Seminar on Models of Intuitionism

Hand-out lecture 9

20 April

1 Muchnik Degrees

Definition 1.1. A *mass problem (mp)* is a subset of $\mathbb{N}^{\mathbb{N}}$.

Definition 1.2. A *solution* to a mass problem $\mathcal{A}$ is a recursive function $f \in \mathcal{A}$.

Definition 1.3. For $g\colon \mathbb{N} \to \mathbb{N}$, the class of *$g$-partial recursive functions* is the smallest class of functions

- containing g and the initial functions;
- closed under composition, primitive recursion and minimalisation.

Definition 1.4. Let $f, g\colon \mathbb{N} \to \mathbb{N}$. We say that f is *Turing reducible* to g (written $f \leq_T g$) if f is g-recursive.

Definition 1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$ (written $\mathcal{A} \leq_w \mathcal{B}$) if for every $g \in \mathcal{B}$ there is some $f \in \mathcal{A}$ with $f \leq_T g$.

Definition 1.6. We write $\equiv_w$ for the equivalence relation generated by $\leq_w$ and we call the equivalence class $[\mathcal{A}]$ of a mass problem $\mathcal{A}$ its *Muchnik degree*. Further, we write $\mathfrak{M}_w$ for poset of Muchnik degrees, where $[\mathcal{A}] \leq [\mathcal{B}]$ iff $\mathcal{A} \leq_w \mathcal{B}$.

Definition 1.7. A *Muchnik mass problem (Mmp)* $\mathcal{A}$ is a mass problem satisfying: if $f \in \mathcal{A}$ and $f \leq_T g$, then $g \in \mathcal{A}$.

Lemma 1.8. *For every mass problem $\mathcal{A}$, there is a unique Muchnik mass problem $C(\mathcal{A})$ such that $\mathcal{A} \equiv_w C(\mathcal{A})$.*

Lemma 1.9. *The set $\mathfrak{M}_w$ is a complete (bounded) lattice with least element the Muchnik degree of any mass problem containing a recursive function and top element $[\emptyset]$. Joins and meets are given by: $\bigwedge_{i\in I}[\mathcal{A}_i] = \left[\bigcup_{i\in I}\mathcal{A}_i\right]$ and $\bigvee_{i\in I}[\mathcal{A}_i] = \left[\bigcap_{i\in I} C(\mathcal{A}_i)\right]$ where $\{A_i \mid i \in I\}$ is a collection of mass problems.*

Remark 1.10. We will extend the notation for joins and meets to mass problems, for a collection of mass problems $\{\mathcal{A}_i \mid i \in I\}$ we will write $\bigwedge_{i\in I}\mathcal{A}_i$ for $\bigcup_{i\in I} A_i$ and $\bigvee_{i\in I}\mathcal{A}_i$ for $\bigcap_{i\in I} C(\mathcal{A}_i)$.

Lemma 1.11. *The Muchnik degree of the set* $0' \coloneqq \{g \colon \mathbb{N} \to \mathbb{N} \mid g >_T Z\}$ *is the least non-zero Muchnik degree.*

Lemma 1.12. *The lattice* $\mathfrak{M}_w$ *is a Heyting algebra.*

2 Logic in Heyting Algebras

Definition 2.1. Let H be a Heyting algebra. A propositional formula φ with n variables is *true in* H if $\varphi(a_1, \dots, a_n) = 1$ for all $a_1, \dots, a_n \in H$. The set of true propositional formulas in H is denoted by $\mathrm{Th}(H)$.

Lemma 2.2. *For any Heyting algebra* H *we have* $\mathbf{IPC} \subseteq \mathrm{Th}(H)$*.*

Lemma 2.3. (Jaśkowski) $\mathbf{IPC} = \bigcap\{\mathrm{Th}(H) \mid H$ *a finite Heyting algebra*$\}$.

Lemma 2.4. *Suppose* H_0 *and* H_1 *are Heyting algebras and suppose* $F \colon H_0 \to H_1$ *is a Heyting algebra homomorphism. We have*

(1) If F *is injective, then* $\mathrm{Th}(H_1) \subseteq \mathrm{Th}(H_0)$*.*

(2) If F *is surjective, then* $\mathrm{Th}(H_0) \subseteq \mathrm{Th}(H_1)$*.*

3 Not Double Diamond-Like Lattices

Definition 3.1. Let L be a lattice. An element $a \in L$ is called *join-irreducible* if for any $b, c \in L$ we have $a = b \vee c$ implies $a = b$ or $a = c$. In a similar fashion we also define *meet-irreducibility.*

Definition 3.2. A lattice L is called *not double diamond-like (not dd-like)* if whenever $a, b \in L$ are join-irreducible, then so is $a \wedge b$.

Lemma 3.3. *Let* H *be a Heyting algebra and let* $a, b \in H$ *be such that* $a < b$*. Then* $H[a, b] := \{x \in H \mid a \leq x \leq b\}$ *is again a Heyting algebra with* $0_{[a,b]} = a$*,* $1_{[a,b]} = b$ *and* $x \to_{[a,b]} y = b \wedge (x \to y)$*. We abbreviate* $H[0, b]$ *by* $H(\leq b)$ *and* $H[a, 1]$ *by* $H(\geq a)$*.*

Lemma 3.4. (Terwijn) *A finite distributive lattice is isomoprhic to an initial segment of* $\mathfrak{M}_w$ *iff it is not dd-like and 0 is meet-irreducible.*

Definition 3.5. We call a lattice *nice* if is finite and not dd-like.

Definition 3.6. We write I_1 for the two element Heyting algebra. Given two Heyting algebras A and B, let $A + B$ be the Heyting algebra by stacking B on top of A and identifying 1_A with 0_B. For any Heyting algebra H, we write $H^+ = H + I_1$ and $H_+ = I_1 + H$.

Lemma 3.7. $\mathbf{IPC} = \bigcap\{\mathrm{Th}(H) \mid H$ *a nice Heyting algebra with 1 join-irreducible*$\}$

4 Logic of Muchnik Degrees

Theorem 4.1. $\mathrm{Th}(\mathfrak{M}_w(\geq 0')) = \mathbf{IPC}$.

Definition 4.2. A proposition formula φ is called *positive* if it does not contain $\bot$ and $\neg$. Given a set I of propositional formulas, we define the *positive fragment* of I as $I^{\mathrm{pos}} := \{\varphi \in I \mid \varphi \text{ is positive}\}$.

Definition 4.3. Let $\mathbf{IPC} + \neg\varphi \vee \neg\neg\varphi$ denote the intermediate propositional logic obtained by adding to $\mathbf{IPC}$ the *weak law of excluded middle* (WLEM), i.e. the axiom scheme $\neg\varphi \vee \neg\neg\varphi$ where φ is any propositional formula.

Lemma 4.4. (Jankov) $\mathbf{IPC} + \neg\varphi \vee \neg\neg\varphi$ *is the* $\subseteq$*-largest intermediate logic* I *containing WLEM and such that* $\mathbf{IPC}^{\mathrm{pos}} = I^{\mathrm{pos}}$.

Lemma 4.5. *Let* H *be a Heyting algebra. Then* $\mathrm{Th}(H_+)^{\mathrm{pos}} \subseteq \mathrm{Th}(H)^{\mathrm{pos}}$.

Lemma 4.6. *If* H *is a Heyting algebra with 0 meet-irreduicble, then for any propositional formula* φ *we have* $\neg\varphi \vee \neg\neg\varphi \in \mathrm{Th}(H)$*, i.e.* $WLEM \subseteq \mathrm{Th}(H)$.

Theorem 4.7. $\mathrm{Th}(\mathfrak{M}_w) = \mathbf{IPC} + \neg\varphi \vee \neg\neg\varphi$.