Seminar on Set Theory

Hand-in exercise 13, model solution

December 18, 2015

Exercise 1

(a) We first notice that $\emptyset \subset \mathcal{F}$, hence $p_\emptyset \in C$, but this is just the empty join, and therefore the zero element. So $0 \in C$. We also notice that C has a supremum, because if $A \subseteq \mathcal{F}$ it is obvious that $p_A \leq p_\mathcal{F}$. We also notice that C is closed under finite joins, because if $A, A' \subseteq \mathcal{F}$ then $A \cup A' \subseteq \mathcal{F}$ and

$$p_A \vee p_{A'} = \bigvee_{f\in A} p_f \vee \bigvee_{f\in A'} p_f = \bigvee_{f\in A\cup A'} p_f = p_{A\cup A'}$$

Now we will show that C is closed under finite meets. Suppose $f, g \in 2^X$ such that $f \neq g$, then there must WLOG be an element $x_i \in X$ such that $f(x_i) = 0$ and $g(x_i) = 1$. We then find that

$$\begin{aligned}
p_f \wedge p_g &= \bigwedge_{y\in X} (p(y, f(y)) \wedge p(y, g(y))) \\
&= p(x_i, 0) \wedge p(x_i, 1) \wedge \bigwedge_{y\in X\setminus\{x_i\}} (p(y, f(y)) \wedge p(y, g(y))) \\
&= 0 \wedge \bigwedge_{y\in X\setminus\{x_i\}} (p(y, f(y)) \wedge p(y, g(y))) \\
&= 0
\end{aligned}$$

In much the same way we can also conclude that $p_f \vee p_g = 1$, so $\{p_f \mid f \in \mathcal{F}\}$ is an antichain, so in fact $p_\mathcal{F} = 1$. We now conclude that if $A, A' \subseteq \mathcal{F}$ then $A \cap A' \subseteq \mathcal{F}$ and

$$\begin{aligned}
p_A \wedge p_{A'} &= \bigvee_{f\in A, g\in A'} (p_f \wedge p_g) \\
&= \bigvee_{f\in A\setminus A', g\in A'} (p_f \wedge p_g) \vee \bigvee_{f\in A\cap A', g\in A'} (p_f \wedge p_g) \\
&= 0 \vee \bigvee_{f\in A\cap A'} (\bigvee_{g\neq f} (p_f \wedge p_g) \vee \bigvee_{g=f} (p_f \wedge p_g)) \\
&= \bigvee_{f\in A\cap A'} p_f \\
&= p_{A\cap A'}
\end{aligned}$$

So C is closed under taking finite meets. Now denote $\mathcal{F}\setminus A$ with A^c for any $A \subseteq \mathcal{F}$. We notice that $A^c \subseteq \mathcal{F}$ for any $A \subseteq \mathcal{F}$. We also notice that for any $A \subseteq \mathcal{F}$: $p_A \wedge p_{A^c} = p_\emptyset = 0$, and $p_A \vee p_{A^c} = p_\mathcal{F}$. So every element p_A of C has an inverse p_{A^c} in C. From these results it follows that C is a Boolean algebra. $\square$

The zero- and one-element is worth $\frac{1}{2}$ *point, and so is the join. The complement is worth* 1 *point and the meet is worth* 2 *points.*

(b) First we will prove that $C \subseteq B'$. Suppose $b \in C$, then we notice that $b = \bigvee_{f \in A} \bigwedge_{y \in X} p(y, f(y))$ for some $A \subseteq \mathcal{F}$. So b is a finite join of a finite meet of elements of B', because every $p(y, f(y))$ is a generator or the complement of a generator. And this implies that $b \in B'$, hence $C \subseteq B'$.
We will now show that $B' \subseteq C$ by showing that $X \subseteq C$. To do this we will prove the following: Suppose $Y \subseteq X$ and $f \in 2^Y$. Let $p_f = \bigwedge_{z \in Y} p(z, f(z))$. Then $p_f = \bigvee\{p_g \mid g \in \mathcal{F},\ g|_Y = f\}$. Because if $g \in \mathcal{F}$ such that $g|_Y = f$, then we notice that if $z \in Y$, we must have $p(z, g(z)) = p(z, f(z))$, hence

$$p_g = \bigwedge_{z \in X} p(z, g(z)) \leq \bigwedge_{z \in Y} p(z, f(z)) = p_f$$

So $p_g \wedge p_f = p_g$ for all such g. Now suppose that $g \in \mathcal{F}$ such that $g|_Y \neq f$, then there must be some $z \in Y$ such that $g(z) \neq f(z)$, hence $p(z, f(z)) \neq p(z, g(z))$, hence the meet of these two elements is $z \wedge z^* = 0$. So if $g|_Y \neq f$ then $p_g \wedge p_f = 0$. So we now combine these two ingredients to find

$$\begin{aligned} p_f &= p_f \wedge \bigvee_{g \in \mathcal{F}} p_g \\ &= \bigvee_{g \in \mathcal{F}} (p_g \wedge p_f) \\ &= \bigvee_{g \in \mathcal{F},\ g|_Y = f} (p_g \wedge p_f) \\ &= \bigvee\{p_g \mid g \in \mathcal{F},\ g|_Y = f\} \end{aligned}$$

So every element of the form p_f if a finite join of elements of C, hence it is an element of C.
We will now apply this to our problem. Let $x_i \in X$, and let $f \in 2^{\{x_i\}}$ such that $f(x_i) = 1$. Then we have just established that $p_f \in C$, but we notice that $p_f = p(x_i, f(x_i)) = x_i$. Hence $x_i \in C$, so $X \subseteq C$, and because C is a Boolean algebra the Boolean algebra generated by X must be contained in C as well, hence $B' \subseteq C$.
So now we have found that $B' = C$. □
The following components are each worth one point: proving that $C \subseteq B'$, proving that $p_f \wedge p_g = p_g$ for g extending f and 0 for g not extending f, proving that p_f is in C, concluding that $B' \subseteq C$.

(c) Suppose that $p_A \in C$, then p_A can be written in a unique way as a join of elements of the form p_f. Also, every join of elements of the form p_f is in C. So the elements of C are determined by the subsets of $\mathcal{F}$, hence the number of elements of C is at most the number of subsets of $\mathcal{F}$. Every element of $\mathcal{F}$ is determined by a function in 2^X, hence $|\mathcal{F}| \leq 2^n$ if $|X| = n$. And therefore $|C| \leq |\mathcal{P}(\mathcal{F})| = 2^{2^n}$. So a Boolean algebra generated by $n > 0$ elements contains at most 2^{2^n} elements. □
Concluding that $|C| \leq |\mathcal{P}(\mathcal{F})|$ is worth one point, and the conclusion is worth one point.