Exercise 1 We check the four points.

1. (1pt) It seems Devlin does not use this property in his proof. We have therefore awarded everyone this point, even if they didn't write anything down.

2. (1pt) In order to even define the relativisation of an LST-formula (with existential quantifiers) to a certain class, we need that class to be definable by an LST-formula. Furthermore, we need to allow the formulas defining the sets in the hierarchy to take parameters (the ordinals are suitable) since there are not enough LST-formulas to, without parameters, define more than countably many sets.

3. (1pt) This property is used to ensure that $(\exists y \in W_\beta)\Phi_j^W(y, \bar{x}_i)$ whenever $(\exists y \in W_{f_i(\bar{x}_i)})\Phi_j^W(y, \bar{x}_i)$, given that $(\forall \bar{x} \in W_\beta)(f_i(\bar{x}_i) < \beta)$.

4. (1pt) This property is *essential* when we employ the Axiom of Collection to find our limit ordinal β.

   One might argue also that it is being *used* when it's not essential, albeit helpful.

Exercise 2 (3 pt)

There are many possible Π_1 formulas that express "x is finite ". Our personal solution was as follows:

$$\Phi(x) = \forall y((\text{"}y \text{ is non-empty"} \wedge y \subseteq \mathcal{P}(x)) \to \text{"}y \text{ contatins a } \subseteq\text{-minimal element"})$$

Assume that x is a finite set. Then we have that $|\mathcal{P}(x)|$ must also be finite, since the power set has cardinality $2^{|x|}$. We now work by contradiction. Assume that y is a non-empty subset of $\mathcal{P}(x)$ and that y contains no $\subseteq$ minimal element. That means that we have an infinite sequence (a_n) of elements of y, such that $a_n \subset a_{n+1}$. We can see that all of the a_n can be chosen to be distinct. Since all of these sets in the sequence (a_n) must be elements of $\mathcal{P}(x)$, we see that $\mathcal{P}(x)$ must be infinite, but that is in contradiction with what we said before.

For the other direction we first prove the following claim that $\Phi(x)$ implies the following: if $q \subseteq \mathcal{P}(x)$ such that $\emptyset \in q$ and $(y \in q \wedge u \in x \to y \cup \{u\} \in q)$, then $x \in q$.

The proof is as follows. Let a q be given with the properties as listed. We look at the set $b = \{x \setminus y | y \in q\}$. Then we have by assumption that $\Phi(b)$ holds, so b has a $\subseteq$-minimal element b'. This however implies that q has a $\subseteq$-maximal element $x \setminus b'$. By the implication $y \in q \wedge u \in x \to y \cup \{u\} \in q$, we must have that $x \setminus b' = x$.

Assume now that $\Phi(x)$ holds. Look at the following collection of sets: $y = \{a \in x | a \text{ finite}\}$. Since we have that $\Phi(x)$ and y fulfils the prerequisites of the claim, we may conclude that $x \in y$. So x must be finite.

Another solution presented was "$\mathcal{P}(\mathcal{P}(x))$ is Dirichlet-finite", where the latter term means that any injective function from a set to itself is also surjective. This prove was awarded full points.

Solutions that used the axiom of choice, explicitly or not, had one point deducted if the proof was still correct.

Exercise 3 (3 pt)

First the proof from right to left. Let f be a function with the given properties and let A be a non-empty subset of X. We can look at the set $c = \{\alpha | \exists a \in A(f(a) = \alpha)\}$. Since the ordinals are well-ordered, we have that c has a least element γ, so $\forall a \in A(f(a) \geq \gamma)$. Since we have that $\exists a \in A(f(a) = \gamma$, take such an a. Now we must have that a is an E-minimal element of A, since xEa implies $f(x) < f(a)$, which is impossible.

For the other direction, assume that E is a well-founded relation. We will recursively construct an f that fulfils the prerequisites. We will say, for example, that $f(x) = \sup\{f(z)|zEx\} + 1$. Since E is well-founded, this recursive definition makes sense. We can now easily see that yEx implies that $f(y) \in \sup\{f(z)|zEx\}$, so $f(y) < \sup\{f(z)|zEx\} = f(x)$.