Models of Intuitionistic Logic, Model Solution 1

1. We write A_i for $A_i(P)$. First note that for any sentence A we have that $A \to A$ is derivable in IPC (by $\to$-introduction). From this it follows that for all $i \in \mathbb{N}$ we have $A_i \to A_i$ and $A_i \to A_\omega$. Next we have that $(A \to B) \wedge (B \to C) \to (A \to C)$ is derivable in IPC. Therefore, we only have to check that every level in the diagram implies the next one. Both $\bot \to P$ and $\bot \to \neg P$ are derivable in IPC (absurdity rule). Now let $n \in \mathbb{N}$. Then $A_{2n+1} \to (A_{2n+2} \to A_{2n+1})$, $A_{2n+1} \to A_{2n+1} \vee A_{2n+2}$ and $A_{2n+2} \to A_{2n+1} \vee A_{2n+2}$ are derivable in IPC. Recall that $A_{2n+2} \to A_{2n+1} = A_{2n+4}$ and $A_{2n+1} \vee A_{2n+2} = A_{2n+3}$ We conclude that $A_i \leq A_j \implies A_i \to A_j$.
   (Note: the deduction trees for all derivable statements are not required, as they are quite elementary).

   Points:
   1 point) $A \to A$ is derivable, concluding $A_i \to A_i$ and $A_i \to A_\omega$.
   1 point) $(A \to B) \wedge (B \to C) \to (A \to C)$ is derivable, concluding that only at each level of the diagram has to be checked.
   1 point) Implications at the bottom of the diagram.
   1 point) Implications for $n \in \mathbb{N}$.

2. a) For this one the domain does not matter, so fix some constant inhabited domain. Let A be a nullary relationsymbol. Consider the following Kripke Model:

   $$\begin{array}{cl} 1 & \Vdash A \\ | & \\ 0 & \end{array}$$

   Then $1 \Vdash A$, so $0, 1 \nVdash \neg A$. So $0 \Vdash \neg\neg A$, but $0 \nVdash A$, so $0 \nVdash \neg\neg A \to A$.
   b) For this one the domain does not matter, so fix some constant inhabited domain. Let A and B be nullary relationsymbols. Consider the following Kripke Model:

   $$\begin{array}{cl} 1 & \Vdash A \\ | & \\ 0 & \end{array}$$

   We have $1 \Vdash A$ and $1 \nVdash B$ hence $0 \nVdash A \to B$. As $1 \Vdash A$ we have $0 \Vdash (A \to B) \to A$. But $0 \nVdash A$, so $0 \nVdash ((A \to B) \to A) \to A$.

c) Let A be a unary relationsymbol. Consider the following Kripke Model:

$$\begin{array}{lll} 1 & \{a, b\} & \Vdash A(b) \\ | & & \\ 0 & \{a\} & \end{array}$$

Then $1 \Vdash \exists x A(x)$, hence $0, 1 \nVdash \neg\exists x A(x)$, hence $0 \Vdash \neg\neg\exists x A(x)$. But $0, 1 \nVdash A(a)$, hence $0 \Vdash \neg A(a)$. So $0 \nVdash \neg\neg A(a)$, and thus $0 \nVdash \exists x \neg\neg A(x)$. We conclude that $0 \nVdash \neg\neg\exists x A(x) \to \exists x \neg\neg A(x)$.

Points:
At each exercise you get 1 point for a correct Kripke Model and 1 point for a correct explanation why it works.