Seminar Constructible Set Theory: Handout 7

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Today, we will look at the Souslin Problem: a question about whether certain conditions are sufficient to characterize the real number up to order-isomorphism. The Souslin problem cannot be solved in $\mathbf{ZFC} + \mathbf{GCH}$, but can be solved in $\mathbf{ZF} + V = L$.

1 Ordered Sets and $\mathbb{R}$

Definition 1. A *densely ordered set* is a linearly ordered set $\langle X, \leq \rangle$ such that whenever $x, z \in X$ and $x < z$, there is a $y \in X$ such that $x < y < z$.

We will sometimes denote $\langle X, \leq \rangle$ simply by X.

Definition 2. An *interval* in a linearly ordered set $\langle X, \leq \rangle$ is a subset of X of the form

$$(x, z) = \{y \in X \mid x < y < z\}.$$

Definition 3. An *ordered continuum* is a densely ordered set such that every non-empty subset of every interval has an infimum and a supremum.

Definition 4. A linearly ordered set is said to be *open* if it has no end-points.

Definition 5. A subset Y of a densely ordered set $\langle X, \leq \rangle$ is said to be *dense* in X if for any $x, z \in X$ such that $x < z$ there is a $y \in Y$ such that $x < y < z$.

Theorem 1 (Cantor)**.** *Every open, ordered continuum containing a countable dense subset is order-isomorphic to* $\mathbb{R}$.

2 The Souslin Property and Souslin's Hypothesis

Definition 6. We say a linearly ordered set X *has the Souslin Property* if every set of pairwise disjoint, non-empty intervals of X is countable.

The Souslin Hypothesis states that every open, ordered continuum with the Souslin Property is order-isomorphic to $\mathbb{R}$.

Lemma 2. *The Souslin Hypothesis holds iff every densely ordered set with the Souslin Property has a countable dense subset.*

Souslin's Hypothesis is independent of ZFC. However, it fails when $V = L$.

3 Trees

Definition 7. A *tree* is a partially ordered set $\mathbf{T} = \langle T, \leq_T \rangle$ such that for every $x \in T$, the set

$$\hat{x} = \{y \in T \mid y <_T x\}$$

is well-ordered by $\leq_T$.

Definition 8. The *height* of an element x in a tree $\mathbf{T}$ is the order-type of $\hat{x}$ under $<_T$; that is, the unique ordinal α such that there is an order-isomorphism between $\hat{x}$ and α.

We denote the height by $ht_{\mathbf{T}}(x)$.

Definition 9. A *level* of a tree $\mathbf{T}$ is the set containing all elements of a certain height.

For any ordinal α, we denote the α-th level of $\mathbf{T}$ by T_α; this is the set

$$T_\alpha = \{x \in T \mid ht_{\mathbf{T}}(x) = \alpha\}.$$

We use $T \restriction \alpha$ to denote $\bigcup_{\beta<\alpha} T_\beta$ and $\mathbf{T} \restriction \alpha$ for the restriction of the structure $\mathbf{T}$ to this set.

Definition 10. A *branch* of $\mathbf{T}$ is a downwards-closed linearly ordered subset b of T. A branch is *maximal* if it is not properly contained in any other branch. For any ordinal α, an *α-branch* is a branch with order-type α.

By the Axiom of Choice, every branch can be extended to a maximal branch.

Definition 11. An *antichain* of $\mathbf{T}$ is a subset c of T such that for all distinct $x, y \in c$, x and y are not comparable. An antichain is *maximal* if it is not properly contained in any other antichain.

By the Axiom of Choice, every antichain can be extended to a maximal antichain.

Definition 12. Let θ be an ordinal and λ a cardinal. A tree $\mathbf{T}$ is a *(θ, λ)-tree* if the following conditions hold:

(i) $(\forall \alpha < \theta)(T_\alpha \neq \emptyset)$;

(ii) $T_\theta = \emptyset$;

(iii) $(\forall \alpha < \theta)(|T_\alpha| < \lambda)$.

Definition 13. A tree $\mathbf{T}$ has *unique limits* if whenever α is a limit ordinal and $x, y \in T_\alpha$, if $\hat{x} = \hat{y}$ then $x = y$.

Definition 14. A (θ, λ)-tree $\mathbf{T}$ is *normal* if it has unique limits and the following conditions hold:

(i) $|T_0| = 1$;

(ii) If $\alpha, \alpha + 1 < \theta$ and $x \in T_\alpha$, there there exist distinct $y_1, y_2 \in T_{\alpha+1}$ such that $x <_T y_1$ and $x <_T y_2$;

(iii) if $\alpha < \beta < \theta$ and $x \in T_\alpha$, there is a $y \in T_\beta$ such that $x <_T y$.

For infinite cardinals κ, a κ-tree is a normal (κ, κ)-tree.

Lemma 3. *Every ω_0-tree has an ω_0-branch.*

Definition 15. An *Aronszajn tree* is an ω_1-tree with no ω_1 branch.

Theorem 4 (1.1 in Devlin)**.** *There exists an Aronszajn tree.*

4 Souslin Trees

Definition 16. A *Souslin tree* is an ω_1-tree with no uncountable antichain.

Theorem 5 (1.2 in Devlin)**.** *Every Souslin tree is an Aronszajn tree.*

Lemma 6 (1.3 in Devlin)**.** *(i) Let $\mathbf{T}$ be an (ω_1, ω_1)-tree with unique limits, having no uncountable branch. Then there is a subset T^* of T such that, under the induced ordering, T^* is an Aronszajn tree.*

(ii) Let $\mathbf{T}$ be an (ω_1, ω_1)-tree with unique limits, having no uncountable antichain. Then there is a subset T^* of T such that, under the induced ordering, T^* is a Souslin tree.

Theorem 7 (1.4 in Devlin)**.** *Souslin's Hypothesis is equivalent to the non-existence of a Souslin tree.*

Exercises

Exercise 1 (a). Let X be a subset of an interval $I \subset \mathbf{R}$, such that for every $q \in I \cap \mathbf{Q}$ and every $k \in \mathbf{N}$, there is an $x(q, k) \in X \cap (q - 2^{-k}, q + 2^{-k})$. Show that there is a countable subset of X that is dense in I.

Exercise 1 (b). Show that if X is a dense subset of an interval $I \subset \mathbf{R}$, then it is contains a countable subset dense in I.

Exercise 2. Prove that the set $\langle X, <_X \rangle$ defined in the proof of the left-to-right implication of Theorem 1.4 of Devlin is a densely ordered set of cardinality 2^ω.