Seminar Constructible Sets

Model solution session 9: Kurepa trees and Inaccessible cardinals

2018-04-25

Exercise 1

For an arbitrary set A, let $\bar{A} = A \cap L[A]$ and prove that for all ordinals α, $L_\alpha[A] = L_\alpha[\bar{A}]$ and thus $L[A] = L[\bar{A}]$.

Solution to exercise 1 (4pt)

We prove this by induction on α.

For $\alpha = 0$, we have that $L_0[A] = \emptyset = L_0[\bar{A}]$.

Next, suppose that $L_\alpha[A] = L_\alpha[\bar{A}]$ for some α, and prove the result for $\alpha + 1$: We start by noting two things:

(*) Since $L_\alpha[A] \subseteq L[A]$, we have that $A \cap L_\alpha[A] = A \cap L[A] \cap L_\alpha[A] = \bar{A} \cap L_\alpha[A]$.

(**) From the way we interpret $\mathring{A}$ in our structures, we get that $Def^A(L_\alpha[A]) = Def^{A\cap L_\alpha[A]}(L_\alpha[A])$ and $Def^{\bar{A}}(L_\alpha[\bar{A}]) = Def^{\bar{A}\cap L_\alpha[\bar{A}]}(L_\alpha[\bar{A}])$.

Thus, we get that:

$$\begin{aligned} L_{\alpha+1}[A] &= Def^A(L_\alpha[A]) \\ {\scriptstyle (**)} &= Def^{A\cap L_\alpha[A]}(L_\alpha[A]) \\ {\scriptstyle (*)} &= Def^{\bar{A}\cap L_\alpha[A]}(L_\alpha[A]) \\ {\scriptstyle (\text{ind. hyp.})} &= Def^{\bar{A}\cap L_\alpha[\bar{A}]}(L_\alpha[\bar{A}]) \\ {\scriptstyle (**)} &= Def^{\bar{A}}(L_\alpha[\bar{A}]) \\ &= L_{\alpha+1}[\bar{A}]. \end{aligned}$$

The limit case follows then immediately from the induction hypothesis as $L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A] = \bigcup_{\alpha<\lambda} L_\alpha[\bar{A}] = L_\lambda[\bar{A}]$ for limit ordinals λ.

In the same way we then get that $L[A] = \bigcup_{\alpha\in\mathbf{On}} L_\alpha[A] = \bigcup_{\alpha\in\mathbf{On}} L_\alpha[\bar{A}] = L[\bar{A}]$.

Exercise 2

We will prove Lemma 10 in detail. Let X be any set, prove the following facts. In each part you may of course use the preceding parts.

(a) Show that if κ is a cardinal in V, then κ is also a cardinal in $L[X]$.

(b) Show that for any ordinal α, we have that $\omega_\alpha^{L[X]}$ is an ordinal in V.

(c) Show that for any ordinal α we have $\omega_\alpha^{L[X]} \leq \omega_\alpha$ as ordinals in V.

Solution to exercise 2 (6pt)

We will make repeated use of Lemma 8.4 in Devlin's book to determine the complexity of a formula (in the Lévy hierarchy). Then because $L[X]$ is transitive and an inner model of **ZFC**, we can use absoluteness (Lemma 8.3 in Devlin's book).

(a) (2pt) Any cardinal κ in V is in particular an ordinal, and thus in $L[X]$. We are thus left to show that it is also a cardinal in $L[X]$.

By definition, an ordinal α is a cardinal if there is no ordinal $\beta < \alpha$ such that there is a surjective function $\beta \to \alpha$. So we can express "x is a cardinal" by the following LST-formula:

$$\forall f \forall b(\mathbf{On}(x) \wedge (\mathbf{On}(b) \wedge b \in x \wedge \text{``}f : b \to x'') \to x \neq \text{ran(f)}).$$

The part after the universal quantifiers is Σ_0, so the entire formula is Π_1 and therefore D-absolute. That means that for any $x \in L[X]$, if "x is a cardinal" holds in V, then it holds in $L[X]$. So we conclude that indeed any cardinal in V must also be a cardinal in $L[X]$.

(b) (1pt) The assertion "x is an ordinal" (denoted $\mathbf{On}(x)$) is Σ_0, so it is absolute. Thus the class of ordinals in $L[X]$ coincides with the class of ordinals in V. Any cardinal $\omega_\alpha^{L[x]}$ in $L[X]$ is in particular an ordinal in $L[X]$, and therefore an ordinal in V.

(c) (3pt) We prove this by induction to α. The case $\alpha = 0$ is trivial, since ω is the same in V and $L[X]$.

The limit step is also easy, because if λ is a limit ordinal and $\omega_\alpha^{L[X]} \leq \omega_\alpha$ for all $\alpha < \lambda$, we have that $\omega_\alpha^{L[X]} \subseteq \omega_\alpha$ for all $\alpha < \lambda$. So we conclude that

$$\omega_\lambda^{L[X]} = \bigcup_{\alpha<\lambda} \omega_\alpha^{L[X]} \subseteq \bigcup_{\alpha<\lambda} \omega_\alpha = \omega_\lambda,$$

and thus $\omega_\lambda^{L[X]} \leq \omega_\lambda$.

Finally, for the successor step, suppose that $\omega_\alpha^{L[X]} \leq \omega_\alpha$. By part (a) we have that $\omega_{\alpha+1}$ is a cardinal in $L[X]$, and we have $\omega_\alpha^{L[X]} \leq \omega_\alpha < \omega_{\alpha+1}$. So the next cardinal after $\omega_\alpha^{L[X]}$ in $L[X]$ can be at most $\omega_{\alpha+1}$, which is saying exactly that $\omega_{\alpha+1}^{L[X]} \leq \omega_{\alpha+1}$.

We conclude that indeed $\omega_\alpha^{L[X]} \leq \omega_\alpha$ for all ordinals α.