1 Beth Models

Note: the numbering for definitions, theorems, etc. directly corresponds to the numbering used in *Constructivism in Mathematics: An Introduction, Volume II*, 1988, ch. 13.

1.1 Introduction

Definition 1.1. A *Beth model* for a relational language $\mathscr{L}$ is a quadruple $\mathscr{B} = (K, \preceq, D, \Vdash)$ such that

(i) $(K, \preceq)$ is a spread,

(ii) D is a domain function assigning to each node $k \in K$ a non-empty set $D(k)$ such that $k \preceq k'$ implies $D(k) \subseteq D(k')$,

(iii) the forcing relation $\Vdash$ is a binary relation between nodes of K and atomic sentences P such that

B1. $k \Vdash P \iff \forall \alpha \in k \exists m : \bar{\alpha}(m) \Vdash P$ and $D(k)$ contains the constants in P,
$k \nVdash \bot$ for all $k \in K$,

B2. $k \Vdash A \wedge B \iff k \Vdash A$ and $k \Vdash B$,

B3. $k \Vdash A \vee B \iff \forall \alpha \in k \exists n : \bar{\alpha}(n) \Vdash A$ or $\bar{\alpha}(n) \Vdash B$,

B4. $k \Vdash A \rightarrow B \iff \forall k' \succeq k : k' \Vdash A$ implies $k' \Vdash B$,

B5. $k \Vdash \exists x A(x) \iff \forall \alpha \in k \exists n \exists d \in D(\bar{\alpha}(n)) : \bar{\alpha}(n) \Vdash A(d)$,

B6. $k \Vdash \forall x A(x) \iff \forall k' \succeq k \forall d \in D(k') : k' \Vdash A(d)$.

In this definition α ranges over the infinite branches of $(K, \preceq)$.

If $(K, \preceq)$ is a fan, we can, instead of B1, B3, B5, use the following, stronger conditions:

B1′ $k \Vdash P \iff \exists z \forall k' \succeq_z k \exists k'' \preceq k' : k'' \Vdash P$

B2′ $k \Vdash A \vee B \iff \exists z \forall k' \succeq_z k : k' \Vdash A$ or $k' \Vdash B$

B3′ $k \Vdash \exists x A(x) \iff \exists z \forall k' \succeq_z k \exists d \in D(k') : k' \Vdash A(d)$

We can also liberalize the definition of Beth models by allowing $(K, \preceq)$ to be an arbitrary tree instead of a spread, i.e. we no longer require each $k \in K$ to have a $\preceq$-successor. This permits Beth models to be finite, with quantification over infinite branches α replaced by quantification over the $\preceq$-maximal nodes in the tree. Let us refer to these as **liberalized Beth models**.

1.2 Relation to Kripke Models

Definition 1.5. Let $\mathscr{K} = (K, \preceq, D, \Vdash)$ be a Kripke model. We associate to this Kripke model a Beth model $\mathscr{K}' = (K', \preceq', D', \Vdash')$ in the following manner:

(i) K′ consists of all finite non-decreasing sequences of $(K, \preceq)$,

(ii) $\preceq'$ is the usual initial segment relation,

(iii) $D'((k_1, \ldots, k_n)) := D(k_n)$,

(iv) $(k_1, \ldots, k_n) \Vdash' P \iff k_n \Vdash P$.

Theorem 1.5. *Let $\mathscr{K}$ be a Kripke model and $\mathscr{K}'$ its corresponding Beth model. For all nodes $k_1, \ldots, k_n \in K$ and $\mathscr{L}(D(k_n))$-sentences A, we have*

$$(k_1, \ldots, k_n) \Vdash' A \iff k_n \Vdash A.$$

By a more elaborate construction, we can show something stronger: we can transform every Kripke model to a Beth model *with constant domain*.

1.3 Completeness

Lemma 2.3. *For all $k \in K$, $\mathscr{L}(\Gamma_k)$-sentences A and $x \in \mathbb{N}$:*

$$\Gamma_k \vdash A \iff \forall k' \succeq_x k : \Gamma_{k'} \vdash A.$$

Lemma 2.5. *For the Beth model $\mathscr{B}^*$, we have for every $k \in K$ and $\mathscr{L}(\Gamma_k)$-sentence A:*

$$k \Vdash A \iff \Gamma_k \vdash A.$$

Theorem 2.8. *For **IQC** there exists a fallible Beth model $\mathscr{B}^*$ such that, for all sentences A,*

$$\mathscr{B}^* \Vdash A \iff \Gamma \vdash A.$$

2 Heyting algebras

Definition. A *lattice* is a poset $(A, \leq)$ such that for each $a, b \in A$ there is a least upper bound $a \vee b$ (the *join* of a and b) and a greatest lower bound $a \wedge b$ (the *meet* of a and b).

Definition. A lattice $(A, \leq)$ is *bounded* if it contains an element $\bot$, called *bottom*, satisfying $\forall a \in A(\bot \leq a)$ and an element $\top$, called *top*, satisfying $\forall a \in A(a \leq \top)$. If existing, top and bottom are unique.

Definition. A lattice $(A, \leq)$ is *distributive* if for all $a, b, c \in A$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Definition. We say that a lattice is *complete* if every subset $X \subseteq A$ has a *join* $\bigvee X := \sup(X)$ and a *meet* $\bigwedge X := \inf(X)$.

Definition. A *(complete) Heyting algebra*, (c)Ha for short, is a (complete) bounded lattice $(A, \leq)$ such that for each $a, b \in A$ the set $\{x \mid x \wedge a \leq b\}$ has a greatest element, which we then denote by $a \to b$.

Properties. The following properties hold for a Ha $(A, \leq)$ and elements $a, b, c \in A$.

1. A is distributive.
2. $(a \wedge b) \leq c \Leftrightarrow (a \leq b \to c)$,
3. $a \to b = \top \Leftrightarrow a \leq b$,

3 Global Ω-models

We work in a fixed one-sorted **IQC**-language $\mathcal{L}$ without equality. Let Ω be a fixed cHa.

Definition. A *global Ω-model* for $\mathcal{L}$ consists of a set M together with:

- an element $[\![c]\!] \in M$ for each constant symbol $c \in \mathcal{L}$,
- a function $[\![R]\!] : M^n \to \Omega$ for each n-ary relation symbol R in $\mathcal{L}$,
- a function $[\![f]\!] : M^n \to M$ for each n-ary function symbol f in $\mathcal{L}$.

Semantics. We extend $[\![\ ]\!]$ to terms in $\mathcal{L}_M$ by taking

$$[\![c_m]\!] := m,$$

$$[\![f(t_1, \ldots, t_n)]\!] := [\![f([\![t_1]\!], \ldots, [\![t_n]\!])]\!] := [\![f]\!]([\![t_1]\!], \ldots, [\![t_n]\!]).$$

Now $[\![\ ]\!]$ is defined for sentences of $\mathcal{L}_M$ by

$$[\![R(t_1, \ldots, t_n)]\!] := [\![R([\![t_1]\!], \ldots, [\![t_n]\!])]\!] := [\![R]\!]([\![t_1]\!], \ldots, [\![t_n]\!]),$$

$$[\![\bot]\!] := \bot,$$

$$[\![A \circ B]\!] := [\![A]\!] \circ [\![B]\!] \text{ for } \circ \in \{\wedge, \vee, \to\},$$

$$[\![\forall x A(x)]\!] := \bigwedge\{[\![A(m)]\!] \mid m \in M]\},$$

$$[\![\exists x A(x)]\!] := \bigvee\{[\![A(m)]\!] \mid m \in M\}.$$

4 Intuitionistic logic with existence

We transform **IQC** (without equality) to a logic with existence as follows. First we add the rule

$$\text{SUB} \quad \frac{A}{A[x/t]},$$

where x is any variable not occurring freely in assumptions of the derivation of A. Furthermore, we add a special relation **E** and adapt the quantifiers deduction rules as follows

$$\frac{\begin{matrix}[\mathbf{E}x]\\ \vdots \\ A\end{matrix}}{\forall y A[x/y]}\,\forall I^{\mathbf{E}} \qquad \frac{\forall x A \quad \mathbf{E}t}{A[x/t]}\,\forall E^{\mathbf{E}},$$

$$\frac{A[x/t] \quad \mathbf{E}t}{\exists x A}\,\exists I^{\mathbf{E}} \qquad \frac{\exists y A[x/y] \quad \begin{matrix}[A][\mathbf{E}x]\\ \vdots \\ C\end{matrix}}{C}\,\exists E^{\mathbf{E}}.$$

To turn it into a logic with equality we add a special relation $=$ and the rules

$$\text{EQEX} \quad \frac{t=t}{\mathbf{E}t} \qquad \text{EXEQ} \quad \frac{\mathbf{E}t}{t=t} \qquad \text{REPL} \quad \frac{A[x/t] \quad \mathbf{E}t \vee \mathbf{E}s \rightarrow t=s}{A[x/s]}.$$

Finally, for a given language $\mathcal{L}$, we add rules for all relation and function symbols representing the assumption of *strictness*:

$$\text{STRR} \quad \frac{R(t_1,\ldots,t_n)}{\mathbf{E}t_i} \qquad \text{STRF} \quad \frac{\mathbf{E}f(t_1,\ldots,t_n)}{\mathbf{E}t_i}.$$

The resulting system is called **IQCE**.

Properties. The following are derivable in **IQCE**.

1. $\mathbf{E}t \leftrightarrow t=t \leftrightarrow \exists x(t=x)$,
2. $t=s \leftrightarrow \exists x(t=x \wedge s=x)$,
3. $f(\vec{t})=x \leftrightarrow \exists \vec{y}(\vec{y}=\vec{t} \wedge f(\vec{y})=x)$.

5 Nonglobal Ω-structures

We work in a fixed one-sorted **IQCE**-language $\mathcal{L}$. let Ω be a fixed cHa. Write .

Definition. A nonglobal Ω-structure for $\mathcal{L}$ is consists of a pair $(M, [\![\cdot=\cdot]\!])$ containing a set M and a function $[\![\cdot=\cdot]\!] : M \times M \rightarrow \Omega$ such that for all $x,y,z \in M$,

$$[\![x=y]\!] = [\![y=x]\!], \qquad [\![x=y]\!] \wedge [\![y=z]\!] \leq [\![x=z]\!],$$
$$E(x) := [\![x=x]\!], \qquad [\![\vec{x}=\vec{y}]\!] := \bigwedge [\![x_i=y_i]\!],$$

together with Ω-interpretations for all symbols in $\mathcal{L}$ such that for all relations R and functions f

$$[\![\vec{a} = \vec{b}]\!] \wedge R(\vec{a}) \leq R(\vec{b}) \qquad\qquad E(f\vec{a}) \wedge [\![\vec{a} = \vec{b}]\!] \leq [\![f\vec{a} = f\vec{b}]\!]$$
$$[\![R(\vec{a})]\!] \leq [\![E(\vec{a})]\!] \qquad\qquad E(f\vec{a}) \leq E\vec{a}.$$

Semantics. We extend $[\![\ ]\!]$ as before, where $[\![\cdot = \cdot]\!]$ is the interpretation of $=$ and E of **E**. The interpretations of the quantifiers are adapted to

$$[\![\forall x A(x)]\!] := \bigwedge\{[\![E(m) \to A(m)]\!] \mid m \in M]\},$$
$$[\![\exists x A(x)]\!] := \bigvee\{[\![E(m) \wedge A(m)]\!] \mid m \in M\}.$$

6 Soundness and completeness

Theorem 1 (Soundness, Troelstra & van Dalen, 6.7)**.** *If* $\textbf{\textit{IQCE}} + \Gamma \vdash A$ *for a set of sentences* Γ *and a sentence* A*, then* $[\![A]\!] = \top$ *in each* Ω*-model for which* $[\![B]\!] = \top$ *for all* $B \in \Gamma$*, we write* $\Gamma \Vdash_{cHa} A$*.*

Definition. Let Θ be a Ha. A Θ-*structure* is defined exactly as a Ω-structure. Of a Θ-structure $(M, [\![\cdot = \cdot]\!])$ for some language $\mathcal{L}$ we say that it is *definitionally complete* w.r.t. $\mathcal{L}$ if for all $\mathcal{L}$-formulae $B(\vec{x})$ such that $[\![B(\vec{m})]\!] \in \Theta$ for all $\vec{m} \in M$, we have

$$\bigvee\{E\vec{m} \wedge [\![B\vec{m}]\!]\} \in \Theta, \qquad\qquad \bigwedge\{E\vec{m} \to [\![B\vec{m}]\!]\} \in \Theta.$$

Theorem 2 (Troelstra & van Dalen, 6.12)**.** *Let* Γ *be and* $\mathcal{L}$*-theory. Then there is a definitionally complete* Θ*-structure in which*

$$\Gamma \vdash A \Leftrightarrow [\![A]\!] = \top.$$

Theorem 3 (Troelstra & van Dalen, 6.13)**.** *Any Ha can be embedded in a cHa preserving* $\wedge, \vee, \to$ *,* $\bot$ *and all existing meets and joins.*

Theorem 4 (Completeness, Troelstra & van Dalen, 6.15)**.** $\Gamma \vdash A \Leftrightarrow \Gamma \Vdash_{cHa} A$*.*