, we see that the elements of I are the same as the possibly improper prime filters of H, i.e. the filters F for which $x \vee y \in F$ implies that either $x \in F$ or $y \in F$. The ordering $i \leq j$ becomes set-theoretic containment $F \subseteq G$. In particular, I has a maximal element, the identically-1 function, or the improper filter $F = H$; this fact, which will turn out to be important, would not be true if we dealt with Heyting algebras and asked for all $i \in I$ to preserve the least element as well.

Now, for a moment taking an arbitrary poset I (although we will need the construction only for the I introduced in the previous paragraph), we consider the poset 2^I of all order-preserving maps $\varphi\colon I \to 2$, with the pointwise ordering (2^I is essentially the same as the poset of all upward closed subsets of I, with set-theoretic containment as ordering). 2^I is a Heyting algebra; the operations of meet and join are computed pointwise, and the Heyting implication (=exponential) $\varphi \to \psi =_{\mathrm{def}} \psi^{\varphi}$ is given by

$$(1) \qquad (\varphi \to \psi)(i) = 1 \Leftrightarrow \forall j \geq i.[\varphi(j) = 1 \text{ implies } \psi(j) = 1]$$

(we leave this, and other, easy-to-prove assertions to the reader to verify).

Returning to I as obtained from H as above, we note that we have the mapping $e\colon H \to 2^I$ defined by $x \mapsto [i \mapsto i(x)]$ (that is, $e(x)(i) = i(x)$), which is called the "evaluation mapping". Clearly e is an aL homomorphism. We claim that it is an aH homomorphism, meaning that it is a c.c. functor, i.e., it also preserves Heyting implications; that is, for any $x, y \in H$, $e(x \to y) = e(x) \to e(y)$.

According to the meanings of the terms involved (see (1)), this means that, for any $i \in I$,

$$\forall j \geq i.[j(x) = 1 \text{ implies } j(y) = 1] \Leftrightarrow i(x \to y) = 1.$$

If $i(x \to y) = 1$, then, for any $j \geq i$, $j(x \to y) = 1$; since $j(x) \wedge j(x \to y) \leq j(y)$, the phrase in brackets on the left of $\Leftrightarrow$ follows; this shows the right-to-left implication. Conversely, by contraposition, let us assume $i(x \to y) \neq 1$. Let $F = \{z \in H\colon i(z) = 1\}$, the filter corresponding to i, and consider the filter generated by F and x. The latter is $F' = \uparrow(F(\wedge)x)$, the upward closure of the set $F(\wedge)x = \{z \wedge x\colon z \in F\}$. We have that $y \notin F'$; otherwise we would have some $z \in F$ with $z \wedge x \leq y$, implying that $z \leq x \to y$, from which $x \to y \in F$ would follow, in contradiction to the assumption $i(x \to y) \neq 1$. Now we have the prime filter existence theorem: for any filter F' and element $y \notin F'$, there exists a prime filter P such that $F' \subset P$ and $y \notin P$ (in fact, any filter P maximal among the ones containing F' and not containing y is a prime filter). Let $j\colon H \to 2$ be the aL homomorphism corresponding to P; we have $j \geq i$ and $j(x) = 1$, but $j(y) \neq 1$, as required.

Another application of the prime filter existence theorem shows (as is well known) that e is one-to-one; it easily follows that $e(x) \leq e(y)$ if and *only if* $x \leq y$.

Let us summarize what we have just shown in a proposition.

3.2. PROPOSITION (A. Joyal). *For any almost Heyting algebra H, the canonical evaluation mapping $e\colon H \to 2^{[H,2]}$ is an almost Heyting isomorphic embedding.*

$\square$ 3.2

In the last proposition, we could just as well talk about Heyting algebras and Heyting embeddings.

This concludes the second step of the proof. With $H = \mathrm{Po}(\mathbf{C})$, we now have the second functor $e\colon H \to 2^I$, and the composite $e \circ \gamma\colon \mathbf{C} \to 2^I$ still preserves the emptiness of $\mathbf{C}(A, B)$.

Recall that a Kripke model for a given language $\mathscr{L}$ is usually defined as a pair $(I, \Vdash)$, where I is a poset and $\Vdash$ is a relation between elements of I and formulas of $\mathscr{L}$ satisfying certain inductive conditions. Let H be the Lindenbaum-Tarski algebra of T (see above). Any Heyting homomorphism of the form $e: H \to 2^I$ gives rise to a Kripke model of T: define $i \Vdash A$ to mean $e([A])(i) = 1$ (here $[A]$ is the equivalence class (an element of H) containing A). For instance, the relation (1) concerning the meaning of implication in 2^I gives us the clause for implication in Kripke models:

$$i \Vdash A \to B \quad \text{iff} \quad \text{for all } j \geq i, \text{ if } j \Vdash A, \text{ then } j \Vdash B.$$

Reflection shows that a Kripke model of T is in fact the same as a Heyting homomorphism of the form $e: H \to 2^I$, with H as before. It is then immediately seen that 3.2 (formulated for Heyting algebras rather than aH algebras) is a form of Kripke's completeness theorem.

Third step: choosing a rooted Kripke model. The poset $I = [H, 2]$ does not have, in general, a least element, a fact that will turn out to be a hindrance. We now correct this.

Note that $\mathbf{C}(X, Y) = \varnothing$ iff not $\gamma X \leq \gamma Y$ iff there is $i_1 \in I$ such that $i_1(\gamma X) = 1$ and $i_1(\gamma Y) = 0$. Apply this to $X = A$ and $Y = B$ for which we have that $\mathbf{C}(A, B) = \varnothing$; let i_1 be chosen accordingly. Pick any $i_0 \leq i_1$, and consider $I_0 = \{i \in I: i_0 \leq i\}$. We would like to substitute 2^{I_0} for 2^I. There is an obvious map $\Phi: 2^I \to 2^{I_0}$ defined by $\Phi\varphi = \varphi \mid I_0$ (φ restricted to I_0). Clearly, Φ preserves meets, joins and 1; it is not hard to see that it also preserves $\to$ as well (the reason is that I_0 is upward closed in I; we will discuss this point shortly in a broader context).

It follows that we have a third c.c. functor $\Phi: 2^I \to 2^{I_0}$. Finally, the composite functor $\Phi e \gamma: \mathbf{C} \to 2^{I_0}$ still preserves the emptiness of $\mathbf{C}(A, B)$, since the element i_1 is included in the set I_0.

Preparations for the remaining steps. The next two steps will consist in gradually "preparing" the Kripke model $\Phi e: H \to 2^{I_0}$ just constructed. We now consider two arbitrary posets N and I, and an order-preserving mapping $f: N \to I$. f induces the order-preserving mapping

$$f^*: 2^I \to 2^N, \qquad \varphi \mapsto \varphi \circ f.$$

It is clear that f^* is a lattice homomorphism. Let us see what it means for f^* to be a morphism of Heyting algebras, that is, to preserve Heyting implication. Consulting the formula (1) above, and applying it in both 2^I and 2^N, we see that f^* preserves the implication $\varphi \to \psi$ in 2^I just in case, for all $p \in N$,

(2) $\quad \forall j \geq f(p).[\varphi(j) = 1 \Rightarrow \psi(j) = 1] \Leftrightarrow \forall q \geq p.[\varphi(fq) = 1 \Rightarrow \psi(fq) = 1].$

It is clear that the left-to-right implication is automatic.

Fourth step: constructing a countable rooted Kripke model. We want to find a countable subset J of I_0 containing the minimal element of I_0 such that, with $k: J \to I_0$ the inclusion, the composite $k^*\Phi e: H \to 2^J$ is still an aH homomorphism, and so that the further composite $k^*\Phi e\gamma$ preserves the emptiness of $\mathbf{C}(A, B)$.

3.3. Proposition. *Let $e: H \to 2^I$ be an aH algebra homomorphism of a countable aH algebra H into 2^I, with I a poset. Assume $a, b \in H$ are such that not $e(a) \leq e(b)$. Then there is a countable subposet J of I such that, for the inclusion $k: J \to I$, the* *induced mapping $k^*e\colon H \to 2^J$ is again an aH algebra homomorphism, and still not $k^*e(a) \leq k^*e(b)$. In fact, J can in addition be chosen to contain any given countable subset of I.*

PROOF. The proof is a downward Löwenheim-Skolem-style argument. For any $J \subset I$, and with the inclusion $k\colon J \to I$, the condition (2) for k^*e to preserve $\to$ becomes, by passing to the contrapositive, the following:

($*$) For any $j \in J$ and $\Phi, \Psi \in \mathrm{Im}(e)$ (with the elements of $\mathrm{Im}(e) \subset 2^I$ considered as upward closed subsets of I),

$$\exists x \in I.[x \geq j \;\&\; x \in \Phi \;\&\; x \notin \Psi] \Rightarrow \exists l \in J.[l \geq j \;\&\; l \in \Phi \;\&\; l \notin \Psi].$$

Fulfilling condition ($*$) requires throwing into J appropriate witnesses l for instances of the statement appearing on the left side of the implication in ($*$), each such statement depending on elements of J already available. Pick $i_1 \in I$ such that $e(a)(i_1) = 1$ and $e(b)(i_1) = 0$, and construct the countable $J \subset I$ containing i_1 and any prescribed subset of I so that J satisfies ($*$). □ 3.3

Apply 3.3 to I_0 of the third step as I, and Φe as e; we obtain a fourth c.c. functor $k^*\colon 2^{I_0} \to 2^J$ so that the composite $k^*\Phi e\gamma\colon \mathbf{C} \to 2^J$ still preserves the emptiness of $\mathbf{C}(A, B)$.

Preparation for the fifth step: calculating the poset reflection of $\mathbf{Set}^{\mathbb{Z}}$. Think of $\mathbb{Z}$-sets as sets with distinguished permutations; let X and Y be $\mathbb{Z}$-sets with distinguished permutations σ and ρ. We want to know when $\mathbf{Set}^{\mathbb{Z}}(X, Y) \neq \varnothing$, i.e., when there is a function $g\colon X \to Y$ that is equivariant in the sense that $g(\sigma x) = \rho g(x)$. This equivariance condition implies that for all $n \in \mathbb{Z}$, if $\sigma^n x = x$, then $\rho^n g(x) = g(x)$ as well. Thus, a necessary condition for $\mathbf{Set}^{\mathbb{Z}}(X, Y) \neq \varnothing$ is the following:

($**$) For all $n \in \mathbb{Z}$, if there is $x \in X$ with $\sigma^n x = x$, then there is also $y \in Y$ with $\rho^n y = y$.

3.4. *Claim. Condition ($**$) is also sufficient for* $\mathbf{Set}^{\mathbb{Z}}(X, Y) \neq \varnothing$.

Proof. For $x \in X$, let the order of x be the least positive integer n such that $\sigma^n x = x$ if there is such n, and 0 otherwise. If $o(x) = \{\sigma^k x\colon k \in \mathbb{Z}\}$, the *orbit of* x, then the order of x is the cardinality of $o(x)$ when $o(x)$ is finite, and 0 if $o(x)$ is infinite. The orbits of elements of X are each closed under the action of σ, and they form a partition of X; thus it suffices to define g on each orbit separately. Let $x \in X$ have order n; if ($**$) holds, then there is $y \in Y$ with $\rho^n y = y$; let $g(x) = y$ and extend this to $o(x)$ by $g(\sigma^k x) = \rho^k y$. The way y was chosen ensures that g is well-defined. □ 3.4

Letting N be the set of all nonnegative integers, define, for any $\mathbb{Z}$-set X,

$$\delta(X) = \{n \in N\colon \sigma^n x = x \text{ for some } x \in X\}. \tag{3}$$

Claim 3.4 can be restated as saying that $\mathbf{Set}^{\mathbb{Z}}(X, Y) \neq \varnothing$ iff $\delta(X) \subseteq \delta(Y)$.

Let us consider N as the poset with ordering relation the divisibility relation $|$; 1 is the least element of $(N, |)$ and 0 is its greatest. Obviously, each $\delta(X)$ is an upward closed subset of $(N, |)$.

For any $n \in N$, let X_n be the set of integers modulo n (hence, for $n = 0$, X_n is $\mathbb{Z}$), with the distinguished permutation σ on X_n defined as adding 1 mod n; then X_n consists of a single orbit, and it is of size n if $n \neq 0$, infinite otherwise. Now, if P is any upward closed subset of $(N, |)$, then for X the disjoint union of the X_n with $n \in P$ we clearly have that $\delta(X) = P$.

We have shown that the poset reflection of $\mathbf{Set}^{\mathbb{Z}}$ is the set of upward closed subsets of $(N, |)$ ordered by inclusion.

The upward closed subsets of $(N, |)$ can be identified, via their characteristic functions, with the elements of 2^N, the set of order-preserving maps from $(N, |)$ into 2. We can sum this up in the following proposition.

3.5. PROPOSITION. *The poset reflection of* $\mathbf{Set}^{\mathbb{Z}}$ *is isomorphic to* 2^N *via the poset reflection functor* δ *defined in* (3). $\square$ 3.5

Fifth step: a quite surjective $f\colon N \to J$. With J constructed in the fourth step, and the poset N obtained just now, our goal is to construct an aH embedding $f^*\colon 2^J \to 2^N$, using a suitable order-preserving function $f\colon N \to J$. To see what f should be like, let us return to condition (2) for f^* being an aH homomorphism. Let us call an order-preserving mapping $f\colon N \to J$ between two arbitrary posets N and J *upward closed* if for all $p \in N$, f maps $\{q \in N\colon q \geq p\}$ onto $\{i \in J\colon i \geq f(p)\}$, and *quite surjective* if it is upward closed and onto J. Now, it is immediate that any upward closed f satisfies (2) at any $p \in N$. Furthermore, it is clear that if f is onto, then f^* is one-to-one. We have proved.

3.6. PROPOSITION. *If* $f\colon N \to J$ *is a quite surjective order-preserving mapping of posets, then* $f^*\colon 2^J \to 2^N$ *is an aH embedding.* $\square$ 3.6

The remarkable property of the poset $(N, |)$ that is the key to the whole proof is that for any countable poset J with a least and a greatest element, there is a quite surjective order-preserving mapping $f\colon N \to J$. We show something more general, in view of a use this has in [M1].

For arbitrary categories N and J, and for a functor $f\colon N \to J$, we say that f is *upward closed* if for any $p \in \mathrm{Ob}(N)$ and any arrow $\iota\colon f(p) \to j$ in J, there is $\theta\colon p \to q$ in N such that $f(\theta) = \iota$ (and hence $f(q) = j$). f is called *quite surjective* if it is closed upward, and surjective on the objects.

In the next proposition, N is the above poset $(N, |)$. A *weak initial object* in a category is one from which there is at least one arrow to every object.

3.7. PROPOSITION. *If* J *is a countable category with a weak initial object and a terminal object, then there is a quite surjective functor* $f\colon N \to J$ *taking the least* (*greatest*) *element of* N *to a weak initial* (*terminal*) *object of* J.

PROOF. Note that our N is a countable poset satisfying the following three conditions:

(a) N has a least and a greatest element (the latter is the number 0).

(b) For every $p \in N - \{0\}$ the set $\{q \in N\colon q \leq p\}$ is finite.

(c) For any finite subset $A \subset N - \{0\}$ and any $a \in A$, there is $b \in N - \{0\}$ such that $b > a$, but b is incomparable with all $c \in A$ for which $\neg(c \leq a)$.

We define f by finite approximations, using finite functors g with a finite domain contained in N, and with codomain J, such that g maps the least (greatest) element of N to a weak initial (terminal) object of J, and such that $\mathrm{dom}(g) - \{0\}$ is closed downward. Given any such g, any $p \in \mathrm{dom}(g) - \{0\}$, and $j \in J$ with an arrow $\iota\colon f(p) \to j$, we can choose $q > p$ such that q is not comparable with any element in $\mathrm{dom}(g) - \{0\}$ except those $\leq p$, and in fact so that q is minimal with this property, and define g' extending g by setting $g'(q) = j$, $g'(p \leq q) = \iota$, and $g'(r \leq q) = \iota \circ g(r \leq p)$ for all $r \in \mathrm{dom}(g)$ with $r \leq p$.

In this way, g' is in fact a functor, and one instance of the upward closedness condition is satisfied. However, $\mathrm{dom}(g') - \{0\}$ may fail to be closed downward, in which case we have to extend g' further.

If r is in the downward closure P of $\mathrm{dom}(g') - \{0\}$ but not in $\mathrm{dom}(g')$ itself, then it must satisfy $r < q$, not $r \leq p$ and r is incomparable with the elements of $\mathrm{dom}(g) - \{0\}$ that are not $\leq p$ (for, if s is an element of $\mathrm{dom}(g) - \{0\} - \{u: u \leq p\}$ then $r \geq s$ would make q comparable with s and $r \leq s$ would make r an element of $\mathrm{dom}(g')$). Define h: $P \to \mathrm{dom}(g') - \{0\}$, an order-preserving map, by $h(s) = s$ for $s \in \mathrm{dom}(g') - \{0\}$, and by $h(r) = p$ for $r \in P - \mathrm{dom}(g')$. Because of what we said about elements r of $P - \mathrm{dom}(g')$, h is an order-preserving retraction. Let g'': $P \to J$ be defined as the composite $g'' = g' \circ h$, and extend g'' by adding $g''(0) = g'(0)$. g'' extends g', and thus g as well; also, g'' is a finite functor into J with $\mathrm{dom}(g'') - \{0\}$ closed downward, extending g.

The other task we have to be able to perform is to extend a given g to some g' mapping an arbitrarily given $p \in N$ to some object of I; but, since $\mathrm{dom}(g) - \{0\}$ is closed downward, if $p \notin \mathrm{dom}(g)$, we can define g' to map p to the terminal object of I already in the range of g; the definition of g' on the arrows involved becomes uniquely determined since only instances of the form $g'(r \leq p)$, $r \in \mathrm{dom}(g)$, are there to be considered. (This last argument is the sole point where we need the terminal object of I.)

It is clear that, with appropriate bookkeeping, we can piece together g's to get f: $N \to J$ so that f is upward closed. Since the construction also makes sure that the minimal element of N is mapped to a weak initial object, the surjectivity of f on objects is a consequence of the upward closure. □ 3.7

Note that, with Propositions 3.6 and 3.7, we have indeed come up with the **aH** embedding f^*: $2^J \to 2^N$, completing the fifth step of the construction.

PROOF OF 3.1. We have constructed the following c.c. functors:

$$\mathbf{C} \xrightarrow{\gamma} H \xrightarrow{e} 2^I \xrightarrow{\Phi} 2^{I_0} \xrightarrow{k^*} 2^J \xrightarrow{f^*} 2^N;$$

their composite $h =_{\mathrm{def}} f^*k^*\Phi e\gamma$: $\mathbf{C} \to 2^N$ preserves the emptiness of $\mathbf{C}(A, B)$; that is, not $hA \leq hB$. Now, the poset reflection γ: $\mathbf{Set}^{\mathbf{Z}} \to 2^N$ is a surjective functor. By the projectivity of $\mathbf{C}$, there is a c.c. functor F: $\mathbf{C} \to \mathbf{Set}^{\mathbf{Z}}$ such that $\delta F = h$:

$$\begin{tikzcd} \mathbf{C} \arrow[d, "F"'] \arrow[dr, "h"] & \\ \mathbf{Set}^{\mathbf{Z}} \arrow[r, "\delta"'] & 2^N \end{tikzcd}$$

It follows immediately that $\mathbf{Set}^{\mathbf{Z}}(FA, FB) = \varnothing$, as desired. □ 3.1

Let us repeat the statement of the uniform extended Läuchli theorem, Theorem 0.3.1(b).

3.8. THEOREM. *If* $\mathbf{C}$ *is a countable free connectionally closed category with the disjunction property, then there is a weakly full, connectionally closed functor from* $\mathbf{C}$ *to* $\mathbf{Set}^{\mathbf{Z}}$.

PROOF. The disjunction property of $\mathbf{C}$ translates into the following property of the poset reflection H of $\mathbf{C}$: for $x, y \in H$, if $x \vee y = 1$, then either $x = 1$, or $y = 1$. This means that $F = \{1\}$ is a prime filter over H, and, of course, it is the least prime filter. In other words, $I = [H, 2]$ has a least element $\perp$. Repeat the proof of 3.1 step by step by picking $i_0 = \perp$ in the third step; this makes $I_0 = I$ and Φ the identity functor (rendering the third step superfluous in this case). The composite functor $h\colon \mathbf{C} \to 2^N$ will be full under these conditions as all its components are, which makes the final F: $\mathbf{C} \to \mathbf{Set}^{\mathbf{Z}}$ weakly full. □ 3.8

REMARK (more on what happens when one chooses intuitionistic logic as the basic calculus). Had we chosen the intuitionistic, rather than minimal, logic as basis, the category of proofs would have been bicartesian closed (cf. the remark in §2), and we would have to consider functors (or morphisms) that are b.c. rather than c.c. A functor $F\colon \mathbf{C} \to \mathbf{Set}^{\mathbf{Z}}$, with $\mathbf{C}$ a b.c. category with initial object $\mathbf{f}$, is b.c. iff it is c.c. and maps $\mathbf{f}$ to the empty set $\varnothing$. Since Läuchli regarded $F(A)$ as the set of "abstract proofs" of the formula A, it seems more than reasonable to restrict ourselves to functors that are b.c., and assign to the false atom $\mathbf{f}$ an empty set of proofs. However, Theorems 3.1 and 3.8 become false if we require the free category and the functor involved to be b.c.! The simplest counterexample is, perhaps, the one (due to Läuchli) related to the formula $\neg A \vee \neg\neg A$, which is not provable intuitionistically, and yet, for any c.c. functor $F\colon \mathscr{F}_{\mathscr{L}} \to \mathbf{Set}^{\mathbf{Z}}$ with $F(\mathbf{f}) = \varnothing$, there is always an arrow $1 \to F(\neg A \vee \neg\neg A)$, where 1 is the terminal object of $\mathbf{Set}^{\mathbf{Z}}$. It is an amusing exercise to see why $A \vee \neg A$ is not a counterexample.

Läuchli's 0.2.1, referring as it does to full intuitionistic logic, is true, and follows from 3.1 applied to the category $\mathbf{C} = \mathscr{F}_{\mathscr{L}}(T)$ with T the set of entailments $\mathbf{f} \Rightarrow A$ for all A; to see this, one invokes the italicized statement of §0.3.

In the context of Läuchli's completeness theorem, we consider c.c. functors $F\colon \mathbf{C} \to \mathbf{Set}^{\mathbf{Z}}$ for which $F\mathbf{f}$ may be nonempty, but for which we always have at least one arrow $F\mathbf{f} \to FA$, for every formula A; such an arrow represents a function that transforms any possible abstract proof of $\mathbf{f}$ into one of A. This corresponds to the point of view that denies the existence of an "absolutely false" statement whose truth is inconceivable, and sees, instead, $\mathbf{f}$ as an idealized statement that is "hardest to prove". Seeing $\mathbf{f}$ as a hardest, rather than impossible, to prove statement removes a reasonable objection against the view that a proof of $\neg A$ is a construction that transforms any proof of A into a proof of absurdity; the objection doubts that this view has a constructive meaning, given that "absurdity" is regarded as something that cannot possibly have any proof. This objection was strong enough to lead to the development of negationless intuitionistic mathematics (cf. [He, §VIII.2]).

One final comment. According to the view just explained, it would be wrong to stipulate, in the definition of a Kripke model, that $i \Vdash \neg A$ iff for all $j \geq i$, not $(j \Vdash A)$; rather, one should say that $i \Vdash \neg A$ iff for all $j \geq i$, if $j \Vdash A$ then $j \Vdash B$ for all B (the structure produced in the proof of 3.2 is a Kripke model only in this sense).

§4. Some definability results.

4.1. Looking at the statements of 3.1 and 3.8 with a fresh eye we see that, in the first place, they say something about *definability of sets* rather than anything concerning proof theory. Notice that the arrows of the free c.c. category $\mathscr{F}_{\mathscr{L}}(X)$ are *c.c. definable* from the indeterminate objects and arrows that generate the category; by this we mean that they are definable by the set-theoretic operations codified in the notion of c.c. category, namely, the one-element set, cartesian product, disjoint sum (=coproduct) and exponentiation of sets. We call the objects of $\mathscr{F}_{\mathscr{L}}(X)$ *c.c. types.* Theorem 3.1 can be paraphrased as follows. If for two c.c. types A and B we always have an equivariant map $A \to B$, no matter how we realize A and B as $\mathbb{Z}$-sets through an interpretation of the indeterminate objects and arrows as $\mathbb{Z}$-sets and equivariant maps between them, then we must have also some c.c. definable functional $A \to B$. From this statement we can derive results of a purely definability-theoretical nature, no longer making any reference to $\mathbb{Z}$-sets. We can show, for instance, that if for c.c. types A and B there is a functional $A \to B$ definable in full classical type theory, then there is also a c.c. definable functional $A \to B$. This terse statement conveys only a rough idea. To make it precise, we first have to elaborate the notions involved.

4.2. Type theory has been extensively described in the literature. The following presentation closely follows [LS, Part II, §1]. There are, in fact, many type theories. The language of any one is determined by a set $\mathscr{L}$ of *primitive type symbols* and a set Y of *primitive term symbols. Types* and *terms* are certain expressions that we now describe.

The *types* are the elements of $\mathscr{L}$ and all expressions of the form 1, $U \times V$ and $\mathscr{P}(U)$, where U and V are types (an inductive definition; $\mathscr{P}$ means power set). We write Ω for $\mathscr{P}(1)$.

A very simple (actually, the intended) semantics for types is obtained by assigning an arbitrary set $P^{\#}$ (i.e. an object of the category **Set**) to every primitive type symbol P and a one-element set $1^{\#}$ to the type symbol 1, and extending this in the natural way to an assignment of an object $U^{\#}$ of **Set** to any type U. We also say that $U^{\#}$ is the interpretation of the type U in the (**Set**-valued) model defined by the assignment $^{\#}$.

We now turn to *terms*. Each term has its type, and we write "t: U" to indicate that the expression t is a term of type U; we write also "$t :\subset U$" for "t: $\mathscr{P}(U)$"; terms of type Ω are called *formulas*. The definition of terms proceeds by induction as follows:

(i) If $R \in Y$ is a primitive term symbol, then R is associated with a definite type U and $R :\subset U$ (we also say that R is a *predicate* over U; we do not need primitive terms other than ones of type of the form $\mathscr{P}(U)$).

(ii) For each type U we have a countably infinite list of *variables* of that type, each of them being a term of type U.

(iii) $*$: 1.

(iv) $\top, \bot$: Ω ($\top$ and $\bot$ are called "true" and "false").

(v) If a: U and b: V then $\langle a, b\rangle$: $U \times V$.

(vi) If a, a': U and $\alpha :\subset U$, then $a = a'$ and $a \in \alpha$ are formulas (the latter being also denoted as "$\alpha(a)$").

(vii) If ϕ and ψ are formulas and u: U is a variable, then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\exists u \in U.\phi$ and $\forall u \in U.\phi$ are formulas as well.

(viii) If ϕ is a formula and u: U a variable, then $\{u \in U \mid \phi\} :\subset U$.

Free and bound variables of terms are defined as usual, with u being bound in $\{u \in U \mid \phi\}$. A *closed term* is one with no free variables.

The natural **Set**-valued semantics for types can be easily extended to terms. Assume that we have an assignment $^\#$ for types. If we further assign a subset $R^\#$ of $U^\#$ to each predicate $R \in Y$ over a type U, and we define $\perp^\# = 0$ and $\top^\# = 1^\#$, then there is an easy natural extension of this to an interpretation that assigns to any term t: U with free variables v_1: $V_1, \ldots, v_n$: V_n a function $t^\#$: $V_1^\# \times \cdots \times V_n^\# \rightarrow U^\#$ (i.e. $t^\#$ is an arrow of the category **Set**; strictly speaking, $t^\#$ is an interpretation of the pair $(t, \vec{v})$, where $\vec{v}$ is a list that contains all free variables of t). In particular, a closed term t: U is interpreted as $t^\#$: $1^\# \rightarrow U^\#$ which is, essentially, an element of $U^\#$. A sentence (i.e. closed formula) ϕ is said to be *true* in the model determined by $\#$ iff $\phi^\# = \top^\#$.

A fact of particular importance for us is that we can define interpretations of types and terms as objects and arrows of the category $\mathbf{Set}^{\mathbb{Z}}$, rather than **Set**.

In fact, this can be done in such a way that if we forget the $\mathbb{Z}$-set structure of the values of the types, we get the corresponding **Set**-valued interpretation. For example, for $\mathbb{Z}$-sets A and B, the exponential B^A can be taken to be the $\mathbb{Z}$-set C whose underlying set $|C|$ is $|C| = |B|^{|A|}$, and for which the action $f \mapsto \sigma f$ ($\sigma \in \mathbb{Z}$, $f \in |B|^{|A|}$) is given by $(\sigma f)(a) = \sigma(f(\sigma^{-1}a))$ (where the inner σ^{-1} acts as on A, the outer one as on B). Below, we will make a statement of this circumstance that we think is better although less direct; for the time being, we leave it to the reader to ponder. Note that our claim includes the assertion that all the functions that arise as interpretations of terms turn out to be equivariant.

Note that, as a consequence of the above, a sentence ϕ is true in the model determined by the $\mathbf{Set}^{\mathbb{Z}}$-valued interpretation $^\#$ iff it is true in the **Set**-model which is $^\#$ followed by the forgetful functor $\mathbf{Set}^{\mathbb{Z}} \rightarrow \mathbf{Set}$. Thus, calling a sentence ϕ *valid* in **Set** if it is true in every **Set**-valued model, and similarly for $\mathbf{Set}^{\mathbb{Z}}$, we obtain that **Set** and $\mathbf{Set}^{\mathbb{Z}}$ are "type-theoretically equivalent": precisely the same sentences are valid in them! [The reader will see that in one direction this assertion uses an additional fact, namely that for interpretations in $\mathbf{Set}^{\mathbb{Z}}$ where the basic types are interpreted as trivial $\mathbb{Z}$-sets (every action is the identity map), truth is just as in **Set**; see also below.] Note that the *axiom of choice* is a sentence of type-theory that is valid in **Set** (and $\mathbf{Set}^{\mathbb{Z}}$).

4.3. A *pure classical type theory* is based on familiar axioms and rules which include the axioms of extensionality and comprehension, axioms describing pairing, and the *Boolean axiom* $\forall p \in \Omega(p \vee (p \rightarrow \perp))$ (which makes the theory classical); see [LS, pp. 130–131] for details.

A general, not necessarily pure, type theory is obtained by allowing also an arbitrary set Σ of extralogical axioms; we use "$\Sigma \vdash \phi$" for denoting that ϕ is provable in the type theory based on the extralogical axioms Σ.

It goes without saying that the deductive system for classical type theory is sound for the **Set**-valued interpretation: if, in any given interpretation, each axiom in Σ is true, and $\Sigma \vdash \phi$, then ϕ is true as well. As a consequence of the above discussion, the same holds for $\mathbf{Set}^{\mathbb{Z}}$.

The adequacy of our semantics for type theory has various easy but significant consequences. One of particular importance in our context is this. For terms $F :\subset U \times V$, $S :\subset U$ and $T :\subset V$ let "$F: S \to T$" be the formula of type theory stating that F is the graph of a function from S (i.e. $\{y \in U \mid y \in S\}$) to T. If F, S and T are closed terms such that $\Sigma \vdash F: S \to T$ then, under any interpretation in **Set** or $\mathbf{Set}^{\mathbf{Z}}$ that makes all Σ axioms true, $F^{\#}$ is the graph of an arrow from $S^{\#}$ to $T^{\#}$.

4.4. The realm of objects described in type theory is much richer than that codified by the notion of c.c. category. In fact one can interpret the latter in the former. To be more precise, given a free c.c. category $\mathscr{F}_{\mathscr{L}}(X)$, we interpret each c.c. type (i.e. object) A of $\mathscr{F}_{\mathscr{L}}(X)$ as a closed term $\langle A \rangle :\subset [A]$ of a type theory based on $\mathscr{L}$ as set of primitive type symbols. We define $\langle A \rangle$ and $[A]$ by simultaneous induction on c.c. types. The clauses of this definition are very natural. For example, if $P \in \mathscr{L}$ then $[P] = P$ and $\langle P \rangle = \{x \in [P] \mid x = x\}$; also, $[A^B] = \mathscr{P}([A] \times [B])$ and $\langle A^B \rangle = \{x \in \mathscr{P}([A] \times [B]) \mid x: \langle B \rangle \to \langle A \rangle\}$, and so on (but notice that the disjoint sum should be interpreted in some unusual way, e.g. we could put

$$[A + B] = \mathscr{P}(\mathscr{P}[A] \times \mathscr{P}[B])$$

and

$$\langle A + B \rangle = \{\langle \{a\}, \varnothing \rangle \mid a \in \langle A \rangle\} \cup \{\langle \varnothing, b \rangle \mid b \in \langle B \rangle\},$$

where in the last expression some obvious abbreviations are used). Next for every $\xi \in X$ with $\xi: d(\xi) \to c(\xi)$, let $\langle \xi \rangle$ be a predicate symbol over $[d(\xi)] \times [c(\xi)]$. Consider the type theory T based on $\mathscr{L}$ as the set of primitive types, $Y = \{\langle \xi \rangle: \xi \in X\}$ as the set of primitive terms and on the set of extralogical axioms $\Sigma = \Sigma_{\mathscr{L},X}$ consisting of the sentences $\langle \xi \rangle: \langle d(\xi) \rangle \to \langle c(\xi) \rangle$ for $\xi \in X$. We have a full interpretation of $\mathscr{F}_{\mathscr{L}}(X)$ in T in the sense that for every arrow $f: A \to B$ there is a term $\langle f \rangle :\subset [A] \times [B]$ such that $\Sigma \vdash \langle f \rangle: \langle A \rangle \to \langle B \rangle$.

Composing this interpretation $\langle - \rangle$ of $\mathscr{F}_{\mathscr{L}}(X)$ in Σ with any **E**-valued model $^{\#}$ of Σ, where **E** is either **Set** or $\mathbf{Set}^{\mathbf{Z}}$, we get a c.c. functor $\langle - \rangle^{\#}$ from $\mathscr{F}_{\mathscr{L}}(X)$ into **E** (we mean, of course, that $\langle - \rangle^{\#}$ maps any object A to $\langle A \rangle^{\#}$ and similarly for arrows). In fact, any c.c. functor F between the said categories is essentially of this form. More precisely, there is a natural isomorphism between F and $\langle - \rangle^{\#}$, where $^{\#}$ is the model generated by the assignment that sends $A \in \mathscr{L}$ to FA and $\langle \xi \rangle$ to $F\xi$ for any $\xi \in X$. This is very easy to verify when **E** is **Set** or $\mathbf{Set}^{\mathbf{Z}}$ (the latter being of the most importance in our context); more about the case of a more general **E** will be said in 4.6 below.

4.5. We are now ready to give a precise statement of a definability result.

4.5.1. THEOREM. *Let Σ be any set of sentences of type theory valid in* **Set**. *If A and B are objects of the free c.c. category $\mathscr{F}_{\mathscr{L}}(X)$, then the following three conditions are equivalent*:

(i) *There is a closed term $f :\subset A \times B$ such that $\Sigma \cup \Sigma_{\mathscr{L},X} \vdash f: \langle A \rangle \to \langle B \rangle$.*

(ii) *There is a c.c. definable functional (i.e. an $\mathscr{F}_{\mathscr{L}}(X)$-arrow) $A \to B$.*

(iii) *$A \Rightarrow B$ is deducible from the entailments $d(\xi) \Rightarrow c(\xi)$ $(\xi \in X)$ in minimal logic.*

PROOF. (iii) $\Rightarrow$ (ii) is the essence of §2. (ii) $\Rightarrow$ (i) follows immediately from the existence of the assignment $f \mapsto \langle f \rangle$ given in 4.4. (i) $\Rightarrow$ (iii) follows from 3.1 with the help of the facts mentioned in the last paragraph of 4.4.

We could extend pure type theory by adding a primitive type symbol N together with axioms stating that N is a "natural number object" which means that N behaves like the natural numbers system; cf. [LS, Part II, §§3 and 4]; the result remains true. Another variant of 4.5.1, which follows from 3.8 rather than from 3.1, states that if $\mathscr{F}_{\mathscr{L}}(X)$ has the disjunction property and $\Sigma_{\mathscr{L},X}$ proves a statement like

$$f\colon \langle A\rangle \to \langle B\rangle \vee g\colon \langle C\rangle \to \langle D\rangle;$$

then either $A \Rightarrow B$ or $C \Rightarrow D$ is provable from $\{d(\xi) \Rightarrow c(\xi)\colon \xi \in X\}$ in minimal logic.

Let us point out that the equivalence of (i) and (iii) implies, via the decidability of intuitionistic propositional logic, that the question of the existence of a functional $U \to V$ definable in classical type theory is decidable if the question is restricted to definable sets U and V of the forms $\langle A\rangle$ and $\langle B\rangle$, with A and B c.c. types.

4.6. As it is pointed out in [LS, Part II], type theories are intimately related to toposes. Actually, the notion of topos can be viewed as a categorical formulation of that of an (intuitionistic) type theory. (Therefore, it is remarkable how simple the definition of topos is; although every topos is in particular bicartesian closed (see §1), the definition of "topos" is shorter than that of "bicartesian closed category".) In particular, the type theory determined by $(\mathscr{L}, X, \Sigma)$ gives rise to a topos $\mathscr{T}(\mathscr{L}, X, \Sigma) = \mathscr{T}(\Sigma)$. The objects of $\mathscr{T}(\Sigma)$ are the closed terms of the theory, with two such terms S and T being identified whenever they are provably equal in Σ (which means that, in particular, they have the same type). The arrows $S \to T$ are the Σ-definable functionals, i.e. the terms R such that Σ proves $R\colon S \to T$ where, again, we identify two such terms whenever Σ proves their equality. The composition of arrows is defined in a straightforward manner that we leave to the reader to spell out. See [LS, Part II, §§11 and 12] for more details including the easy identification of the topos structure of $\mathscr{T}(\Sigma)$. Now we can view $\langle - \rangle\colon \mathscr{F}_{\mathscr{L}}(X) \to \mathscr{T}(\Sigma_{\mathscr{L},X})$ as a c.c. functor and $^{\#}\colon \mathscr{T}(\Sigma_{\mathscr{L},X}) \to \mathbf{E}$ as a *logical* functor (a functor between toposes is called *logical* if it preserves the topos structure).

We have thus associated a topos $\mathscr{T}(\Sigma_{\mathscr{L},X})$ with every free c.c. category $\mathscr{F}_{\mathscr{L}}(X)$. This state of affairs can be generalized: there is a purely category-theoretical operation that associates with *any* c.c. category $\mathbf{C}$ a topos $\Lambda(\mathbf{C})$ such that in the particular case of $\mathbf{C} = \mathscr{F}_{\mathscr{L}}(X)$, $\Lambda(\mathbf{C})$ is isomorphic to $\mathscr{T}(\Sigma_{\mathscr{L},X})$. A description of this construction is as follows. Let **BooleTop** be the category of Boolean toposes and **Concl** be that of c.c. categories. Every topos is a c.c. category, hence we have the forgetful functor Ψ: **BooleTop** $\to$ **Concl** that associates with each Boolean topos the same category as a c.c. one and leaves morphisms alone. Ψ has a left adjoint Λ: **Concl** $\to$ **BooleTop** (see [CWM], for instance). $\Lambda(\mathbf{C})$ may be described as the Boolean topos *freely c.c.-generated* by the c.c. category $\mathbf{C}$; by this we mean that $\Lambda(\mathbf{C})$ satisfies and is characterized up to isomorphism by the following universal property: it is a Boolean topos that comes with a c.c. morphism $F = F_{\mathbf{C}}\colon \mathbf{C} \to \Lambda(\mathbf{C})$ such that for any c.c. morphism $G\colon \mathbf{C} \to \mathbf{E}$ into a Boolean topos $\mathbf{E}$, there is a unique logical morphism H: $\Lambda(\mathbf{C}) \to \mathbf{E}$ such that $G = H \cdot F$. A remark: the reason for our saying "c.c.-generated" rather than, simply, "generated" is that the topos *freely generated* by $\mathbf{C}$ is something entirely different; in the "c.c.-generated" case, the connecting morphism $F\colon \mathbf{C} \to \Lambda(\mathbf{C})$ is c.c. whereas in the other case it is a mere functor and the c.c. structure of $\mathbf{C}$ plays no role in $\Lambda(\mathbf{C})$ whatsoever.

That this construction generalizes the one described before for free c.c. categories is seen from the following fact.

4.6.1. PROPOSITION. *If* $\mathbf{C} = \mathscr{F}_{\mathscr{L}}(X)$ *is a free c.c. category, then* $\Lambda(\mathbf{C})$ *and* $F_{\mathbf{C}}$ *are isomorphic to* $\mathscr{T}(\Sigma_{\mathscr{L},X})$ *and* $\langle - \rangle$ *respectively.*

The existence of the functor Λ and the assertion of 4.6.1 are not shown in [LS], but closely related facts are. In Example 16.5 on p. 209, the free topos generated by a graph is described in similar terms (in this case, the category of graphs replaces **Concl** and the category of toposes replaces **BooleTop**). We ask the reader to accept 4.6.1 without proof (those who feel ambitious may work it out analogously to the reference we just gave).

The following purely category-theoretical statement can be seen as a reformulation of 4.5.1.

4.6.2. THEOREM. *If* $\mathbf{C}$ *is a free c.c. category freely c.c.-generating the Boolean topos* $\Lambda(\mathbf{C})$ *with canonical c.c. functor* $F\colon \mathbf{C} \to \Lambda(\mathbf{C})$, *then* F *is weakly full.*

PROOF. Suppose first that $\mathbf{C}$ is countable. By 3.1 there is a set I and a weakly full c.c. functor $G\colon \mathbf{C} \to (\mathbf{Set}^{\mathbb{Z}})^I$. Since $(\mathbf{Set}^{\mathbb{Z}})^I$ is a Boolean topos, the universal property of $\Lambda(\mathbf{C})$ implies the existence of a (logical) functor H such that $G = H \cdot F$. The weak fullness of G immediately implies that of F. The general statement for not necessarily countable $\mathbf{C}$ follows from the special case just proved, by general arguments on the nature of free constructions. $\square$ 4.6.2

Notice that 4.5.1 with $\Sigma = 0$ immediately follows from 4.6.1 and 4.6.2.

Referring back to the last three paragraphs of §4.2, we recall that the relevant topos-theoretic facts are, first, that $\mathbf{Set}^{\mathbb{Z}}$ is a Boolean topos, with the forgetful functor $\mathbf{Set}^{\mathbb{Z}} \to \mathbf{Set}$ being logical (this fact is mentioned e.g. in Example (ii) on p. 65 of [BW]), and secondly, that the functor $\mathbf{Set} \to \mathbf{Set}^{\mathbb{Z}}$ mapping a set to the $\mathbb{Z}$-set with the trivial action on the given set is also logical ($\mathbf{Set}^{\mathbb{Z}}$ is atomic; see [BD]).

4.7. Theorem 4.6.2 has an interesting significance for proof theory. This becomes evident if we keep in mind the syntactical character of the categories $\mathscr{F}_{\mathscr{L}}(X)$ and $\mathscr{T}(\Sigma_{\mathscr{L},X})$. The objects of $\mathscr{F}_{\mathscr{L}}(X)$ are actually sentences, and its arrows are denoted by (terms representing) deductions of entailments in a theory T (in minimal propositional logic), two deductions denoting the same arrow precisely when their equality is provable from the c.c. identities of §2. Analogously, the objects and arrows of $\mathscr{T}(\Sigma_{\mathscr{L},X})$ are denoted by formal expressions, namely, certain terms of type theory with two such denoting the same object or the same arrow iff they are provably equal in $\Sigma_{\mathscr{L},X}$. Finally, $\langle - \rangle$, viewed as a syntactical operation on formal expressions, establishes a link between the two categories as it associates, with any sentence A and deduction f, terms $\langle A \rangle$ and $\langle f \rangle$ denoting an object and an arrow of $\mathscr{T}(\Sigma_{\mathscr{L},X})$. We may say that sentences and deductions originally used to construct $\mathscr{F}_{\mathscr{L}}(X)$ "live" in $\mathscr{T}(\Sigma_{\mathscr{L},X})$ as well. The following terminology is, therefore, justified.

4.7.1. DEFINITION. If A and B are sentences and $F :\subset [A] \times [B]$ a constant term such that $\Sigma_{\mathscr{L},X} \vdash F\colon \langle A \rangle \to \langle B \rangle$, then we say that F is a deduction of $A \Rightarrow B$ *in the generalized sense* or, for short, a *generalized deduction.*

Now 4.6.2 can be paraphrased as follows:

4.7.2. COROLLARY. *If an entailment* $A \Rightarrow B$ *has a generalized deduction* F, *then it has also a(n ordinary) deduction.*

All the considerations described so far can be extended to predicate calculus in the framework of fibrations; cf. [M1] and [M2]. The extended version of Corollary 4.7.2 answers in the affirmative a generalized version of a question asked by Howard [Ho, p. 489]. We should mention that, when we only had some vague feeling that the present work might be connected to this, Professor Howard kindly informed us that he had answered in the mean-time his own question but had not published the solution yet. Howard regards formulae as *types* (analogously, we regarded sentences as c.c. types) and defines, for every formula ϕ, a family of terms of type ϕ, each such family denoting a proof of ϕ in Heyting arithmetic (HA). He interprets his terms set-theoretically. In particular, proofs of existential and of disjunctive formulas are interpreted as ordered pairs (a proof of $\exists x.\phi(x)$ is a pair $\langle n, f\rangle$ with n a natural number and f a proof of $\phi(n)$, and a proof of $\phi_0 \vee \phi_1$ is $\langle i, f\rangle$ with $i = 0$ or 1 and f a proof of ϕ_i). The two projection operations that map an ordered pair to its components do not have a clear logical meaning, but, remarks Howard, they can be used in building more terms. One gets an extended class of terms, some of them of types that are not formulas. Howard's question is: if there is a closed term G in this extended class such that G has as type a formula ϕ, then must ϕ be provable in HA? The positive answer follows from 4.7.2 because the extended terms of Howard can be identified with terms that describe arrows in $\mathscr{T}(\Sigma_{\mathscr{L},X})$ (some cumbersome details have to be worked out to bridge the gap between Howard's working in natural deduction and set theory vs. our use of Lambek's system and type theory). Corollary 4.7.2 actually generalizes Howard's positive answer to his question in two respects: first, it applies to arbitrary theories, not just to HA, and second, the family of extended terms is closed under all type-theoretical operations, not just some, as in [Ho]. However, Howard's solution answers more than originally asked (cf. the next paragraph).

It is natural to ask about further properties of the functor $\mathbf{C} \to \Lambda(\mathbf{C})$ of 4.3, and about related properties of other similarly constructed functors. First of all, for the same reasons as we could have Σ in 4.5.1, we may "impose" the "relations" $\phi = T$ on $\Lambda(\mathbf{C})$ for any set of sentences ϕ valid in the standard interpretation. This fact could be expressed quite elegantly in categorical terms; but we shall not do so. As to the natural question of strengthening "weakly full" to "full" in the conclusion, the answer is no, as George Cubric (McGill University) has pointed out to us. However, if in place of $\Lambda(\mathbf{C})$ we consider the free topos, rather than the free Boolean topos, the resulting extension-functor might in general be full and in fact also faithful at the same time. In fact, Howard's full answer to his own question amounts to the fullness of the appropriate functor in a special case.

If $F\,(=\langle - \rangle)$ turns out to be faithful, this will mean that whenever two deductions f and g of the same entailment are provably equal in $\Sigma_{\mathscr{L},X}$ (i.e., $\Sigma_{\mathscr{L},X} \vdash \langle f\rangle = \langle g\rangle$), the equality $f = g$ is deducible from the c.c. identities. D. Prawitz advanced the thesis that two deductions in natural deduction represent the same proof iff they are inter-reducible in a suitable lambda-calculus (see [Pr, p. 257]). We believe that, after a suitable and natural link is established between natural deduction and the Lambek calculus, Prawitz's inter-reducibility will turn out to be equivalent to equality deducible under the c.c. identities (note, in particular, the absence of any special treatment of "false"). Thus, the question of the faithfulness of the functor F, or any of its variants, is relevant to the question of how stable the notion of equivalence of proofs is.

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