Models of Intuisionism

Hand-in exercise 7: Model solution

Sven Bosman

(1a.) Let $F(P)$ be a given assignment of possible solutions to P. Since $F(P)$ is nonempty, we know that there is some element $a \in F(P)$, and hence the constant a-function $f_a : F(\neg\neg P) \to F(P)$ is an element of $F(\neg\neg P \to P)$. We now define the constant a-function for every $a \in F(P)$, and we fix a specific $a \in F(P)$. Also note that we can easily define a well-ordering on $F(P)$, since it is finite. We will define a function $g \in F(\varphi)$ by:

$$g(h) = \begin{cases} h(f_b) & \text{if } b \text{ is the least element of } F(P) \text{ such that } h(f_b) \in \{1\} \times F(\neg\neg P) \\ h(f_a) & \text{if such an element } b \text{ does not exist} \end{cases}$$

We claim that this function g is an element of $X(\varphi)$ for every assignment $X(P)$. In order to prove this we need a case distinction between $X(P) = \emptyset$ and $X(P) \neq \emptyset$.
Suppose first that $X(P)$ is empty. Then we notice that $X(\neg P) = F(\neg P)$ and $X(\neg\neg P) = \emptyset$. And by this last observation we notice that $f_a \in X(\neg\neg P \to P)$ for all $a \in F(P)$. Now if $h \in X((\neg\neg P \to P) \to (\neg P \vee \neg\neg P))$ is given, then we always have $h(f_a) \in X(\neg P \vee \neg\neg P)$, and hence we find that $g(h) \in X(\neg P \vee \neg\neg P)$ for all $h \in X((\neg\neg P \to P) \to (\neg P \vee \neg\neg P))$. So indeed in this case we have that $g \in X(\varphi)$.
Now suppose that $X(P)$ is not empty. Then let $b \in X(P)$, so we notice that $f_b(X(\neg\neg P)) \subseteq X(P)$. Hence $f_b \in X(\neg\neg P \to P)$, which means that if $h \in X((\neg\neg P \to P) \to (\neg P \vee \neg\neg P))$ then $h(f_b) \in X(\neg P \vee \neg\neg P)$. We easily notice that in this case we have that $X(\neg P)$ is empty so $h(f_b) \in \{1\} \times X(\neg\neg P)$. Since there is such an element b, we know that $g(h) = h(f_c)$ with c the smallest element in $F(P)$ such that $h(f_c) \in \{1\} \times F(\neg\neg P)$. Since $X(\neg\neg P) = F(\neg\neg P)$ we now know that $g(h) \in X(\neg P \vee \neg\neg P)$, so indeed we see that $g \in X(\varphi)$. $\square$

The case distinction on $X(P)$ was crucial in this exercise, so using this was awarded 1 point. Working out $X(\varphi)$ in the different cases was also worth 1 point. 1 point was awarded for giving a correct function, and 1 point for the rest of the proof.

(1b.) We prove that φ is not provable in intuitionistic logic by giving a Kripke counter model. Consider the following model:

$$\begin{tikzcd} 2 & & \\ 1 \arrow[u] & & 3 \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Here P is forced in world 2, and we see that $\neg P$ is forced in world 3. We see that $\neg\neg P$ is forced in worlds 1 and 2. So worlds 2 and 3 are the only ones where $\neg\neg P \to P$ is forced, and in both of these worlds we see that $\neg P \vee \neg\neg P$ is forced. So in fact we see that $(\neg\neg P \to P) \to (\neg P \vee \neg\neg P)$ is forced in all the worlds of this Kripke model. However, since

$\neg P$ is forced in world 3 and P in world 2, we see that $\neg P \vee \neg\neg P$ is not forced in world 0. So φ is not forced in world 0. It follows that φ is not provable in intuitionistic logic. Combining exercises a and b we conclude that the Medvedev model of finite problems is not complete with respect to intuitionistic propositional logic. $\square$

1 point was awarded for giving a correct Kripke model. $1\frac{3}{4}$ *point for explaining why this model works, and* $\frac{1}{4}$ *point for the conclusion on completeness.*

(2.) Let $J = \bigwedge_{i<n}((P_i \to Q_i) \to Q_i) \to R$ is a critical implication. Define $F(x) = \{*\}$ for every elementary x occurring in J. We now show that for every $f \in F(J)$, we can find an assignment X to the elementary problems in J such that $f \notin X(J)$. So suppose we are given an f in $F(J)$. We first notice that $F(P_i) = \{\langle *, *, ..., * \rangle\} = \star$ for every i, so define for every i: $g_i : F(P_i \to Q(i)) \to F(Q_i)$ by $g_i(h) = h(\star)$. Let $x_1, ..., x_r$ be the elementary problems occurring in R. Notice that $f(\langle g_0, ..., g_{n-1} \rangle) = \langle j, * \rangle$ for some $j \leq r$. Now let $X(x_j) = \emptyset$ and $X(x) = F(x)$ for all $x \neq x_j$. We notice that if we can prove that $g_i \in X((P_i \to Q_i) \to Q_i)$ for every i, we would find that $f \notin X(J)$. We will show this using a case distinction. So we fix an $i < n$.
First suppose that x_j does not occur in Q_i. Then $X(Q_i) = F(Q_i)$, and hence we find that $F((P_i \to Q_i) \to Q_i) = X((P_i \to Q_i) \to Q_i)$. So clearly $g_i \in X((P_i \to Q_i) \to Q_i)$.
Now suppose that x_j does occur in Q_i. Then clearly is does not occur in P_i, so $X(P_i) = F(P_i)$. For any $a \in X(P_i \to Q_i)$, we know that $a(X(P_i)) \subseteq X(Q_i)$. So since $X(P_i) = \{\star\}$, we know that $g_i(a) = a(\star) \in X(Q_i)$. So we indeed see that $g_i(X(P_i \to Q_i)) \subseteq X(Q_i)$. And hence $g_i \in X((P_i \to Q_i) \to Q_i)$. $\square$

1 point was awarded for giving a possible solution assignment and to every f *an assignment* X *such that* $f \notin X(J)$*. Proving that this* X *is correct was worth 2 points. A solution which only works in the case that* R *is disjoint from all the* Q_i *was awarded with at most 1 point.*