Notes on QM for logicians

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1 Classical Mechanics

Any physical system is described by three basic ingredients: *states*, *observables*, and *dynamics*. In classical mechanics, the state of a system is characterised by the position $\vec{x}(t)$ and the momentum $\vec{p}(t)$ (i.e. the product of mass and velocity, $\vec{p}(t) = m\vec{v}(t)$) of the system. Observables are measurable quantities like energy or momentum. The dynamics of a physical system are governed by Newton's three laws of motion:

- Law 1 Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
- Law 2 If a net external force is applied to an object, the acceleration of the object is proportional to the applied force.
- Law 3 For every action there is an equal and opposite reaction.

Let $\vec{F}$ be the net external force acting on ab object. Newton's second law then reads

$$\vec{F} = m\vec{a} = m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}. \tag{1}$$

If we introduce the *momentum* $\vec{p} = m\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$, Newton's second law reads

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \tag{2}$$

Newton's second law provides us with a system of second order linear differential equation in $\vec{x}(t)$. Together with the initial conditions

$$\vec{x}(t_0), \quad \left.\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right|_{t=t_0} \tag{3}$$

the time-evolution of the system is determined completely. In classical mechanics we visualize the time-evolution of the system in the *phase-space* of the system. For mechanical systems, e.g. a particle, the phase-space consists of all possible position and momentum variables.

2 The quantum principles

In quantum mechanics we also deal with states and observables and the dynamics thereof. However, in quantum mechanics we face a different situation altogether. The difference between the quantum world and the classical world are described by the two following fundamental quantum principles:

Principle 1 (Superposition principle) *Two (or more) quantum states can be added together ("superposed") and the result will be another valid quantum state.*

Principle 2 (Uncertainty principle of Heisenberg) *If the same preparation procedure on a*

physical system is repeated many times, and is followed either by a measurement of the position x, or by a measurement of the momentum p, the various results obtained for x and for p have standard deviations, Δx and Δp, whose product cannot be less than $\hbar/2$:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{4}$$

2.1 Implication of the quantum principles

Both quantum principles point us towards the new structure of quantum mechanics. The superposition principle leads us to consider linear spaces . These will turn out to be Hilbert spaces. Stated will now be interpreted as vectors in a suitable Hilbert space

The uncertainty principle can be deduced once when reinterprets the meaning of an observable; in classical mechanics one can in principle, if the measuring device is accurate enough, measure the system to infinite accuracy. The product of both standard deviations would be zero. However, in quantum mechanics this is no longer the case. There is a *fundamental* uncertainty build in. And we can deduce this uncertainty if we reinterpret observables. Observables are no longer functions of phase space but linear operators on Hilbert space. Moreover the uncertainty principle follows from non-commuting pairs of observables.

3 Postulates of Quantum Mechanics

Definition 3.1 *Quantum mechanics is a physical theory that deals with the structure and behaviour of elementary particles, or quanta.*

It is governed by the following (physical) postulates:

Postulate 1 *Each physical system corresponds to a complex Hilbert space $\mathscr{H}$. A state of the system is associated with a subspace of that Hilbert space.*

Postulate 2 *Physical observables are given by densely defined (i.e. defined on a dense linear subspace of $\mathscr{H}$) self-adjoint operators on the Hilbert space. The expected value of the observable A for the system in a state represented by the unit vector $\psi \in \mathscr{H}$ is given by $\langle \psi, A\psi \rangle$. One can show that the possible values for the observable must belong to the spectrum of the operator A. If the spectrum is discrete, the possible outcomes of measuring A are exactly its eigenvalues. After the measurement, the system will be in the eigenstate of A corresponding to the measured eigenvalue.*

Postulate 3 *The dynamics of the system is given by the time-evolution operator $U(t) \equiv e^{-itH}$, where H is the self-adjoint Hamilton operator corresponding to the total energy of the system. If $\psi(t)$ is the state at time t, then the state at time $t+s$ is given by*

$$\psi(t+s) = U(s)\psi(t) \tag{5}$$

for all $t, s \in \mathbb{R}$.

More generally the time-evolution of a quantum system is described by the time-dependent Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t). \tag{6}$$

4 Mathematical structure of Quantum Mechanics

4.1 Hilbert space

Definition 4.1 (Hilbert Space) *A Hilbert space is a $\mathbb{C}$-vector space $\mathscr{H}$ equipped with an inner product $\langle f, g\rangle$ such that the norm defined by*

$$|f| \equiv \sqrt{\langle f, f\rangle}$$

turns $\mathscr{H}$ into a complete metric space*, i.e. a metric space in which every Cauchy sequence is convergent. Recall that the norm $d(f,g) \equiv \sqrt{|f-g|}$ is induced by the inner product $\langle\ ,\ \rangle$ and satisfies*

1. $d(f,g) = 0 \iff f = g$.
2. $d(f,g) = d(g,f)$.
3. $d(f,g) + d(g,h) \geq d(f,h)$.

In quantum mechanics one often takes as a Hilbert space the infinite-dimensional Hilbert space $L^2(\mathbb{R}^n)$, i.e. the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the integral of $|f|^2$ over the whole space is finite. In this case, the inner product is defined by (* denotes complex conjugation)

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f^* g \, \mathrm{d}^n x. \tag{7}$$

Let $\mathscr{M}$ denote a subspace of a Hilbert space $\mathscr{H}$. Then the orthogonal complement $\mathscr{M}^\perp$ of $\mathscr{M}$ is defined to be

$$\mathscr{M}^\perp = \{f \in \mathscr{H} \mid \langle f, g\rangle = 0 \forall g \in \mathscr{M}\}.$$

Lemma 4.1 *A subspace $\mathscr{M}$ of a Hilbert space $\mathscr{H}$ is called dense if and only if $\mathscr{M}^\perp = \{0\}$.*

4.2 Self-adjoint operators

A linear operator is a function from a subspace $\mathscr{M} \subset \mathscr{H}$ to the Hilbert space $\mathscr{H}$.

The adjoint operator A^* of a densely defined operator A is defined by

$$\begin{aligned} \mathcal{D}(A^*) &= \{\psi \in \mathscr{H} \mid \exists \tilde{\psi} \in \mathscr{H} : \langle \psi, A\phi\rangle = \langle \tilde{\psi}, \phi\rangle \, \forall \, \phi \in \mathcal{D}(A)\}, \\ A^*\psi &= \tilde{\psi}. \end{aligned} \tag{8}$$

If the operator A is symmetric, i.e.

$$\langle \psi, A\psi\rangle = \langle A\psi, \psi\rangle \ \forall \psi \in \mathcal{D}(A) \tag{9}$$

and $A = A^*$ then we call A *self-adjoint.*

Theorem 4.2 (Stone's theorem) *Let $\{U(t) \mid t \in \mathbb{R}\}$ be a strongly[1] continuous one-parameter unitary group on a Hilbert space $\mathscr{H}$. Then there exists a unique self-adjoint operator A on $\mathscr{H}$ such that $U(t) = e^{iAt}$. The operator iA is called the infinitesimal* generator *of $U(t)$.*

If we take a closer look at the third postulate, the time-evolution of a quantum state:

$$\psi(t+s) = U(s)\psi(t),$$

then the set $U(s)$ constitutes a one-parameter unitary group. Stone's theorem then guarantees the existence of a unique

[1] Strongly continuous: $\forall t_0 \in \mathbb{R}, \psi \in \mathscr{H} : \lim_{t \to t_0} U_t\psi = U_{t_0}\psi$

self-adjoint operator H such that $U(s) = e^{-isH}$.

5 Derivation of the uncertainty principle

Let A and B be two non-commuting operators on a Hilbert space $\mathscr{H}$. The expectation values $\langle A\rangle$ and $\langle B\rangle$ of A and B in a given state ψ are given by

$$\langle A\rangle = \langle \psi, A\psi\rangle, \quad \langle B\rangle = \langle \psi, B\psi\rangle. \tag{10}$$

We then define the standard deviation ΔA (or uncertainty) of A to be

$$\Delta A = \sqrt{\langle (A - \langle A\rangle)^2\rangle} \tag{11}$$

Likewise for B. One can then show that

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B], \rangle\right| \tag{12}$$

where the brackets $[\,,\,]$ denote the *commutator* of A and B:

$$[A, B] \equiv AB - BA. \tag{13}$$

In quantum mechanics the fundamental commutator for position and momentum is

$$[x, p] \equiv i\hbar. \tag{14}$$

We take $\mathscr{H} = L^2(\mathbb{R})$ in this example. The operator p is taken to be the closure of $i\frac{\mathrm{d}}{\mathrm{d}x}$ with domain $\mathcal{S}(\mathbb{R})$, the Schwartz space or the space of rapidly decreasing functions. This operator is self-adjoint on $\mathcal{S}(\mathbb{R})$. The operator x is the operator T_x, which multiplies with the function $f(x) = x$.

The Schwartz space $\mathcal{S}(\mathbb{R})$, is the set off all Schwartz functions. A function $f \in C^\infty(\mathbb{R}^n)$ is called a Schwartz function if it goes to zero as $|x| \to \infty$ faster than any inverse power of x, as do all of its derivatives. That is, a function is a Schwartz function if there exist real constants $C_{\alpha\beta}$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta f(x)| \leq C^{\alpha\beta}, \tag{15}$$

where multi-index notation has been used for α and β. (i.e. $\alpha = (\alpha_1, \ldots, \alpha_n)$)