Seminar on Set Theory

Hand-in Lecture 11
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As always, let M be a transitive $\in$-model of ZFC and let $B \in M$ be a complete Boolean algebra in the sense of M.

Problem 1: Atoms in B[1]

An *atom* in B is an element $a \in B$ such that $a \neq 0$ and for all $x \in B$, $x \leq a$ implies $x = 0$ or $x = a$.

1. (1 point) Show that the set $U_a = \{x \in B \mid a \leq x\}$ is an ultrafilter in B.
2. (1 point) Show that U_a is M-generic.
3. (1 point) Show that $U_a \in M$ and deduce that $M[U_a] = M$.
4. (2 points) Let U be an ultrafilter in B such that $U \in M$ and put $a = \bigwedge U$. Show that the following are equivalent: (a) $a \neq 0$, (b) a is an atom, (c) $U = U_a$.

Problem 2: The Reflection Principle

A *cumulative hierarchy* is a sequence of sets indexed by ordinals, M_α, that satisfies

1. $M_\alpha \subseteq M_{\alpha+1} \subseteq \mathcal{P}(M_\alpha)$ for all α, and
2. $\bigcup_{\alpha<\lambda} M_\alpha = M_\lambda$ for λ a limit ordinal.

If the sequence M_α is a cumulative hierarchy we write $M(x)$ to denote $\exists\alpha.\, x \in M_\alpha$.

A class of ordinals C is *closed unbounded* (abbreviated *club*) iff

- It is closed: For every limit ordinal λ, $C(\lambda)$ holds whenever for every $\beta < \lambda$, there is a γ such that $\beta < \gamma < \lambda$ and $C(\gamma)$; and
- It is unbounded: For every ordinal α there is some ordinal $\beta > \alpha$ such that $C(\beta)$.

Part 1, 1 point: Show that the intersection of two closed unbounded classes is itself closed unbounded.

You may from now on assume that for any map F which sends ordinals to ordinals, the class

$$C = \{\alpha \mid \forall\beta.\, \beta < \alpha \Rightarrow F(\beta) < \alpha\}$$

[1] Problem 4.28 from the book.

is closed unbounded.

We wish to prove: For every cumulative hierarchy M and formula $\phi(x_1, \ldots, x_n)$, there is a closed unbounded class C of ordinals such that for every $\alpha \in C$,

$$\forall \vec{x} \in M_\alpha.\, \phi^{M_\alpha}(\vec{x}) \Leftrightarrow \phi^M(\vec{x}). \tag{1}$$

Here it is reasonable to regard ϕ^M as "ϕ with all quantification restricted to M", and idem for ϕ^{M_α}.

Since disjunction and universal quantification can be expressed in terms of the other connectives, you may assume they do not occur in ϕ. We prove by induction.

Part 2, 1 point: Prove the cases for atomic formulae, negation, and conjunction.

We now consider the existential case. Suppose $\phi = \exists y.\, \psi(\vec{x}, y)$; let C_ψ reflect $\psi(\vec{x}, y)$ by the inductive hypothesis.

Define $G(\vec{x})$ to be the least α such that there is some $y \in M_\alpha$ with $\psi^M(\vec{x}, y)$, or 0 if there is no such α. Now define

$$F(\beta) = \sup\{G(\vec{x}) \mid \vec{x} \in M_\beta\}.$$

Let $C_\phi = C_\psi \cap \{\alpha \mid \lim(\alpha)\} \cap \{\alpha \mid \forall \beta.\, \beta < \alpha \Rightarrow F(\beta) < \alpha\}$. Note that C_ϕ is club.

Part 3, 2 points: Show that C_ϕ satisfies (1).

Part 4, 1 point: Conclude that for every formula $\phi(x_1, \ldots, x_n)$ there is a closed unbounded class C of ordinals such that for every $\alpha \in C$,

$$\forall \vec{x} \in V_\alpha.\, \phi^{V_\alpha}(\vec{x}) \Leftrightarrow \phi(\vec{x}).$$