Seminar on Constructible Sets
Exercises Session 7

4th April 2018

Exercises

Exercise 1 (a). Let X be a subset of an interval $I \subset \mathbb{R}$, such that for every $q \in I \cap \mathbb{Q}$ and every $k \in \mathbb{N}$, there is an $x(q,k) \in X \cap (q - 2^{-k}, q + 2^{-k})$. Show that there is a countable subset of X that is dense in I.

Answer: The subset $X' = \{x(q,k) \,|\, q \in \mathbb{Q}, k \in \mathbb{N}\}$ is countable, being an image of $\mathbb{Q} \times \mathbb{N}$. It is also dense: let $a, b \in I$ with $a < b$ and let $k = \lceil -\log(b-a)/\log(2) \rceil + 1$ if $b - a < 1$ and 1 otherwise. Let $q = \frac{a+b}{2}$. Clearly, $a \leq q - 2^{-k} < q + 2^{-k} \leq b$ hence $a < x(q,k) b <$, as desired.

Exercise 1 (b). Show that if X is a dense subset of an interval $I \subset \mathbb{R}$, then it is contains a countable subset dense in I.

Answer: Since I is an interval, for every $q \in I$ and $k \in \mathbb{N}$, the set $I \cap (q - 2^{-k}, q + 2^{-k})$ is a non-empty interval and thus contains some element $x(q,k) \in X$. The result then follows by Exercise 1(a).

Exercise 2. Prove that the set $\langle X, <_X \rangle$ defined in the proof of the left-to-right implication of Theorem 1.4 of Devlin is a densely ordered set of cardinality 2^ω.

Answer: Given any two maximal branches $b, d \in X$, there is α such that $b(\alpha) \neq d(\alpha)$ and $b(\beta) = d(\beta)$ for all $\beta < \alpha$. Since T_α is linearly ordered, we have that either $b(\alpha) <_\alpha d(\alpha)$, and so $b <_X d$, or $d(\alpha) <_\alpha b(\alpha)$ and then $d <_X b$. Hence X is linearly ordered. Now, consider the successors of $b(\alpha)$ in $T_{\alpha+1}$, which are order-isomorphic to the rationals. Then we can find some $c(\alpha+1) \in T_{\alpha+1}$ satisfying $b(\alpha+1) <_{\alpha+1} c(\alpha+1)$. We can now consider a maximal branch c such that $b(\beta) = c(\beta)$ for all $\beta \leqslant \alpha$ and $c(\alpha+1)$ is the element of $T_{\alpha+1}$ we just picked, and this branch satisfies that $b <_X c <_X d$, so X is densely ordered.

For the cardinality, note that X has at least 2^ω elements, since we can easily find 2^ω different ω-branches by starting at 0 and picking a different successor at each level up to ω, and then each of these branches can be extended to a maximal branch. So this gives $2^\omega \leqslant |X|$. On the other hand, we are working with a Souslin tree, so every maximal branch is countable. Given any level of a maximal branch, there are ω possible sucessors to pick from in the next level. Hence, for any $\alpha < \omega_1$, we have at most $\omega^\omega = 2^\omega$ different branches with order type α. Hence there are at most $\sum_{\alpha<\omega_1} 2^\omega = \omega_1 \cdot 2^\omega = 2^\omega$ maximal branches, which gives $|X| \leqslant 2^\omega$.