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FINITE PROBLEMS

Ju. T. MEDVEDEV

The completeness of the intuitionist propositional calculus H, formulated by Heyting [1], has been studied by many authors. An algebraic model with respect to which H is complete was constructed by Jaśkowski [2]. McKinsey and Tarski [3] proved the completeness of H in a topological sense. From a significant point of view, these and related results still have not solved the problem of the completeness of H because they have been obtained for models having the character of ad hoc interpretations. A. N. Kolmogorov [4] proposed a constructive interpretation of intuitionist logic, the idea of which consists in identifying true logical formulas with identically "solvable" problems. In accordance with this idea, the completeness of the calculus interpreted as a calculus of problems means the derivability of any formula which answers an "identically solvable" problem. Rose [5] discovered the incompleteness of H interpreted as a calculus of problems related to the recursive realizability of formulas [6].

The present paper is devoted to the problem of the completeness of Π, the positive part of H (i.e., that part in which negation is not considered). The models are formed by problems of the Kolmogorov type which we call finite, because we can study them by means of the theory of finite sets. For such models one can prove the completeness of Π in the sense indicated above (Theorem 1). An analogous result (Theorem 2) holds for the infinite analogue of finite problems, mass problems in the sense of earlier papers by the author [7,8]. In the final section some remarks are made about formulas containing negation.

1. By a finite problem we mean, in the present paper, any problem the solution of which is an element of some previously known nonempty finite set F of admissible possibilities. As examples of finite problems we may use the problems of determining the truth or falsity of propositions: here the set F consists of two elements, "true" and "false", and that one of the two which is the truth value of the proposition serves as the solution. Chess problems also may be considered as finite under the appropriate interpretation.

We remark that to every precisely formulated (not necessarily finite) problem $\mathfrak{A}$ one can associate a finite "approximating" problem $\mathfrak{A}_n$. To do this, let us consider a language R about which it is known that the solution of the problem $\mathfrak{A}$ can be expressed in it. As the set F_n of admissible possibilities for $\mathfrak{A}_n$, we take the set of all phrases of the language R having length $\leq n$, and as solutions those elements of F_n which express a solution of $\mathfrak{A}$. (For sufficiently large n the finite problem $\mathfrak{A}_n$ becomes equivalent to the problem $\mathfrak{A}$. However, it can turn out that the determination of such an n can be realized only after the solution of $\mathfrak{A}$.)

The set of solutions of a finite problem may be empty (in contradistinction to the set of admissible possibilities).

If F is the set of admissible possibilities for $\mathfrak{A}$, and X is the set of all solutions, then we shall write

$$F = \phi(\mathfrak{A}), \quad X = \chi(\mathfrak{A}).$$

The adjective "finite", modifying the word "problem", will sometimes be omitted for the sake of brevity.

2. Let us introduce the operations of conjugation, implication, and disjunction of problems. As a preliminary step, let us agree on some notation. Let E_1 and E_2 be arbitrary sets. Then:

1) By $E_1 \times E_2$ we denote the Cartesian product of E_1 and E_2, i.e., the set of all ordered pairs (x, y), where $x \in E_1$, $y \in E_2$.

2) By $E_2^{E_1}$ we designate the set of all mappings of E_1 into E_2.

3) By $\{E_1|E_2\}$ we denote the union of the sets $E_1 \times \{1\}$ and $E_2 \times \{2\}$, where $\{1\}$ and $\{2\}$ are single-element sets consisting, respectively, of the numbers 1 and 2. It is natural to call $\{E_1|E_2\}$ the ordered union of E_1 and E_2.

Now, let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be arbitrary finite problems, where

$$\phi(\mathfrak{A}_1) = F_1, \quad \chi(\mathfrak{A}_1) = X_1, \quad \phi(\mathfrak{A}_2) = F_2, \quad \chi(\mathfrak{A}_2) = X_2.$$

Each of the operations on problems introduced below yields the indicated set F of possibilities and the set X of solutions of the problem $\mathfrak{A}$ which is the result of the application of this operation.

I. For the conjunction $\mathfrak{A} = \mathfrak{A}_1 \,\&\, \mathfrak{A}_2$, we set $F = F_1 \times F_2$, $X = X_1 \times X_2$.

II. For the implication $\mathfrak{A} = \mathfrak{A}_1 \supset \mathfrak{A}_2$, we set $F = F_2^{F_1}$, and $X =$ the set of all those mappings of F_1 into F_2 which transform X_1 into X_2.

III. For the disjunction $\mathfrak{A} = \mathfrak{A}_1 \vee \mathfrak{A}_2$, we set

$$F = \{F_1|F_2\}, \quad X = \{X_1|X_2\}.$$

It is easy to verify that our definitions agree with the constructive standpoint given in the paper by A. N. Kolmogorov mentioned above.

3. Let us define the notion of an identically solvable composite problem. A logical formula $U(z_1, \cdots, z_m)$ containing no connectives other than &, $\supset$, and $\vee$ can be considered as a composite problem which is a function of the problems $z_1, \cdots, z_m$. The latter play the role of independent variables in place of which one can substitute concrete problems $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ obtaining as a result the problem $U = U(\mathfrak{A}_1, \cdots, \mathfrak{A}_m)$. From the definition of the operations on problems it follows that, in addition, $F = \phi(\mathfrak{A})$ depends upon $F_1 = \phi(\mathfrak{A}_1), \cdots, F_m = \phi(\mathfrak{A}_m)$ and does not depend upon $X_1 = \chi(\mathfrak{A}_1), \cdots, X_m = \chi(\mathfrak{A}_m)$. Let us at first consider fixed sets $F_1, \cdots, F_m$. To the choice of different systems of sets X_i ($X_i \subseteq F_i$, $i = 1, \cdots, m$) will correspond various $U(\mathfrak{A}_1, \cdots, \mathfrak{A}_m)$ with one and the same set F and different sets X. If there is an element of F belonging to all such sets X (i.e., the intersection of all such X is nonempty), then we shall say that the given composite problem U is solvable for the system of sets $F_1, \cdots, F_m$. If $U(z_1, \cdots, z_m)$ is solvable for any system of

(finite) sets $F_1, \cdots, F_m$, then we call it identically solvable. This definition has a transparent meaning: the identical solvability of U means that we are able to solve any problem $U(\mathfrak{A}_1, \cdots \cdots, \mathfrak{A}_m)$, knowing only the sets of admissible possibilities of the problems $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$.

4. The axioms of the positive calculus Π are the following eight formulas:

1. $x \supset (y \supset x)$.
2. $(x \supset (y \supset z)) \supset ((x \supset y) \supset (x \supset z))$.
3. $x \supset (y \supset (x \,\&\, y))$.
4. $(x \,\&\, y) \supset x$.
5. $(x \,\&\, y) \supset y$.
6. $x \supset (x \vee y)$.
7. $y \supset (x \vee y)$.
8. $(x \supset z) \supset ((y \supset z) \supset ((x \vee y) \supset z))$.

If Δ is a finite (possibly empty) sequence of formulas and A is a formula, then the expression $\Delta \vdash A$ means that A is derivable from Δ and axioms 1-8 using the rule of modus ponens and the rule of substitution.

In the sequel we shall use the term "identically solvable formula", understanding by this the identical solvability of the corresponding composite problem. Formulas which are not identically solvable are called refutable.

Lemma 1. *If $\Delta \vdash A$ and every formula of the sequence Δ is identically solvable, then the formula A is also identically solvable.*

From this, for empty Δ, follows

Corollary. *Every derivable formula of the calculus Π is identically solvable.*

In order to describe the essential factors in the proof of the converse proposition, we introduce some terminology. First, we shall write $\Pi_{i<n}\ C_i$ instead of the conjunction $C_1 \,\&\cdots\&\, C_n$, and $\Sigma_{i<n}\ C_i$ instead of the disjunction $C_1 \vee \cdots \vee C_n$, and each formula C_i is called a member of the given conjunction or disjunction. If all the C_i are variables (i.e., simple letters), then the given conjunction or disjunction is called elementary. We say that the formula J is a critical implication if J has the form

$$(\Pi_{i<n}\ ((P_i \supset Q_i) \supset Q_i)) \supset R,$$

where the formulas P_i are elementary conjunctions, and the formulas Q_i and R are elementary disjunctions, and if, in addition, for all $i = 1, \cdots, n$ the following condition is satisfied: no member of the conjunction P_i is a member of the disjunction Q_i. One easily proves

Lemma 2. *If J is a critical implication, then J is a refutable formula.*

The basic difficulty is presented by the proof of the following fact.

Lemma 3. *For every formula A one of the following assertions is valid:* 1) *A is derivable, or* 2) *there is a critical implication J such that $A \vdash J$.*

From Lemmas 1, 2, and 3, our basic result follows.

Theorem 1. *Any identically solvable formula of the calculus Π is derivable.*

This result permits of strengthening, leading to an algorithm for derivability of formulas of the calculus Π. Namely, for any formula U one can effectively determine a number $k = k(U)$ such that the derivability of U is equivalent to the solvability of U for a system of sets, each consisting of $l \geq k$ elements. The expression for the function $k(U)$, which one can find by analysis of the proofs of Lemmas 1, 2, and 3, we shall not give here. We mention only that it gives too large a value for k for the algorithm of derivability based upon it to have practical value.

5. We shall indicate the application of Theorem 1 to the problem of the completeness of Π as a

positive calculus of mass problems [7] (in fact, the solution of just this problem suggested to the author of this paper the consideration of finite problems). The following theorem is obtained as a quite simple consequence of Theorem 1, if we use the construction of operations on mass problems indicated in the author's dissertation [8].

Theorem 2. *Let the formula* $U(z_1, \cdots, z_m)$ *of the calculus* Π, *upon substitution of any degrees of difficulty* $a_1, \cdots, a_m$ *for* $z_1, \cdots, z_m$, *give (as a result of the application to* $a_1, \cdots, a_m$ *of the operations indicated in* U) *the degree of difficulty* 0. *Then* $U(z_1, \cdots, z_m)$ *is derivable in* Π.

Thus, Π turns out to be complete also as a calculus of mass problems.

6. One can introduce the operation of negation $\neg \mathfrak{A}$, defined for any finite problem $\mathfrak{A}$, by setting $\neg \mathfrak{A} = \mathfrak{A} \supset \mathfrak{A}_0$, where $\mathfrak{A}_0$ is a fixed problem with an empty set of solutions. This permits us (as in the case of formulas of the calculus Π) to introduce the notion of identical solvability for formulas of the calculus H. It is easy to see that every formula derivable in the calculus H will be identically solvable, but that there exist identically solvable nonderivable formulas. Among the latter belongs, for example, the formula considered for an analogous purpose by Rose [5] (and the formula $((\neg\neg D \supset D) \supset (\neg\neg D \vee \neg D)) \supset (\neg\neg D \vee \neg D)$, where D is any formula), and the formula $(\neg x \supset (y \vee z)) \supset ((\neg x \supset y) \vee (\neg x \supset z))$.

Thus, H is incomplete with respect to the given interpretation.

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INTERPRETATION OF LOGICAL FORMULAS BY MEANS OF FINITE PROBLEMS AND ITS RELATION TO THE REALIZABILITY THEORY

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In [1] we investigated a concept of the validity of logical formulas that was based on their interpretation as being composite problems of the Kolmogorov type. Theorems 1 and 2 of the present paper and the previous results show that for those formulas of Heyting's calculus H that contain none of the connectives $\neg$, $\vee$, or $\supset$, this validity is equivalent to deducibility in H. From Theorem 7 it follows that this assertion remains true if we replace validity in our sense by the realizability of Kleene and Rose. The latter fact was known at least in part with respect to the connective $\supset$ ([2], Theorem 7.5).

1. The class of formulas of the calculus H that are valid in the sense of [1] (i.e., the class of identically solvable formulas) will be denoted by T.

Theorem 1. *If a formula A does not contain the disjunction* $\vee$, *then it follows from* $A \in \mathrm{T}$ *that A is deducible in* H.

The proof is by the same method as in Theorem 1 in [1]. (It is based on a lemma analogous to Lemma 3 in [1] which is formulated in exactly the same way except that the term "critical implication" is defined differently.)

Theorem 2. *If a formula A does not contain the implication* $\supset$, *then it follows from* $A \in \mathrm{T}$ *that A is deducible in* H.

This theorem results from the following three easily proven theorems that are also of interest in themselves.

Theorem 3 (Rose, [2], Theorem 7.1). *If a formula A of the calculus* H *is not deducible in* H *and does not contain implication, then A is equivalent in* H *to the disjunction of a certain number of formulas that are not deducible in the classical propositional calculus.*

Theorem 4. *If a formula A has the form of a disjunction, then it follows from* $A \in \mathrm{T}$ *that at least one of the members of this disjunction* $\in \mathrm{T}$.

Theorem 5. *If* $A \in \mathrm{T}$, *then A is deducible in the classical propositional calculus.*

2. We denote the class of realizable formulas of the calculus H by P.

Theorem 6. *It follows from* $U \in \mathrm{P}$ *that* $U \in \mathrm{T}$.

In view of the comparative simplicity of the proof of this theorem, it will be given below (with some abbreviation).

Definition 1. Let $E_1, \cdots, E_n$ be some system of sets of natural numbers. We will say that this system *has an effectively empty intersection* if there exists a general recursive function p for which it follows from $p(x) = y$ that $1 \leq y \leq n$ and $x \in \overline{E}_y$.

Remark. In analogy with the concept of multiple separability in descriptive set theory [3], it is possible to make the following definition: sets of natural numbers $E_1, \cdots, E_n$ will be said to be *recursively multiply separable* if there exist recursive sets $B_1, \cdots, B_n$ such that $E_i \subseteq B_i$ for $1 \leq i \leq n$, and the intersection of all the B_i is empty. It is easy to show that the property of the system of sets

$E_1, \cdots, E_n$ having an effectively empty intersection is equivalent to the property of the sets of this system being recursively multiply separable.

Lemma 1. *For any natural number $n \geq 2$ there exists a system of n mutually disjoint recursively enumerable sets that does not have an effectively empty intersection.*

We consider for the proof the partial recursive function w in two variables that is universal for the class of all partial recursive functions in one variable. We denote by q the partial recursive function $q(x) \simeq w(x, x)$, and we let E_i $(1 \leq i \leq n)$ be the set of all x for which $q(x) = i$. We will show that the system of sets $E_1, \cdots, E_n$ has the desired properties. It is first clear that the sets of this system are recursively enumerable and mutually disjoint. We will show that the system has an effectively empty intersection and that p is the general recursive function required by Definition 1. In view of the universality of w, there exists an e such that $p(x) = w(e, x)$ for all x. Denoting $p(e)$ by k, we will have $k = w(e, e) = q(e)$. It follows from the definition of the function p that $1 \leq k \leq n$ and $e \,\overline{\in}\, E_k$, from which it follows that $q(e) \neq k$. The contradiction thus obtained shows that the system $E_1, \cdots, E_n$ does not have an effectively empty intersection.

Lemma 2. *For any natural number $N \geq 2$ there exists a system of $N-1$ mutually disjoint recursively enumerable sets $E_1, \cdots, E_{N-1}$ that possesses the following property: the system of the N sets $E_1, \cdots, E_{N-1}, E_N$, where E_N is the complement of the set $\bigcup_1^{N-1} E_i$, does not have an effectively empty intersection.*

It is sufficient to take for $E_1, \cdots, E_{N-1}$ the sets constructed for the proof of Lemma 1. Then for $N = 2$, the proposition to be proved follows from the fact that E_1 is not a recursive set. For $N > 2$, if one assumes that there exists a general recursive function p satisfying the condition of Definition 1 with respect to the system $E_1, \cdots, E_{N-1}, E_N$, one can easily construct (using the recursive enumerability of the sets $E_1, \cdots, E_{N-1}$) a general recursive function p satisfying that condition with respect to the system $E_1, \cdots, E_{N-1}$. This would contradict Lemma 1, and the proof of Lemma 2 is thus complete.

Now let the formula $U(z_1, \cdots, z_m)$, constructed from the propositional symbols $z_1, \cdots, z_m$, not be identically solvable, i.e., $U \,\overline{\in}\, \mathrm{T}$. Our goal is to show that then $U \,\overline{\in}\, \mathrm{P}$. In view of the refutability of U this formula is not solvable for some system of finite sets $F_1, \cdots, F_m$. One may assume that all the F_j are identical and coincide with the set I_n of natural numbers k satisfying the inequality $1 \leq k \leq n$ for some natural number $n \geq 1$. This follows from the following proposition, the proof of which we omit: if a formula is solvable in a system of sets $G_1, \cdots, G_m$, then it is also solvable in a system of sets $F_1, \cdots, F_m$, for which, for all $i = 1, \cdots, m$, the number of the elements in the set $F_i \leq$ the number of elements in the set G_i.

We enumerate in some definite manner all the possible ordered sequences of m elements having the form $c = (X_1, \cdots, X_m)$, where X_j are the subsets of I_n: $X_j \subseteq I_n$ for $1 \leq j \leq m$. We obtain the sequence $c_1, \cdots, c_N$, where $c_s = (X_1^{(s)}, \cdots, X_m^{(s)})$ for $1 \leq s \leq N$, where $N = 2^{mn} \geq 2$.

Let $E_1, \cdots, E_N$ be a system of sets, the existence of which was established in Lemma 2. There exist logical-arithmetic formulas $\alpha_1(t), \cdots, \alpha_N(t)$ containing a unique free number variable t and possessing the property that $\alpha_i(t)$ expresses the predicate $k \in E_i$. This means that if for any natural number k one denotes by k^* the term representing k in the formula system of arithmetic indicated, then for any s, $1 \leq s \leq N$, the relationship $k \in E_s$ is equivalent to the truth of the proposition $\alpha_k(k^*)$ under the intensional interpretation of the predicate $\alpha_s(t)$. We denote by ϵ_0 and ϵ_1 the arithmetical formulas expressing the propositions $0 = 0$ and $0 = 1$ respectively, and by $\beta_j(t)$ the arithmetic formula

$$\Sigma_{r\leq n}\Pi_{s\leq N}(\delta_{j,r,s} \vee \neg\, \alpha_s(t)),$$

where $\delta_{j,r,s}$ coincides with ϵ_0 for $r \in X_j^{(s)}$ and with ϵ_1 for $r \overline{\in} X_j^{(s)}$.

We will show that the arithmetic formula $\omega(t)$, obtained from $U(z_1, \cdots, z_m)$ by substituting $\beta_1(t), \cdots, \beta_m(t)$ for $z_1, \cdots, z_m$, is not realizable. With this it will also be shown that $U \overline{\in} \mathrm{P}$. We denote by $\mathfrak{A}_j(k)$ the finite problem for which $\phi(\mathfrak{A}_j(k)) = I_n$, $\chi(\mathfrak{A}_j(k)) = X_j^{(s)}$, where s $(1 \leq s \leq N)$ is defined from the condition $k \in E_s$ (in view of the properties of the system $E_1, \cdots, E_n$, such an s exists and is unique).

Lemma 3. *For the realizability of the formula* $\Pi_{s\leq N}(\delta_{j,r,s} \vee \neg\, \alpha_s(k^*))$ *i.e., of the member with the number r of the disjunction* $Bj(k^*)$, *it is necessary and sufficient that the condition* $r\in\chi(\mathfrak{A}_j(k))$ *be satisfied.*

We note that a formula of the form $\alpha_s(k^*)$ is realizable if and only if the predicate expressed by the formula $\alpha_s(t)$ is true for $t = k$. This follows from well-known results in realizability theory [4] and from the fact that $\alpha_s(k^*)$ has the form $(\exists u)\, \gamma(u)$ (for $1 \leq s \leq N-1$) or the form $(\forall u)\, \gamma(u)$ (for $s = N$), where the formula $\gamma(u)$ expresses a primitive recursive predicate. Thus, for the realizability of the given conjunction, it is necessary and sufficient that for all those s for which $\delta_{j,r,s}$ is ϵ_1 the formula $\alpha_s(k^*)$ expresses a false proposition. In other words, it must follow from $r \overline{\in} X_j^{(s)}$ that $k \overline{\in} E_s$. This is equivalent to the condition $r \overline{\in} \chi(\mathfrak{A}_j(k))$.

Lemma 4. *Let* $A(z_1, \cdots, z_m)$ *be any formula of the calculus* H. *We denote by* $\omega_A(k^*)$ *the formula* $A(\beta_1(k^*), \cdots, \beta_m(k^*))$ *and by* $\mathfrak{A}_A(k)$ *the finite problem* $A(\mathfrak{A}_1(k), \cdots, \mathfrak{A}_m(k))$. *Then there exist computable functions* f_A *and* g_A *such that for all k, e, and h the following conditions are satisfied:*

1) *If e realizes* $\omega_A(k^*)$, *then* $f_A(e) \in \chi(\mathfrak{A}_A(k))$.

2) *If* $h \in \chi(\mathfrak{A}_A(k))$, *then* $g_A(h)$ *realizes* $\omega_A(k^*)$.

In this formulation the domain of definition of the function g_A and the range of the function f_A are assumed to coincide with $\phi(\mathfrak{A}_A(k))$ (clearly, it does not depend on k).

The proof is by induction on the number of logical connectives in the formula A. If this number is equal to zero, then A coincides with the letter z_j for some j, $1 \leq j \leq m$. Then $\omega_A(k^*)$ coincides with $\beta_j(k^*)$ and $\mathfrak{A}_A(k)$ coincides with $\mathfrak{A}_j(k)$. In this case the existence of the functions f_A and g_A follows almost directly from Lemma 3 and the definition of the realizability of formulas (with consideration of the form of the formulas $\beta_j(k^*)$). We now assume that the number of logical connectives of the formula A is not equal to zero and that for formulas with a smaller number of connectives the assertion of the lemma is valid. Then A is a formula of one of four types: $B \,\&\, C$, $B \vee C$, $B \supset C$, or $\neg B$. We restrict ourselves to an analysis of the third case, i.e., to a formula A of the form $B \supset C$. As for the proof of Lemma 1, let w be a universal partial recursive function and let x realize $\omega_A(k^*)$. For every $b \in \phi(\mathfrak{A}_B(k))$ let $h(b) = f_C(w(x, g_B(b)))$. It is easy to show that if $b \in \chi(\mathfrak{A}_B(k))$ then $h(b) \in \chi(\mathfrak{A}_C(k))$. Thus, with each x realizing $\omega_A(k^*)$ the above expression associates (clearly in a computable way) a function $h \in \chi(\mathfrak{A}_A(k))$ and at the same time defines some function f_A with the required property. The function g_A is constructed analogously with the use of the functions f_B and g_C.

We turn directly to the proof of Theorem 6. Let us assume that the formula $\omega(t)$ is realizable. This implies the existence of a general recursive function r such that for any natural number k the number $r(k)$ realizes $\omega(k^*)$. We will designate $\mathfrak{A}_U(k)$ simply by $\mathfrak{A}(k)$. In view of Lemma 4, there exists a computable function f such that for any natural number k and for any x realizing $\omega(k^*)$, it is true that $f(x) \in \chi(\mathfrak{A}(k))$. Consequently, for a computable function $f_1(k) = f(r(k))$ for any k we have $f_1(k) \in \chi(\mathfrak{A}(k))$. Let $\mathfrak{A}^{(s)}$ $(1 \leq s \leq N)$ be the finite problem $U(\mathfrak{A}_1^{(s)}, \cdots, \mathfrak{A}_m^{(s)})$, where $\phi(\mathfrak{A}_j^{(s)}) = I_n$,

$\chi(\mathfrak{A}_j^{(s)}) = X_j^{(s)}$. Clearly, it follows from $k \in E_s$ that $\mathfrak{A}^{(s)}$ coincides with $\mathfrak{A}(k)$. Since $U(z_1, \cdots, z_m)$ is not solvable in the system of sets $F_1, \cdots, F_m$, where each F_j is I_n, therefore for any k there exists a smallest s, $1 \leq s \leq N$, for which $f_1(k) \overline{\in} \chi(\mathfrak{A}^{(s)})$, and thus $k \overline{\in} E_s$. Using the computability of f_1, it is easy to find a general recursive function which will locate the s for a given k. Denoting this function by p, we have $1 \leq p(k) \leq N$, where it follows from $p(k) = s$ that $k \overline{\in} E_s$. Thus the system $E_1, \cdots$ $\cdots, E_N$ has an effectively empty intersection, contrary to the original assumption. We must thus assume that the formula $\omega(t)$ is not realizable, which is what was required to be shown.

From Theorems 1, 2, 6, and the fundamental result of [1] (Theorem 1) we obtain the following result.

Theorem 7. *Let a formula* U *of the calculus* H *contain none of the connectives* $\neg, \vee, \supset$. *Then it follows from* $U \in$ P *that* U *is deducible in* H.

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INTERPRETATION OF LOGICAL FORMULAS BY MEANS OF FINITE PROBLEMS

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The present work is a continuation of [1, 2]. Logical formulas that are valid in the sense of those articles (formulas always decidable) will also be called *finitely valid* formulas in the future. The first section of this article contains a description of the finitely valid formulas in algebraic terms (Theorems 1 and 2). The second section introduces a new operation over finite problems, namely *weak disjunction*. The third section is concerned with the extension of the concept of finite validity to the class of formulas of the restricted predicate calculus.

1. Let $n \geq 1$ be a natural number. We denote by I_n the set of natural numbers i such that $1 \leq i \leq n$, and by σ^n the class of all nonempty subsets of the set I_n such that the condition $E \in \sigma^n$ is equivalent to the two conditions: $E \subseteq I_n$ and $E \neq \Lambda$, where Λ is the empty set.

Definition 1. Let $\sigma \subseteq \sigma^n$. The *closure* of the class σ with respect to σ^n is the class $\sigma^* \subseteq \sigma^n$ for which $E \in \sigma^*$ if and only if $E_0 \subseteq E \subseteq I_n$ for at least one $E_0 \in \sigma$.

The operation of closure with respect to σ^n just introduced is defined for each class $\sigma \subseteq \sigma^n$ and possesses the following properties: 1) $\lambda^* = \lambda$ for the empty class λ; 2) $\sigma^n \supseteq \sigma$; 3) $\sigma^{**} = \sigma^*$; 4) $(\sigma_1 \cup \sigma_2)^* = \sigma_1^* \cup \sigma_2^*$. This operation, considered in a collection with the set-theoretic operations of union, intersection, and complementation with respect to σ^n, turns the set of all $\sigma \subseteq \sigma^n$ into a Boolean closure algebra in the sense of McKinsey and Tarski [3], which we will denote by Σ^n.

Let Φ^n be the set of all closed $\sigma \in \Sigma^n$. We define on Φ^n the following operations (it is convenient to use for the symbols of these operations the logical connectives): $\sigma_1 \& \sigma_2 = \sigma_1 \cup \sigma_2$, $\sigma_1 \vee \sigma_2 = \sigma_1 \cap \sigma_2$, $\neg \sigma_1 = (\bar{\sigma}_1)^*$, $\sigma_1 \supset \sigma_2 = (\bar{\sigma}_1 \cap \sigma_2)^*$, where the upper line denotes complementation with respect to σ^n. These operations turn the set Φ^n into a Brauer algebra [4, 5], which we will designate with the same symbol Φ^n for the sake of simplicity. The element $\sigma_1 \supset \sigma_2$ can be defined as the smallest element $\sigma \in \Phi^n$ for which $\sigma_1 \& \sigma \geq \sigma_2$. The element $\neg \sigma$ the same as $\sigma \supset \sigma^n$. We will denote the smallest element of each of the algebras Σ^n and Φ^n (corresponding to the empty class λ) by the symbol 0.

Definition 2. An ordered system S of finite problems $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ $(n \geq 1)$ is said to be *reduced* if all the problems of the set have the same set of permissible possibilities: $\phi(\mathfrak{A}_1) = \cdots = \phi(\mathfrak{A}_n)$. The number n is called the *order* of S.

Definition 3. Let S be a reduced system of finite problems whose order is equal to n. Let F be the set of permissible possibilities, and let $X_1, \cdots, X_n$ be the corresponding sets of solutions. The *characteristic* of the system S is the class $\sigma(S)$ of sets defined by the conditions: 1) $\sigma(S) \subseteq \sigma^n$; 2) if $E \in \sigma^n$, then $E \in \sigma(S)$ if and only if the intersection of all the X_i for which $i \in E$ is the empty set.

Lemma 1. *In order for a class $\sigma \subseteq \sigma^n$ to be the characteristic of some reduced system S of problems with order n, it is necessary and sufficient that σ be a closed element of the algebra Σ^n.*

Definition 4. Let $\sigma_1, \sigma_2 \subseteq \sigma^n$. We will say that the class σ_1 is a *basis* of the class σ_2 if each element of σ_2 contains some element of σ_1.

Clearly the class σ_1 is a basis of the class σ_2 if and only if $\sigma_1^* \geq \sigma_2$ in the algebra Σ^n.

Definition 5. Let $S_1 = \{\mathfrak{A}_i\}$ and $S_2 = \{\mathfrak{B}_i\}$ be two reduced systems of problems of the same order n. We introduce *logical operations over* S_1 *and* S_2, yielding new systems of order n, by means of the stipulations: $S_1 \,\&\, S_2 = \{\mathfrak{A}_i \,\&\, \mathfrak{B}_i\}$, $S_1 \vee S_2 = \{\mathfrak{A}_i \vee \mathfrak{B}_i\}$, $\neg S_1 = \{\neg \mathfrak{A}_i\}$, $S_1 \supset S_2 = \{\mathfrak{A}_i \supset \mathfrak{B}_i\}$.

It turns out that the characteristics of the new systems are completely determined by the characteristics of the systems S_1 and S_2. Moreover, we have the following lemma.

Lemma 2. *Let S_1 and S_2 be two reduced systems of problems of the same order n, and let σ_2 respectively be bases of their characteristics. Then for the system S obtained as a result of one of logical operations over S_1 and S_2, a basis σ of the characteristic can be defined in the following manner:*

1) $S = S_1 \,\&\, S_2$: $\sigma = \sigma_1 \,\&\, \sigma_2$;

2) $S = S_1 \vee S_2$: *$E \in \sigma$ if and only if $E = E_1 \cup E_2$ for some $E_1 \in \sigma_1$ and $E_2 \in \sigma_2$;*

3) $S = \neg S_1$: $\sigma = \tilde{\sigma}_1$;

4) $S = S_1 \supset S_2$: *$E \in \sigma$ if and only if $E \in \sigma_2$ and there does not exist an $E_1 \in \sigma_1$ such that $E_1 \subseteq E$.*

The following assertion follows easily from the above lemma.

Lemma 3. *The following relations hold for reduced systems S_1 and S_2 of the same order n: $\sigma(S_1 \,\&\, S_2) = \sigma(S_1) \,\&\, \sigma(S_2)$, $\sigma(S_1 \vee S_2) = \sigma(S_1) \vee \sigma(S_2)$, $\sigma(\neg S_1) = \neg\sigma(S_1)$, $\sigma(S_1 \supset S_2) = \sigma(S_1) \supset \sigma(S_2)$; where the operations in the right-hand sides are understood in the sense of the algebra Φ^n.*

Let $U(z_1, \cdots, z_m)$ be a logical formula containing no logical connectives other than &, $\vee$, $\neg$, and $\supset$, and let $F_1, \cdots, F_m$ be nonempty finite sets. We consider a sequence X_k^i of sets $(k = 1, \cdots, m;\ i = 1, \cdots, n)$ for which $X_k^i \subseteq F_k$. We denote by $\mathfrak{A}_k^i$ a finite problem for which $\phi(\mathfrak{A}_k^i) = F_k$, $\chi(\mathfrak{A}_k^i) = X_k^i$. For fixed k we have a reduced system S_k of order n of the problems $\mathfrak{A}_k^i$ $(i = 1, \cdots n)$. Let $S = U(S_1, \cdots, S_m)$ be the reduced system obtained by applying to $S_1, \cdots, S_m$ the logical operations found in the formula U. Let us assume that the formula U is refutable in some system of sets $F_1, \cdots, F_m$. From the definition of refutability it follows that for some collection $S_1, \cdots, S_m$ of the described form the system S has as a characteristic a nonempty class $\sigma(S) \subseteq \sigma^n$. By Lemma 3 we have $\sigma(S) = U(\sigma(S_1), \cdots, \sigma(S_m))$, where the operations in the right-hand side of the equation are understood in the sense of the algebra Φ^n. Thus, U does not vanish identically as a function on Φ^n. Assume, conversely, that $U(\sigma_1, \cdots, \sigma_m) = 0$ in the algebra Φ^n. Then it is easy to establish by Lemma 1 that U is refutable in some system of sets $F_1, \cdots, F_m$. We arrive at the following result.

Theorem 1. *In order for a formula U to be finitely valid it is necessary and sufficient that U vanish identically on the Brauer algebra Φ^n for some $n \geq 1$.*

This theorem may be given in another form. Every finite distributive structure L can be considered as a Brauer algebra, if for each pair $a, b \in L$, one assumes the definitions: $a \,\&\, b$ = the upper bound of a and b, $a \vee b$ = the lower bound of a and b, $a \supset b$ = the least $c \in L$ for which $a \,\&\, c \geq b$ (such a c exists because of the finiteness and distributivity of L), $\neg a = a \supset e$, where e is the largest element of L. We denote by L_n the free distributive structure with n generators $x_1, \cdots, x_n$ and the additional symbol for the least element 0. It is easy to see that L_n, considered as a Brauer algebra, is isomorphic to Φ^n. The following variant of Theorem 1 is thus valid.

Theorem 2. *The class of finitely valid formulas coincides with the class of formulas that vanish* *identically on* L_n *for some natural number* n.

It is interesting to compare this result with the well-known fact that for a formula to be intuitionistically deducible it is necessary and sufficient that the formula be identically zero on some finite distributive structure. We note further that the structure L_n is isomorphic to the structure of monotone functions of n variables (two-valued) of the algebra of logic.

The most convenient way of establishing the refutability of a formula comes from Lemma 2. In this lemma there are indicated four (corresponding to the logical connectives) very simple operations over bases of characteristics of systems of order n. Since every $\sigma \subseteq \sigma^n$ is a basis of a characteristic (namely σ^*) for some system of order n, these operations are defined for all such σ. We thus have some algebra B^n. We denote the element of B^n corresponding to the empty family by 0. It follows from Lemma 2 that the formula U vanishes identically on Φ^n if and only if it vanishes identically on B^n.

2. According to A. N. Kolmogorov [6], the theory of a constructive solution of problems contains the propositional logic. It is natural that the language of this theory must also be more expressive. The necessity for the extension of the logical language arises, for example, in the following situation. Let A and B be problems in the sense of A. N. Kolmogorov. One can assume that we have succeeded in finding some object a having some relation to the solution of the problem A, and an object b having some relation to the solution of the problem B, and that, moreover, we have established that the statement "a is not a solution of A and b is not a solution of B" is false. To express this situation it is convenient to assume that the pair (a, b) serve as a solution of the *weak disjunction* $A \sqcup B$ of the problems A and B.

It is possible to be more precise in the case of finite problems. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be finite problems, where $\phi(\mathfrak{A}_1) = F_1$, $\phi(\mathfrak{A}_2) = F_2$, $\chi(\mathfrak{A}_1) = X_1$, $\chi(\mathfrak{A}_2) = X_2$. We define the problem $\mathfrak{A}$, which will be called the weak disjunction of the problems $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and denote it by $\mathfrak{A}_1 \sqcup \mathfrak{A}_2$, by the following stipulations: $F = \phi(\mathfrak{A}) = F_1 \times F_2$, $\chi(\mathfrak{A}) = X$, where $x = (f_1, f_2) \in F$ belongs to X if and only if it is false that $f_1 \in X_1$ and $f_2 \in X_2$. The concept of finite validity extends in an obvious way to the case of formulas containing the new connective $\sqcup$. Here, in particular, the following formula turns out to be finitely valid: $((x \sim u) \& (y \sim v)) \supset ((x \sqcup y) \sim (u \sqcup v))$(here $a \sim b$ denotes the conjunction $(a \supset b)$ & $(b \supset a)$). This shows the "correctness" of the operation of weak disjunction. Some other examples of finitely valid formulas are: $x \supset (x \sqcup y)$, $y \supset (x \sqcup y)$, $((x \vee y) \sqcup z) \supset ((x \sqcup z) \vee (y \sqcup z))$, $(\neg x) \sqcup (\neg\neg x)$.

The operation of weak disjunction can also be defined correctly for mass problems in the sense of [7,8].

3. We now extend the concept of finite validity to the class of formulas of the restricted predicate calculus. In the usual interpretation of such formulas in the case of a finite object domain D containing $k \geq 1$ individuals, the universal quantifier is considered a k-membered conjunction and the existential quantifier a k-membered disjunction. Here it turns out that each closed formula W is intensionally equivalent to some formula U of the propositional calculus (the variables of the formula U are taken from the predicate symbols of W, provided with indices from the elements of D). Assuming that D is the set of natural numbers $1, 2, \cdots, k$, we will denote the operation of passing from W to U by the symbol Γ_k, so that $U = \Gamma_k(W)$. A formal description of the operation Γ_k can be found, for example, in [9].

Definition 6. A closed formula W of the restricted predicate calculus is said to be *finitely valid* if for all $k \geq 1$, the formula $\Gamma_k(W)$ of the propositional calculus is finitely valid.

Theorem 3. *If a closed formula W of the restricted predicate calculus is deducible in the intuitionistic predicate calculus, then W is finitely valid.*

A nontrivial example of a finitely valid formula that is not deducible intuitionistically is the distributivity law for the universal quantifier: $(x)(A(x) \vee B) \supset ((x) A(x) \vee B)$, where $A(x)$ and B are closed formulas.

Theorem 4. *There does not exist an algorithm for recognizing finitely valid closed formulas of the restricted predicate calculus.*

This result is obtained from the following theorem, which is of independent interest.

Theorem 5. *In order for a formula of the propositional calculus of the form $\neg U$ to be finitely valid, it is necessary and sufficient that it be classically true (i.e. deducible in the classical propositional calculus).*

Theorem 4 is now obtained in the following manner. By Theorem 5, the finite validity of a closed formula $\neg\neg W$ of the restricted predicate calculus is equivalent to having, for each $k \geq 1$, the formula $\Gamma_k(\neg\neg W) = \neg\neg\Gamma_k(W)$ be classically true, i.e. to having the formula $\Gamma_k(W)$ be classically true for every k. The latter is equivalent to having W belong to the class Ω of closed formulas that are classically identically true in any finite domain. If there existed an algorithm deciding finite validity, then it would also decide membership of a formula in the class Ω. B. A. Trahtenbrot has proved, however, that such an algorithm is impossible [10].

We note in conclusion that the concept of finite validity extends in an obvious manner to the class of formulas containing in addition to the usual connectives $\&, \vee, \neg, \supset$, the universal quantifier (x) and the existential quantifier (E, x), and also the connective $\sqcup$ and the "weak existential quantifier corresponding to it $[x]$.

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