# Seminar on Set Theory

Solutions to exercise 12

January 8, 2016

## $\mathcal{P}\omega \cap L$ can be countable

(a) We claim that $V^{(B)} \models P\widehat{\lambda} \cap L \subseteq \widehat{P\lambda}$. To prove our claim, we show that

$$V^{(B)} \models \forall x[L(x) \to [x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]].$$

In other words, we want

$$\bigwedge_{x \in V^{(B)}} [\![L(x) \to [x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]]\!] = 1.$$

So take some $x \in V^{(B)}$. Then we need that

$$[\![L(x)]\!] \Rightarrow [\![x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]\!] = 1.$$

By Theorem 1.46, this expression equals

$$\left[\bigvee_{y \in L} [\![x = \hat{y}]\!]\right] \Rightarrow [\![x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]\!].$$

In order for this to equal 1, we need that the inequality

$$\left[\bigvee_{y \in L} [\![x = \hat{y}]\!]\right] \leq [\![x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]\!]$$

holds. So the proof of our claim is done, once we manage to show that

$$[\![x = \hat{y}]\!] \leq [\![x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]\!],$$

for all $y \in L$. So take $y \in L$. Then

$$[\![x = \hat{y}]\!] = [\![x = \hat{y}]\!] \wedge [\![\hat{y} \in P\widehat{\lambda} \to \hat{y} \in \widehat{P\lambda}]\!] \leq [\![x \in P\widehat{\lambda} \to x \in \widehat{P\lambda}]\!],$$

proving the claim. Here we used that $[\![\hat{y} \in P\hat{\lambda} \to \hat{y} \in \widehat{P\lambda}]\!] = 1$, which demands some explanation. This is because $v_1 \subseteq v_2 \to v_1 \in v_3$ is a restricted formula and $y \subseteq \lambda \to y \in P\lambda$ is a true statement. Theorem 1.23 (v) then tells us that $[\![\hat{y} \subseteq \hat{\lambda} \to \hat{y} \in \widehat{P\lambda}]\!] = 1$ and hence $[\![\hat{y} \in P\hat{\lambda} \to \hat{y} \in \widehat{P\lambda}]\!] = 1$.

Now, $|P\lambda| = |2^\lambda|$, so $V^{(B)} \models |\widehat{P\lambda}| = |\widehat{2^\lambda}|$, by (1.48). By Corollary 5.2 we have $V^{(B)} \models \widehat{2^\lambda}$ *is countable*, which therefore implies that $V^{(B)} \models \widehat{P\lambda}$ *is countable*. Since $V^{(B)} \models P\hat{\lambda} \cap L \subseteq \widehat{P\lambda}$ by our claim, it follows that $V^{(B)} \models P\hat{\lambda} \cap L$ *is countable.* ■

*Proving "$V^{(B)} \models P\hat{\lambda} \cap L \subseteq \widehat{P\lambda}$" was worth 4 points. Using this to conclude "$V^{(B)} \models P\hat{\lambda} \cap L$ is countable" was worth the remaining 2 points of this part of the exercise.*

(b) In our model $M$, we have $\aleph_1 = 2^{\aleph_0}$, so that $B$ is the collapsing $(\aleph_0, 2^{\aleph_0})$ algebra in $M$. So, by part (a), we have $M^{(B)} \models$ "$\mathcal{P}\hat{\omega} \cap L$ is countable". But the formula $x = \omega$ is restricted, so $M^B \models$ "$\mathcal{P}\omega \cap L$ is countable". By Corollary 4.2, this means $M^{(B)}/U \models$ "$\mathcal{P}\omega \cap L$ is countable". Since $M^{(B)}/U$ and $M[U]$ are isomorphic (by construction), we have $M[U] \models$ "$\mathcal{P}\omega \cap L$ is countable", as desired. ■

*Showing that $M^{(B)} \models$ "$\mathcal{P}\omega \cap L$ is countable" was worth 2 points. Proving that $M[U] \models$ "$\mathcal{P}\omega \cap L$ is countable" (by using Lemma 4.11 or the above) earned you another 2 points.*