Exercise 1

a) Suppose $n \in \mathbb{N}$ realizes the sentence. Applying the realization rule for $\forall$, we find that for n:

$$\text{for all } m : (\varphi_n(m) \text{ realizes } S(m) = 0 \to \perp) \text{ and } \varphi_n(m) \downarrow$$

By the realization rule for $\to$, we next have: for all m, m' :

$$(m' \text{ realizes } S(m) = 0 \text{ implies } \varphi_{\varphi_n(m)}(m') \text{ realizes } \perp \text{ and } \varphi_{\varphi_n(m)}(m') \downarrow) \text{ and } \varphi_n(m) \downarrow$$

Note that $S(m) = 0$ is not realized by any number since it is false for the natural numbers. Hence, the above implication holds true for any n, m and m'. The only requirement on n that remains is that $\varphi_n(m)$ is defined for every m. We can thus take any n for which φ_n is everywhere defined. For example, we might take n to be the index of the constant zero function.

Grading:
(0.5 points) Working out the realizability rules for the given sentence.
(0.5 points) Giving a correct realizer.

b) Suppose $n \in \mathbb{N}$ realizes the sentence. Applying the realization rule for $\vee$, we find that for n:

$$(\text{fst}(n) = 0 \text{ implies } \text{snd}(n) \text{ realizes } P) \text{ and } (\text{fst}(n) \neq 0 \text{ implies } \text{snd}(n) \text{ realizes } P \to \perp)$$

By the realization rule for $\to$, we next have:

$$(\text{fst}(n) = 0 \text{ implies } \text{snd}(n) \text{ realizes } P)$$

and

$$(\text{fst}(n) \neq 0 \text{ implies for all } m : m \text{ realizes } P \text{ implies } (\varphi_{\text{snd}(n)}(m) \text{ realizes } \perp \text{ and } \varphi_{\text{snd}(n)}(m) \downarrow))$$

Assume that P is true. In this case, $n = \langle 0, n' \rangle$ satisfies the above condition, where n' is an arbitrary natural number. On the other hand, if P is not true, then there exists no number m such that m realizes P. Hence, the above condition is satisfied by any natural number n. In either case, we see that the sentence $P \vee \neg P$ is always realizable.

Grading:
(0.5 points) Working out the realizability rules for the given sentence.
(0.5 points) Giving a correct realizer.

c) Suppose $n \in \mathbb{N}$ realizes the sentence. Applying the realization rules for $\forall, \vee$, and $\exists$ we find that for n: for all $m : \varphi_n(m) \downarrow$ and

$$\text{fst}(\varphi_n(m)) = 0 \text{ implies } \text{snd}(\varphi_n(m)) \text{ realizes } m = 0$$

and

$$\text{fst}(\varphi_n(m)) \neq 0 \text{ implies } \text{snd(snd}(\varphi_n(m))) \text{ realizes } m = S(\text{fst(snd}(\varphi_n(m)))$$

Now, suppose that $m = 0$. Then we can take $\varphi_n(m) = \langle 0, k \rangle$, with k an arbitrary natural number. Next, suppose $m \neq 0$. Then we can take $\varphi_n(m) = \langle 1, \langle m-1, 0 \rangle \rangle$. The defined function is recursive and, in particular, everywhere defined. Hence, the sentence is realizable.

Grading:
(1 point) Working out the realizability rules for the given sentence.
(0.5 points) Giving a correct realizer.

Exercise 2

If CT_0 were to hold true, this would imply that every total function is recursive. Thus, we might take the formula $A(x, y)$ to represent the statement that $\chi_H(x) = y$, where χ_H is the characteristic function of the Halting set. For this A, it is clear that the sentence obtained from the schema CT_0 is not provable in **PA**. Hence, in particular, it is not provable in **HA**.

Grading:
(1 point) Linking the problem to the undecidability of the Halting Problem and giving a correct instantiation of CT_0.
(1 point) Showing the obtained instantiation is not derivable in **HA**.

Exercise 3

a) Let's spell out what it means for e **rn** $\forall x(A(x) \to \exists y B(x, y))$:

$$\forall x(\forall m(m \textbf{ rn } A(x) \to \text{snd}(\varphi_{\varphi_e(x)}(m)) \textbf{ rn } B(x, \text{fst}(\varphi_{\varphi_e(x)}(m))) \wedge \varphi_{\varphi_e(x)}(m) \downarrow) \wedge \varphi_e(x) \downarrow).$$

Hence, from this we can deduce

$$\forall x(\psi_A(x) \textbf{ rn } A(x) \wedge \psi_A(x) \downarrow \to \text{snd}(\varphi_{\varphi_e(x)}(\psi_A(x))) \textbf{ rn } B(x, \text{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \wedge \varphi_{\varphi_e(x)}(\psi_A(x)) \downarrow).$$

Now, since $A(x)$ is almost negative, we can apply Proposition 1.8 to conclude that for any n we have n **rn** $A(x) \to A(x)$ and $A(x) \to \psi_A(x)$ **rn** $A(x) \wedge \psi_A(x) \downarrow$ So we conclude that

$$\forall x, n(n \textbf{ rn } A(x) \to \text{snd}(\varphi_{\varphi_e(x)}(\psi_A(x))) \textbf{ rn } B(x, \text{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \wedge \varphi_{\varphi_e(x)}(\psi_A(x)) \downarrow)$$

as desired.

Grading
(1 point): correct unfolding definition of **rn** .
(0.5 points): As A is almost negative, we can apply proposition 1.8.

b) We define $t_2(e) := [\lambda x.[\lambda n. \left\langle \mu z.T([t_1(e)], x, z), \left\langle 0, \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \right\rangle \right\rangle]]$. Suppose we have an x and n such that n **rn** $A(x)$. Then by $a)$ we conclude that $\varphi_{\varphi_e(x)}(\psi_A(x)) \downarrow$, hence also $\mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x))) \downarrow$ and $\mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x))) \downarrow$. Notice then that $\mu z.T([t_1(e)], x, z)$ terminates (as $\varphi_{[t_1(e)]}(x) \simeq \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))$ and we have that $T([t_1(e)], x, \mu z.T([t_1(e)], x, z))$ holds. On the other hand, by $a)$ we also have that $\mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)))$ **rn** $B(x, \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x))))$ holds. Hence by definition of **rn** and using the hint we conclude

$$T([t_1(e)], x, \mu z.T([t_1(e)], x, z)) \wedge \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \;\mathbf{rn}\; B(x, \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \equiv$$
$$0 \;\mathbf{rn}\; T([t_1(e)], x, \mu z.T([t_1(e)], x, z)) \wedge \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \;\mathbf{rn}\; B(x, U(\mu z.T([t_1(e)], x, z))) \equiv$$
$$\left\langle 0, \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \right\rangle \;\mathbf{rn}\; T([t_1(e)], x, \mu z.T([t_1(e)], x, z)) \wedge B(x, U(\mu z.T([t_1(e)], x, z)))) \equiv$$
$$\left\langle \mu z.T([t_1(e)], x, z), \left\langle 0, \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \right\rangle \right\rangle \;\mathbf{rn}\; \exists z(T([t_1(e)], x, z) \wedge B(x, U(z))) \equiv$$
$$\left\langle \mu z.T([t_1(e)], x, z), \left\langle 0, \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \right\rangle \right\rangle \;\mathbf{rn}\; B(x, \varphi_{[t_1(e)]}(x)) \wedge \varphi_{[t_1(e)]}(x) \downarrow \equiv$$
$$\left\langle \mu z.T([t_1(e)], x, z), \left\langle 0, \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A(x)) \right\rangle \right\rangle \;\mathbf{rn}\; B(x, \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \wedge \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x))) \downarrow.$$

As $[\lambda n. \left\langle \mathrm{snd}(\varphi_{\varphi_e(x)}(\psi_A), \langle 0, \mu z.T([t_1(e)], x, z) \rangle \right\rangle]$ is just a code of a partial recursive function, it is defined. So we deduce that

$$t_2(e) \;\mathbf{rn}\; \forall x(A(x) \rightarrow B(x, \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \wedge \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x))) \downarrow).$$

Grading:
(0.5 points): Finding the correct term.
(0.5 points): Argumenst that some of the terms are defined.
(1 point): Show that the presented term indeed realises the statement.

c) We define $x := [\lambda e. \langle [t_1(e)], t_2(e) \rangle]$ and claim that x **rn** F. Spelling out the definitions:

$$\forall e(e \;\mathbf{rn}\; (\forall x(A(x) \rightarrow \exists y B(x, y))) \rightarrow \langle [t_1(e)], t_2(e) \rangle \;\mathbf{rn}\; \exists e \forall x(A(x) \rightarrow B(x, \varphi_e(x)) \wedge \varphi_e(x) \downarrow)) \equiv$$
$$\forall e(e \;\mathbf{rn}\; (\forall x(A(x) \rightarrow \exists y B(x, y))) \rightarrow t_2(e) \;\mathbf{rn}\; \forall x(A(x) \rightarrow B(x, \varphi_{[t_1(e)]}(x)) \wedge \varphi_{[t_1(e)]}(x) \downarrow)) =$$
$$\forall e(e \;\mathbf{rn}\; (\forall x(A(x) \rightarrow \exists y B(x, y))) \rightarrow t_2(e) \;\mathbf{rn}\; \forall x(A(x) \rightarrow B(x, \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x)))) \wedge \mathrm{fst}(\varphi_{\varphi_e(x)}(\psi_A(x))) \downarrow)).$$

So by $b)$ we conclude that x **rn** F and we thus have $\mathbf{HA} \vdash \exists x(x \;\mathbf{rn}\; F)$.

Grading:
(0.5 points) Finding the correct term.
(0.5 points) Show that it works.