Seminar on Constructible Sets
Handout Session 1

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1 Review of Set Theory

This first part gives a brief review of some of the important concepts from an introduction to set theory, such as was taught during the Grondslagen course. This is by no means a complete abstract, so if any of this section seems unfamiliar, we would like to advise to (re)read the syllabus for Grondslagen, sections $I.1$ to $I.4$ in [Devlin] or chapters 1 and 2 in [Jech] for more in-depth information.

1.1 Language

The language of Zermelo-Fraenkel Set Theory (ZFC) consists of the language first-order logic, complemented with a symbol for set membership. That is, the language has logical connectives $\wedge$ (and), $\neg$ (not), the quantifier $\exists$ (there exists), predicates $\in$ (membership) and $=$ (equality), variables $v_0, v_1, \ldots$, and brackets (,).
From these we can define other symbols, such as $\vee$, $\forall$, $\cup$, $\subset$, etc.
Devlin's book uses primarily:

- lowercase latin letters $(a, b, c, \ldots)$ to denote sets,
- uppercase latin letters $(A, B, C, \ldots)$ to denote (proper) classes
- lowercase greek letters $(\alpha, \beta, \gamma, \ldots)$ from the start of the alphabet for ordinals
- lowercase greek letters $(\kappa, \lambda, \mu, \nu, \ldots)$ from the middle of the alphabet for cardinals
- uppercase greek letters $(\Phi, \Psi, \ldots)$ for formulas.

Sometimes an arrow is used to signify strings of variable length, such as $\forall \vec{a}$ being an abbreviation of $\forall a_1 \forall a_2 \ldots, \forall a_n$.
One particular thing to note is that Devlin quantifies over ordinals and cardinals without defining the domain. In other words, when you encounter $\forall \alpha(\phi(\alpha, \vec{p}))$, this has On (the ordinals) as domain of discourse: $\forall \alpha(\alpha \in \mathrm{On} \rightarrow \phi(\alpha, \vec{p}))$. Similarly the notation $\forall \kappa(\phi(\kappa, \vec{p}))$ will quantify over the cardinals.

1.2 Axioms

Devlin presents eight axioms for ZFC, written in FOL. In English, these axioms mean as much as the following:

i) Axiom of Extensionality: "If two sets have the same elements, they are equal."

ii) Axiom of Union: "If x is a set, then $\bigcup x$ is a set."

iii) Axiom of Power set: "If x is a set, then $\mathcal{P}(x)$ is a set."

iv) Axiom of Infinity: "There is a nonempty set x such that for every set $y \in x$ there is a set that contains it $(y \in z \in x)$ in x." With induction it can be shown that this set has an infinite cardinality.

v) Axiom Schema of Comprehension: "If x is a set, then the subset of elements of x satisfying some property is a set."

vi) Axiom Schema of Collection: "If R is a serial relation on the universe V, then for a set x there is a set y that contains for any $a \in x$ at least one element reachable by R (so $a\ R\ b$ and $b \in y$ for some b)."

vii) Axiom of Foundation: "If x is a set, then there is an $\in$-minimal element y in x, that is, for each $z \in y$ we have $z \notin x$."

viii) Axiom of Choice: "There is a function f from a family of nonempty sets $\mathcal{A} = \{A_i \mid i \in I\}$ to its union $\bigcup \mathcal{A}$ such that $f(A_i) \in A_i$ for all $i \in I$." This is equivalent to saying the cartesian product of nonempty sets is nonempty.

The Axiom Schema of Comprehension is also known as Axiom Schema of Separation / Specification / Subset, while the Axiom of Foundation is sometimes called Axiom of Regularity. The Axiom Schema of Collection is equivalent to the Axiom Schema of Replacement under the other axioms of ZFC, but without this context it is slightly stronger than Replacement.
Using these axioms we can define common notions such as pairs, n-tuples, cartesian products, relations, functions, intersections, etc.

1.3 Ordinals

Definition. A set is *transitive* if $\forall x, y, z(x \in y \in z \to x \in z)$.

Definition. An ordinal is a set that is transitive and linearly ordered by $\in$.

The ordinals form a (proper) class On (in other literature also denoted as Ord or Ω). Every ordinal, as well as On, is a well-ordered class, so every subset contains a minimal element. It therefore always makes sense to talk about "the least ordinal such that ..."
For ordinals we have $\alpha = \{\beta \mid \beta < \alpha\}$, where $<$ is an alternative notation for $\in$.

Definition. An ordinal α is a *successor* if $\alpha = \beta \cup \{\beta\} = \beta + 1$ for some ordinal β. Otherwise it is a *limit*: $\gamma = \bigcup\{\beta \mid \beta < \gamma\} = \sup\{\beta \mid \beta < \gamma\}$.

Definition. An *isomorphism* between $(X, <)$ and $(Y, \prec)$ is a bijective function $f : X \to Y$ that is order preserving.

If X is well-ordered, it is isomorphic to an ordinal α. This ordinal is called the *order-type* of X.

Theorem 1 (Transfinite induction)**.** *Property $\phi(x)$ holds for all $x \in$ On if all of the following are provable:*

- $\phi(\varnothing)$,
- $\phi(\alpha) \to \phi(\alpha + 1)$,
- $\forall \beta < \gamma(\phi(\beta)) \to \phi(\gamma)$ *for limit* γ.

Theorem 2 (Transfinite recursion)**.** *Let G be a function defined on all sets, then there is a (unique) function F defined on* On *such that $F(\alpha) = G(F \restriction \alpha)$ for all $\alpha \in$* On.

1.4 Von Neumann Universe

Theorem 3 ($\in$-induction)**.** *The following is a theorem of ZF:*

$$\forall x(\forall y(y \in x \to \Phi(y)) \to \Phi(x)) \to \forall x \Phi(x)$$

Theorem 4 ($\in$-recursion)**.** *Let G be a function defined on all sets, then there is a (unique) function F defined on all sets such that $F(x) = G(F \restriction x)$ for all x in a set X.*

This shows that we can recursively define a function on X by defining it in terms of the elements $y \in x$ for every $x \in X$.

Definition. The *transitive closure* of x is defined recursively as:

$$TC(x) = x \cup \bigcup\{TC(y) \mid y \in x\}$$

$TC(x)$ is the smallest transitive superset of x.

Definition. The *cumulative hierarchy of sets* is defined recursively as:

$$\begin{aligned} V_0 &= \varnothing \\ V_{\alpha+1} &= \mathcal{P}(V_\beta) \\ V_\gamma &= \bigcup_{\beta<\gamma} V_\beta \quad \text{for } \gamma \text{ limit} \end{aligned}$$

Lemma 5. *The (Von Neumann) universe of sets V is equal to $\bigcup\{V_\alpha \mid \alpha \in \mathrm{On}\}$.*

Definition. The *rank* of set x is the least ordinal α such that $x \in V_{\alpha+1}$.

Lemma 6. *If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.*

2 Cardinals

This chapter makes use heavily of the Axiom of Choice. Without it, many of these results become false.

Definition. For two sets A and B, the *cardinality* of A is less than or equal to B, denote $|A| \leq |B|$ if there exists an injection $f : A \hookrightarrow B$. If f is a bijection, then the sets are of equal cardinality, $|A| = |B|$.

Definition. An ordinal κ is a *cardinal (number)* if for every $\alpha < \kappa$ we have $|\kappa| \not\leq |\alpha|$.

Lemma 7. *For every set A there is a cardinal κ such that $|A| = |\kappa|$.*

Definition. The successor of κ is the least cardinal larger than κ, denoted κ^+. If $\kappa = \lambda^+$ for some λ, we call κ a successor cardinal. Otherwise κ is a limit ordinal, and $\kappa = \bigcup_{\lambda<\kappa} \lambda$.

The class of cardinals is well-ordered, so it can be put in one-one correspondence with On:

$$\begin{aligned} \aleph_0 &= |\omega| = |\omega_0| \\ \aleph_{\alpha+1} &= \aleph_\alpha^+ = |\omega_{\alpha+1}| = |\omega_\alpha|^+ \\ \aleph_\gamma &= \sup_{\alpha<\gamma} \aleph_\alpha = \left| \bigcup_{\alpha<\gamma} \omega_\alpha \right| = |\omega_\gamma| \end{aligned}$$

Interpreted as ordinals $\aleph_\alpha$ and ω_α are equal. $\aleph_\alpha$ is used whenever it is supposed to be interpreted as a cardinal. In Devlin, the convention is to use ω_α mostly as a cardinal and eventual interpretation as an ordinal will be made explicit.

Definition. Cardinal arithmetic is defined as follows. Let $|K| = \kappa$ and $|L| = \lambda$ for sets K, L and infinite cardinalities κ and λ.

$$\begin{aligned} \kappa + \lambda &= |K \sqcup L| \quad \text{(disjoint union of } K \text{ and } L) \\ \kappa \cdot \lambda &= |K \times L| \quad \text{(cartesian product of } K \text{ and } L) \\ \kappa^\lambda &= \left|{}^L K\right| \quad \text{(set of function from } L \text{ to } K) \end{aligned}$$

For infinite sums and products, let $\langle \kappa_\alpha \rangle_{\alpha<\lambda}$ be a λ-sequence of nonzero cardinals and let $|K_\alpha| = \kappa_\alpha$ for all $\alpha < \lambda$. **Note that the infinite sum is defined incorrectly in Devlin!**

$$\begin{aligned} \sum_{\alpha<\lambda} \kappa_\alpha &= \left| \bigsqcup_{\alpha<\lambda} K_\alpha \right| = \lambda \cdot \sup_{\alpha<\lambda} \kappa_\alpha \\ \prod_{\alpha<\lambda} \kappa_\alpha &= \left| \prod_{\alpha<\lambda} K_\alpha \right| = \left(\sup_{\alpha<\lambda} \kappa_\alpha \right)^\lambda \end{aligned}$$

Lemma 8 (Devlin I.5.3; Jech 5.10: König's lemma)**.** *If $\kappa_\alpha < \lambda_\alpha$ for all $\alpha < \beta$, then $\sum \kappa_\alpha < \prod \lambda_\alpha$.*

Corollary 9 (Cantor's theorem)**.** *For any κ we have $\kappa < 2^\kappa$.*

Definition. The *Continuum Hypothesis* (CH) states that $\omega_1 = 2^{\omega_0}$. The *Generalised Continuum Hypothesis* (GCH) states that $\omega_{\alpha+1} = 2^{\omega_\alpha}$.

Definition. Let α be a limit ordinal. A function $f : \gamma \to \alpha$ is *cofinal* in α if f is order-preserving and the range of f has supremum α. The *cofinality* of α is the least ordinal γ such that there is a cofinal function $f : \gamma \to \alpha$, denoted $\gamma = \mathrm{cf}(\alpha)$.

Definition. A limit ordinal α is called *regular* if $\mathrm{cf}(\alpha) = \alpha$. It is called *singular* if $\mathrm{cf}(\alpha) < \alpha$.

Lemma 10. *Every regular ordinal is a cardinal. For every α we have* $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

For many results about cardinal arithmetic and cofinality, refer to the literature.

3 Closed Unbounded Sets

Definition. Let α be a limit ordinal.

- A set $A \subseteq \alpha$ is *unbounded* in α if, and only if, $(\forall \nu \in \alpha)(\exists \tau \in \alpha)(\tau \geqslant \nu)$.
- A limit ordinal γ is called a *limit point* of a set A if, and only if, $A \cap \gamma$ is unbounded in γ. Equivalently, if $\sup(A \cap \gamma) = \gamma$.
- A set $A \subseteq \alpha$ is *closed* in α if, and only if, A contains all its limit points below α.

We say that $A \subseteq \alpha$ is "club" in α if it is closed and unbounded in α.

Lemma 11 (Devlin I.6.1; Jech 8.2)**.** *Let κ be an infinite cardinal with cf(κ)>ω. If $A, B \subseteq \kappa$ are club, then $A \cap B$ is club in κ.*

For the rest of this section we shall assume that κ is a uncountable regular cardinal.

The previous lemma can be generalised to intersections of uncountable families of club subsets:

Lemma 12 (Devlin I.6.2; Jech 8.3)**.** *Let $\lambda < \kappa$ and let $A_\nu, \nu < \lambda$, be club subsets of κ. Then $\bigcap_{\nu<\lambda} A_\nu$ is club in κ.*

Definition. Let α be an infinite ordinal. Let $f\colon \alpha \to \mathrm{On}$ be a non-decreasing function. We say f is *continuous* if for every limit ordinal $\gamma < \alpha$ we have

$$f(\gamma) = \bigcup_{\beta<\gamma} f(\beta).$$

We say $f\colon \alpha \to \mathrm{On}$ is *normal* if it is (strictly) increasing and continuous.

Some results concerning clubs:

Lemma 13 (Devlin I.6.3)**.** *Let α be a limit ordinal, and let $f\alpha \to \mathrm{On}$ be an increasing function. Then $f(\alpha) \geqslant \alpha$.*

Lemma 14 (Devlin I.6.5)**.** *Let $f\colon \kappa \to \kappa$ be a normal function. Then the set $\{\alpha \in \kappa \mid f(\alpha) = \alpha\}$ is club in κ.*

Lemma 15 (Devlin I.6.6)**.** *Let $h\colon \kappa \to \kappa$. Then the set $\{\gamma \in \kappa \mid (\forall \nu < \gamma)(h(\nu) < \gamma)\}$ is club in κ.*

4 The Collapsing Lemma

Definition. A set X is *extensional* if we have that $(\forall u, v \in X)(u \neq v \to (\exists x \in X)(x \in u \leftrightarrow x \notin v))$.

The following theorem is called the Mostowski-Shepherdon Collapsing Lemma. Check [Jech, pg. 67-69] for a more general treatment using classes and well-founded binary relations. The case of sets and the usual membership relation $\in$ is covered as a particular instance of the general results.

Theorem 16 (Devlin I.7.1; Jech 6.15)**.** *Let X be an extensional set.*

1. *There exists a unique transitive set M and a unique bijection $\pi\colon X \leftrightarrow M$ such that*

$$\pi\colon \langle X, \in\rangle \cong \langle M, \in\rangle,$$

i.e. π is an $\in$-isomorphism.

2. *If $Y \subseteq X$ is transitive, then $\pi \restriction Y = id \restriction Y$.*

Definition. The transitive set M in the Collapsing Lemma is called the *transitive collapse* or *transitivisation* of X.

Exercises

Please hand in your solutions on Wednesday, February 28th.

Exercise 1. Lemma 6.2 states that if κ is *regular* and *uncountable*, $\lambda < \kappa$, and $\mathcal{A} = \{A_i \mid i < \lambda\}$ is a family of club subsets of κ, then $\bigcap \mathcal{A}$ is club in κ. Explain, for each case below, whether this also holds:

(i) if κ is *singular*;

(ii) if κ is *countable*;

(iii) if $\lambda = \kappa$.

Exercise 2. (Devlin I.6.4) Let κ be an uncountable regular cardinal.

(i) If $A \subseteq \kappa$ is club, then the enumeration of A in increasing order (as ordinals) is a normal function from κ to κ.

(ii) If $f\colon \kappa \to \kappa$ is a normal function, then $\mathrm{ran}(f)$ is a club subset of κ.

Exercise 3. Determine whether $\langle \omega, \in \rangle$ and $\langle V_\omega, \in \rangle$ are models for the Axiom of Extensionality. Justify your answers.

References

- Devlin, K. J. *Constructibility.* Berlin [etc.]: Springer-Verlag, 1984. (Perspectives in mathematical logic).

- Jech, T. J. *Set Theory.* 3rd ed. Berlin [etc.]: Springer, 2002. (Springer monographs in mathematics).