Seminar on Models of Intuitionism

Solutions to hand-in exercise 5

23 March

Exercise 1.

(a) By primitive recursion:

$$0! = 1 = S0$$
$$(n+1)! = n! \cdot (n+1) = H(n!, n),$$

where $H(x,y) = x \cdot (y+1)$ is clearly primitive recursive. ■

(b) By primitive recursion:

$$\begin{aligned} \mathrm{pd}(0) &= 0 \\ \mathrm{pd}(n+1) &= n = \pi^2_2(\mathrm{pd}(n), n); \\ x - 0 &= x = \pi^1_1(x) \\ x - (n+1) &= \mathrm{pd}(x-n) = \mathrm{pd}(\pi^3_1(x-n, x, n)). \end{aligned}$$

Furthermore, put $(x \leq y) = \mathrm{sg}(x - y)$ and $(x = y) = \mathrm{sg}((x-y) + (y-x))$. ■

(c) Let $\forall_{y<z}[F(\vec{x}, y) = 0] = \mathrm{sg}(\Sigma_{y<z} F(\vec{x}, y))$ and $x \nmid y = \forall_{z<y}[(1 - (x \cdot z = y)) = 0]$. ■

(d) Let $\mathrm{prime}(x) = \mathrm{sg}(x \geq S(S(Z(x))) + \forall_{y<x}[(y \leq S(Z(x))) \cdot (y \nmid x) = 0])$ (i.e. x is prime iff $x \geq 2$ and for any $y < x$, we have $y \leq 1$ or y does not divide x). ■

(e) By primitive recursion:

$$\begin{aligned} p_0 &= 1 = S0 \\ p_{n+1} &= \mu y < (p_n! + 2)[\mathrm{prime}(y) + (p_n + 1 \leq y) = 0] = H(\pi^2_2(p_n, n)), \end{aligned}$$

where $H(x) = \mu y < (x! + 2)[\mathrm{prime}(y) + (x + 1 \leq y) = 0]$ is a composition of primitive recursive functions and therefore primitive recursive.

Note that this works, because the least divisor $y > 1$ of $p_n! + 1$ is prime and must be unequal to $p_1, \ldots, p_n$; for if $y = p_i$, then $y \mid p_n!$, so $y \mid 1$, contradicting that $y > 1$. ■

Half a point for a right bound; half a point for an explanation of why this bound works; one point for an otherwise correct definition.

Exercise 2. By the Recursion Theorem applied to the primitive recursive function $\lambda xy.(x < y)$ (i.e. $\lambda xy.1 - (y \leq x)$), we have e such that $\varphi_e(x) \simeq (x < e)$ for all x. Note that φ_e is recursive and that it is self-describing as the least x with $(x < e) \neq 1$ is exactly e. ■

Exercise 3.

(a) Let $G(x, y, z) \simeq \Phi(1, x, y) \simeq \varphi_x(y)$ be partial recursive (it is so, since Φ is). By the Enumeration Theorem, it has an index c. Put $F(x, y) = S^2_1(c, x, y)$ and observe that F is recursive (in fact, even primitive recursive, since S^2_1 is). Further,

$$\begin{aligned}(x, y) \in H &\Leftrightarrow \varphi_x(y) \text{ is defined}\\ &\Leftrightarrow G(x, y, F(x, y)) \text{ is defined}\\ &\Leftrightarrow \varphi_c(x, y, F(x, y)) \text{ is defined}\\ &\Leftrightarrow \varphi_{S^2_1(c,x,y)}(F(x, y)) \text{ is defined}\\ &\Leftrightarrow \varphi_{F(x,y)}(F(x, y)) \text{ is defined}\\ &\Leftrightarrow F(x, y) \in K.\end{aligned}$$

■

Half a point for defining G and explaining that it is partial recursive; one point for applying the Enumeration Theorem, defining F and mentioning that F is recursive; one point for showing that F works and completing the proof.

(b) Suppose for a contradiction that χ_K were recursive. Then so would $xy.\chi_K(F(x, y))$. But by (a) this function is exactly χ_H, contradicting the undecidability of the Halting Problem. ■